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SEMIGROUPS AND COSINE FUNCTIONS OF NORMAL OPERATORS IN HILBERT SPACES

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0. INTRODUCTION

In this paper, we are concerned with properties of semigroups and cosine functions of normal operators in Hilbert spaces. There are four sections: 1) Some Hilbert spaces notions and lemmas; 2) Growth of semigroups and cosine functions of normal operators; 3) Generators of semigroups and cosine functions of normal operators; 4) Generation of semigroups and cosine functions of normal operators.

We shall employ frequently notations and results of [3] (and partially of [1]). All Hilbert spaces are assumed to be real, but results are provable also in complex Hilbert spaces.

1. SOME HILBERT SPACES NOTIONS AND LEMMAS

1,0. Orientation. In this section, we give an outline of results of general character which are necessary in the sequel. These results are mostly known, only some complements are proved.

1,1. Definition. If E is a (real) Banach space, then we define so called *scalar product* in E as follows: $\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$ for every $x, y \in E$. Evidently $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in E$.

1,2. Definition. A (real) Banach space E is called *Hilbert space* if $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for every $x, y \in E$. In this paper E will denote always a Hilbert space.

1,3. Lemma. *If E is Hilbert space, then the scalar product $\langle x, y\rangle$ is continuous bilinear function on $E \times E$.*

1,4. Lemma. *If $l \in E'$, then there exists one and only one $x \in E$ such that $l(y) = \langle x, y \rangle$ for every $x \in E$. Element x is called representant of l.*

1,5. Lemma. *Every closed subspace of a Hilbert is again a Hilbert space.*

1,6. Definition. If $A \in \mathfrak{L}^+(E)$, then we define the adjoint of A, denoted A^*, as follows: $x \in \mathfrak{D}(A^*)$ if and only if $\langle x, A(y) \rangle$ is continuous on $\mathfrak{D}(A)$ (in y). Then by 1,4 and 1,5 there exists one $z \in \overline{\mathfrak{D}(A)}$ such that $\langle z, y \rangle = \langle x, A(y) \rangle$ for every $y \in \mathfrak{D}(A)$ and we take $A^*(x) = z$.

Remark. If A is densely defined, then this definition coincides with usual one ([1], p. 28).

1,7. Lemma. *If $A \in \mathfrak{L}^+(E)$, then A^* is densely defined.*

Proof is easy.

1,8. Lemma. *If $A \in \mathfrak{L}^+(E)$ and A is densely defined and closed, then $(A^*)^* = A$.*

Proof see [1] p. 29.

1,9. Lemma. *If $A \in \mathfrak{L}^+(E)$ is one-to-one, A and A^{-1} are densely defined, then A^* is one-to-one and $(A^*)^{-1} = (A^{-1})^*$.*

Proof see [1] p. 29.

1,10. Definition. An operator $A \in \mathfrak{L}^+(E)$ is called *normal* if $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $\|A(x)\| = \|A^*(x)\|$ for every $x \in \mathfrak{D}(A)$.

1,11. Lemma. *An operator $A \in \mathfrak{L}^+(E)$ is normal if and only if A is densely defined, closed and $AA^* = A^*A$.*

Proof see [1] p. 33.

1,12. Lemma. *If $A \in \mathfrak{L}^+(E)$ is normal, then A^* is also normal.*

Proof use 1,11.

1,13. Lemma. *If $A_1 \in \mathfrak{L}^+(E)$, $A_2 \in \mathfrak{L}^+(E)$ are one-to-one, then A_1A_2 is one-to-one and $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$.*

Proof see [1] p. 28.

1,14. Lemma. *If an operator $A \in \mathfrak{L}^+(E)$ is one-to-one and normal and if A^{-1} is densely defined, then A^{-1} is also normal.*

Proof. By 1,11 A is closed. Hence A^{-1} is also closed. Since we assume that A^{-1} is densely defined, it suffices by 1,11 to prove only $A^{-1}(A^{-1})^* = (A^{-1})^* A^{-1}$. One sees easily that all assumptions of 1,9 are fulfilled and therefore $(A^*)^{-1} = (A^{-1})^*$. Thus the desired identity is valid if and only if $A^{-1}(A^*)^{-1} = (A^*)^{-1} A^{-1}$. But by 1,13 this equality is equivalent with $(A^*A)^{-1} = (AA^*)^{-1}$, i.e. with $A^*A = AA^*$. But this is true by 1,11 and hence our proof is complete.

1,15. Lemma. *If $A \in \mathfrak{L}^+(E)$ is normal, then αA and $\alpha j + A$ are normal for every $\alpha \in R$.*

Proof see [1] p. 34.

1,16. Lemma. *If $L \in \mathfrak{L}(E)$, then $\|L\| = \|L^*\|$.*

1,17. Lemma. *If $L_1, L_2 \in \mathfrak{L}(E)$ and are normal, then $\alpha_1 L_1 + \alpha_2 L_2$ is normal for every $\alpha_1, \alpha_2 \in R$. If L_1 commutes with L_2^*, then $L_1 L_2$ is normal. Especially if L is normal, then L^n is normal for all $n = 0, 1, 2, \ldots$*

Proof see [1] p. 18.

1,18. Lemma. *If $L \in \mathfrak{L}(E)$ is normal, then $\|L^2\| = \|L\|^2$.*

Proof [1] p. 18.

1,19. Lemma. *If $L_k, L \in \mathfrak{L}(E)$, L_k are normal and if $L_k(x) \to L(x)$ and $L_k^*(x) \to L^*(x)$ for every $x \in E$ as $k \to \infty$, then L is also normal.*

Proof. By 1,11 it is only to prove that $LL^* = L^*L$. But since this is true for L_k, we see easily that this is true also for L.

1,20. Definition. An operator $A \in \mathfrak{L}^+(E)$ is called *symmetric*, if $\langle A(x), y\rangle = \langle A(y), x\rangle$ for every $x, y \in \mathfrak{D}(A)$ and antisymmetric if $\langle A(x), y\rangle = -\langle A(y), x\rangle$ for $x, y \in \mathfrak{D}(A)$. A symmetric normal operator is called selfadjoint.

1,21. Lemma. *An operator $A \in \mathfrak{L}^+(E)$ is selfadjoint if and only if $A = A^*$.*

Proof is easy.

1,22. Lemma. *If $A \in \mathfrak{L}^+(E)$, A is symmetric, one-to-one and $A^{-1} \in \mathfrak{L}(E)$, then A is selfadjoint.*

Proof by the use of 1,14.

1,23. Lemma. *If $L \in \mathfrak{L}(E)$ and L is symmetric, then L is selfadjoint.*

Proof is easy.

1,24. Lemma. *If $L \in \mathfrak{L}(E)$, L is symmetric, then*

$$\|L\| = \sup_{x \neq 0} \frac{|\langle L(x), x\rangle|}{\|x\|^2}.$$

Proof see [1], p. 14.

1,25. Definition. An operator $A \in \mathfrak{L}^+(E)$ is called *bounded above*, if there exists a constant ω such that $\langle A(x), x\rangle \leqq \omega\langle x, x\rangle$ for every $x \in \mathfrak{D}(A)$.

2. GROWTH OF SEMIGROUPS AND COSINE FUNCTIONS OF NORMAL OPERATORS

2,0.. Orientation. Theorem 2,1 on growth of semigroups is known. In 2,3 we prove analogous assertion for cosine functions.

2,1. Theorem. *If $\mathcal{T}$ is a regular semigroup of normal operators, then there exists a constant ω such that*

(1) $\|\mathcal{T}(t)\| = e^{\omega t}$ *for every* $t \in R^+$.

(2) $\langle \mathcal{T}^{\cdot}(x), x\rangle \leqq \omega\langle x, x\rangle$ *for every* $x \in \mathfrak{D}(\mathcal{T}^{\cdot})$.

Proof see [2], 22.4.1.

2,2. Lemma. *If $\mathscr{C}$ is a regular cosine function of normal operators and ω a nonnegative constant such that there exists a constant M for which $\|\mathscr{C}(t)\| \leqq M \operatorname{coh}(\omega t)$ for every $t \in R^+$, then $\|\mathscr{C}(t)\| \leqq \operatorname{coh}(\omega t)$ for every $t \in R^+$.*

Proof. 1° If p is a nonnegative entire number such that $\|\mathscr{C}(t)\| \leqq M^{1/2^p} \operatorname{coh}(\omega t)$ for every $t \in R^+$ then $\|\mathscr{C}(t)\| \leqq M^{1/2^{p+1}} \operatorname{coh}(\omega t)$ for $t \in R^+$.

Proof. Regularity of $\mathscr{C}$ implies $M \geqq 1$. Now by the use of 2,3 in [3] and 1,18 we obtain

$$\|\mathscr{C}(t)\| = (\|\mathscr{C}(t)\|^2)^{1/2} = \|\mathscr{C}(t)^2\|^{1/2} = \|\tfrac{1}{2}(\mathscr{C}(2t) + j)\|^{1/2} \leqq$$

$$\leqq [\tfrac{1}{2}(M^{1/2^p} \operatorname{coh}(2\omega t) + 1]^{1/2} \leqq [M^{1/2^p}(\tfrac{1}{2}(\operatorname{coh}(2\omega t) + 1))]^{1/2} =$$

$$= M^{1/2^{p+1}} \operatorname{coh}(\omega t).$$

2° $\|\mathscr{C}(t)\| \leqq M^{1/2^p} \operatorname{coh}(\omega t)$ for every $t \in R^+$ and $p = 0, 1, 2, \ldots$

Proof by induction under use of 1°.

3° Final part: Our lemma follows from 2° since $M \geqq 1$ and therefore $M^{1/2^p} \to 1$ as $p \to \infty$.

2,3. Theorem. *If $\mathscr{C}$ is a regular cosine function of normal operators, then there exists a constant $\omega \geqq 0$ such that*

(1) $\|\mathscr{C}(t)\| \leqq \operatorname{coh}(\omega t)$ *for every* $t \in R^+$;

(2) *if* N, χ *are two nonnegative constants such that* $\|\mathscr{C}(t)\| \leqq N \operatorname{coh}(\chi t)$ *for every* $t \in R^+$, *then* $\chi \geqq \omega$;

(3) $\langle \mathscr{C}^{\cdot\cdot}(x)\, x\rangle \leqq \omega^2 \langle x, x\rangle$ *for every* $x \in \mathfrak{D}(\mathscr{C}^{\cdot\cdot})$.

Proof. 1° There exist two nonnegative constants M, σ such that $\|\mathscr{C}(t)\| \leqq$ $\leqq M \operatorname{coh}(\sigma t)$ for every $t \in R^+$.

Proof by the use of 2,5 in [3].

2° $\|\mathscr{C}(t)\| \leqq \operatorname{coh}(\sigma t)$ for $t \in R^+$.

Proof from 1° and 2,2.

3° There exists a nonnegative constant ω such that (1), (2) hold.

Proof is easy by the use of 2° and 2,2.

4° $\langle \mathscr{C}^{\cdot\cdot}(x), x\rangle \leqq \omega^2 \langle x, x\rangle$ for every $x \in \mathfrak{D}(\mathscr{C}^{\cdot\cdot})$.

Proof. If $x \in \mathfrak{D}(\mathscr{C}^{\cdot\cdot})$

$$\langle \mathscr{C}^{\cdot\cdot}(x), x\rangle = \lim_{t\to 0+} 2 \frac{\langle \mathscr{C}(t)(x), x\rangle - \langle x, x\rangle}{t^2} \leqq \lim_{t\to 0+} 2 \frac{\|\mathscr{C}(t)\|\, \|x\| - \langle x, x\rangle}{t^2} \leqq$$

$$\leqq \lim_{t\to 0+} 2 \frac{\operatorname{coh}(\omega t) - 1}{t^2} \langle x, x\rangle = \omega^2 \langle x, x\rangle .$$

5° Final part: 3° and 4° imply our theorem.

3. GENERATORS OF SEMIGROUPS AND COSINE FUNCTIONS OF NORMAL OPERATORS

3,0. Orientation. In this section we prove that operators constituting a regular semigroup or a regular cosine function are normal if and only if corresponding generator is normal. These main results are contained in 3,10 and 3,16 (for selfadjoint case, see 3,13 and 3,19). Spectral resolution of normal operators is nowhere used.

3,1. Lemma. *If* $A \in \mathfrak{L}^+(E)$ *and* $\lambda \in R$ *so that*

(I) *A is densely defined,*
(II) $\lambda j - A$ *is one-to-one and* $(\lambda j - A)^{-1} \in \mathfrak{L}(E)$, *then*

(1) $\lambda j - A^*$ *is one-to-one and* $(\lambda j - A^*)^{-1} \in \mathfrak{L}(E)$,
(2) $(\lambda j - A^*)^{-1} = ((\lambda j - A)^{-1})^*$.

Proof follows immediately from 1,11.

3,2. Lemma. *If $A \in \mathfrak{L}^+(E)$ and $\varkappa$ is a constant such that*

(I) *$\lambda j - A$ is one-to-one and $(\lambda j - A)^{-1} \in \mathfrak{L}(E)$ for every $\lambda > \varkappa$.*

(II) *$\lambda(\lambda j - A)^{-1}(x) \to x$ for every $x \in E$ as $\lambda \to \infty$, $\lambda > \varkappa$, then*

(1) *$\lambda j - A^*$ is one-to-one and $(\lambda j - A^*)^{-1} \in \mathfrak{L}(E)$ for every $\lambda > \varkappa$,*

(2) *$\lambda(\lambda j - A^*)^{-1}(x) \to x$ for every $x \in E$ as $\lambda \to \infty$, $\lambda > \varkappa$.*

Proof. 1° A is densely defined.

Proof by the use of 1,3 in [3].

2° $\lambda j - A^*$ is one-to-one, $(\lambda j - A^*)^{-1} \in \mathfrak{L}(E)$ and $(\lambda j - A^*)^{-1} = ((\lambda j - A)^{-1})^*$ for $\lambda > \varkappa$.

Proof. Use 1° and 3,1.

3° $\lambda(\lambda j - A^*)^{-1}(x) \to x$ for every $x \in E$ as $\lambda \to \infty$, $\lambda > \varkappa$.

Proof. One sees easily that (II) implies the existence of a constant c such that $\|(\lambda j - A)^{-1}\| \leqq c/\lambda$ for every $\lambda > \varkappa + 1$. Hence by 1,16 $\|(\lambda j - A^*)^{-1}\| \leqq c/\lambda$. On the other hand we see that by 1° and 1,7 A^* is densely defined. Now it suffices to use 1,2 (5) of [3].

4° Final part: 2° and 3° give our lemma.

3,3. Definition. Let $\mathscr{H}$ be an operator function defined on R^+ with values in $\mathfrak{L}(E)$. Then we define a new operator function $\mathscr{H}^*$ as follows: $\mathscr{H}^*(t) = \mathscr{H}(t)^*$ for every $t \in R^+$.

3,4. Theorem. *If $\mathscr{T}$ is a regular semigroup, then $\mathscr{T}^*$ is also a regular semigroup and $\mathscr{T}^{*\cdot} = \mathscr{T}^{\cdot *}$. Moreover if M, ω are two nonnegative constants such that $\|\mathscr{T}(t)\| \leqq \leqq Me^{\omega t}$ for every $t \in R^+$, then $\|\mathscr{T}^*(t)\| \leqq Me^{\omega t}$.*

Proof. 1° Let us assume that $\|\mathscr{T}(t)\| \leqq Me^{\omega t}$ for every $t \in R^+$. Such constants M, ω always exist by 4,3 in [3].

2° (α) $\lambda j - \mathscr{T}^{\cdot *}$ is one-to-one and $(\lambda j - \mathscr{T}^{\cdot *})^{-1} \in \mathfrak{L}(E)$ for every $\lambda > \omega$.

(β) $\lambda(\lambda j - \mathscr{T}^{\cdot *})^{-1}(x) \to x$ for every $x \in E$ as $\lambda \to \infty$, $\lambda > \omega$,

(γ) $\|(\lambda j - \mathscr{T}^{\cdot *})^{-n}\| \leqq M/(\lambda - \omega)^n$ for every $\lambda > \omega$, and $n = 1, 2, \ldots$

Proof. By the use of 1° and 4,6 in [3] we prove the properties (α), (β), (γ) for $\mathscr{T}^{\cdot}$. Now it suffices to use 3,1, 3,2, 1,16 and 1,17.

3° There exists a regular semigroup $\mathscr{S}$ such that $\mathscr{S}^{\cdot} = \mathscr{T}^{\cdot *}$ and $\|\mathscr{S}(t)\| \leqq Me^{\omega t}$ for every $t \in R^+$.

Proof. Use 2° and 4,7 in [3].

4° $\mathscr{S} = \mathscr{T}^*$.

Proof. If $x, y \in E$ and $\lambda > \omega$, then we may write using 3° and 3,1

$$\int_0^\infty e^{-\lambda\tau} \langle \mathscr{S}(\tau)(x), y\rangle \, d\tau = \left\langle \int_0^\infty e^{-\lambda\tau} \mathscr{S}(\tau)(x) \, d\tau, y \right\rangle =$$
$$= \langle (\lambda j - \mathscr{T}^{\cdot *})^{-1}(x), y\rangle = \langle ((\lambda j - \mathscr{T}^{\cdot})^{-1})^*(x), y\rangle = \langle x, (\lambda j - \mathscr{T}^{\cdot})^{-1}(y)\rangle =$$
$$= \left\langle x, \int_0^\infty e^{-\lambda\tau} \mathscr{T}(\tau)(y) \, d\tau \right\rangle = \int_0^\infty e^{-\lambda\tau} \langle x, \mathscr{T}(\tau)(y)\rangle \, d\tau .$$

Now lemma 1,20 in [3] and theorem 4,4 in [3] give $\langle \mathscr{S}(t)(x), y\rangle = \langle x, \mathscr{T}(t)(y)\rangle$ for every $t \in R^+$, $x, y \in E$, i.e. $\mathscr{S}(t) = \mathscr{T}(t)^* = \mathscr{T}^*(t)$.

5° Final part: 3° and 4° prove our theorem.

Remark. This theorem due to Phillips ([3]) is known in an much more general form. We give here a special case with essential simplifications, since this special form is sufficient for our purposes. This theorem (and also the following for cosine functions) plays only an auxiliary role in the present paper.

3,5. Theorem. *If $\mathscr{C}$ is a regular cosine function, then $\mathscr{C}^*$ is also a regular cosine function and $\mathscr{C}^{*:} = \mathscr{C}^{:*}$. Moreover if M, ω are two nonnegative constants such that $\|\mathscr{C}(t)\| \leqq M \operatorname{coh}(\omega t)$ for every $t \in R^+$, then $\|\mathscr{C}^*(t)\| \leqq \operatorname{coh}(\omega t)$.*

Proof. 1 ° Let us assume that $\|\mathscr{C}(t)\| \leqq M \operatorname{coh}(\omega t)$ for every $t \in R^+$. Such constants M, ω always exist by 2,5 in [3].

2° (α) $(\mu j - \mathscr{C}^{:*})$ is one-to-one and $(\mu j - \mathscr{C}^{:*})^{-1} \in \mathfrak{L}(E)$ for every $\mu > \omega^2$.

(β) $\mu(\mu j - \mathscr{C}^{:*})^{-1}(x) \to x$ for every $x \in E$ as $\mu \to \infty$, $\mu \leqq \omega^2$.

(γ) $\|(d^n/d\lambda^n)(\lambda(\lambda^2 j - \mathscr{C}^{:*})^{-1})\| \leqq (Mn!/2)((1/(\lambda + \omega)^{n+1}) + (1/(\lambda - \omega)^{n+1}))$ for every $\lambda \leqq \omega$ and $n = 0, 1, 2, \ldots$

Proof. By the use of 1° and 3,1 in [3] we establish the properties (α), (β), (γ) for $\mathscr{C}^{:}$. Now it suffices to use 3,1, 3,2, 1,16 and 1,17.

3° There exists a regular cosine function $\mathscr{D}$ such that $\mathscr{D}^{:} = \mathscr{C}^{:*}$ and $\|\mathscr{D}(t)\| \leqq$ $\leqq M \operatorname{coh}(\omega t)$ for every $t \in R^+$.

Proof use 2° and 3,2 in [3].

4° $\mathscr{D} = \mathscr{C}^*$.

Proof. If $x, y \in E$ and $\lambda > \omega$, then we may write by 3°, 3,1

$$\int_0^\infty e^{-\lambda\tau} \langle \mathscr{D}(\tau)(x), y\rangle \, d\tau = \left\langle \int_0^\infty e^{-\lambda\tau} \mathscr{D}(\tau)(x) \, d\tau, y \right\rangle = \langle \lambda(\lambda^2 j - \mathscr{C}^{:*})^{-1}(x), y\rangle =$$
$$= \langle (\lambda(\lambda^2 j - \mathscr{C}^{:})^{-1})^*(x), y\rangle = \langle x, \lambda(\lambda^2 j - \mathscr{C}^{:})^{-1}(y)\rangle =$$
$$= \left\langle x, \int_0^\infty e^{-\lambda\tau} \mathscr{C}(\tau)(y) \, d\tau \right\rangle = \int_0^\infty e^{-\lambda\tau} \langle x, \mathscr{C}(\tau)(y)\rangle \, d\tau .$$

Now lemma 1,20 in [3] and theorem 2,7 in [3] give $\langle \mathscr{D}(t)(x), y\rangle = \langle x, \mathscr{C}(t)(y)\rangle$ for every $t \in R^+$, $x, y \in E$. Hence $\mathscr{D}(t) = \mathscr{C}(t)^* = \mathscr{C}^*(t)$.

5° Final part: 3° and 4° establish our theorem.

3,6. Lemma. *If $A \in \mathfrak{L}^+(E)$ is a normal operator and $\lambda \in R^+$ such that $\lambda j - A$ is one-to-one and $(\lambda j - A)^{-1} \in \mathfrak{L}(E)$, then $(\lambda j - A)^{-1}$ is normal operator.*

Proof by the use of 1,15 and 1,14.

3,7. Lemma. *If $A \in \mathfrak{L}^+(E)$ is a selfadjoint operator and $\lambda \in R$ such that $\lambda j - A$ is one-to-one and $(\lambda j - A)^{-1} \in \mathfrak{L}(E)$, then $(\lambda j - A)^{-1}$ is selfadjoint.*

Proof by the use of 3,6 and 1,20.

3,8. Theorem. *If $\mathscr{T}$ is a regular semigroup of normal operators, then $\mathscr{T}^\bullet$ is normal operator.*

Proof. 1° There exist two constants $M \geqq 0$, $\omega \geqq 0$ such that $\|\mathscr{T}(t)\| \leqq Me^{\omega t}$ for every $t \in R^+$.

Proof by 4,3 in [3].

2° $\lambda j - \mathscr{T}^\bullet$ is one-to-one, $(\lambda j - \mathscr{T}^\bullet)^{-1} \in \mathfrak{L}(E)$ and $(\lambda j - \mathscr{T}^\bullet)^{-1}(x) = \int_0^\infty e^{-\lambda\tau} \mathscr{T}(\tau)(x)\, d\tau$ for every $\lambda > \omega$ and $x \in E$.

See the proof of 11.5.1 in [2].

3° Let us define a sequence of operator function $\mathscr{Q}_p$ defined for $\lambda > \omega$ with values in $\mathfrak{L}(E)$ as follows:

$$\mathscr{Q}_p(\lambda)(x) = \sum_{k=1}^{p^2} e^{-\lambda k/p^2} \mathscr{T}\left(\frac{k}{p^2}\right)(x) \quad \text{for every} \quad x \in E\,,$$

$\lambda > \omega$ and $p = 1, 2, \ldots$

4° $\mathscr{Q}_p(\lambda)(x) \to (\lambda j - \mathscr{T}^\bullet)^{-1}(x)$ for every $x \in E$ and $\lambda > \omega$ as $p \to \infty$.

Proof is easy by the use of 2° and 4,4 in [3].

5° $\mathscr{Q}_p(\lambda)^*(x) \to ((\lambda j - \mathscr{T}^\bullet)^{-1})^*(x)$ for every $x \in E$ and $\lambda > \omega$.

Proof. By 3,1 $((\lambda j - \mathscr{T}^\bullet)^{-1})^* = (\lambda j - \mathscr{T}^{\bullet *})^{-1}$ for $\lambda > \omega$. Further by 3,3 $\mathscr{T}^*$ is regular semigroup and $\mathscr{T}^{*\bullet} = \mathscr{T}^{\bullet *}$. Since evidently $\mathscr{Q}_p(\lambda)^*(x) = \sum_{k=1}^{p^2} e^{-\lambda k/p^2} \mathscr{T}^*(k/p^2)\,.\,(x)$, we may use 4° and obtain our result.

6° $\mathscr{Q}_p(\lambda)$ is normal operator for every $\lambda > \omega$ and $p = 1, 2, \ldots$

Proof. Use 1,17.

7° $(\lambda j - \mathscr{T}^\bullet)^{-1}$ is normal for every $\lambda > \omega$.

Proof is based on 2°, 5° and 1,19.

8° Final part: Since $\mathscr{T}^{\cdot}$ is densely defined we obtain by the use of 1,14 and 7° that $\lambda j - \mathscr{T}^{\cdot}$ is normal for $\lambda > \omega$. Now 1,15 gives the desired result.

Remark. Preceding theorem can be deduced from 22,4,2 in [2] under use of spectral resolution of normal operators. Our proof is essentially simpler and uses only elementary properties of normal operators.

3,9. Theorem. *If $\mathscr{T}$ is a regular semigroup such that $\mathscr{T}^{\cdot}$ is a normal operator, then $\mathscr{T}(t)$ is normal operator for every $t \in R^+$.*

Proof. 1° and 2° as in the proof of 3,8.

3° Let us define a sequence of operator functions $\mathscr{V}_p$ defined on R^+ with values in $\mathfrak{L}(E)$ as follows:

$$\mathscr{V}_p(t) = \left(\frac{p}{t}\right)^{p+1} \left(\frac{p}{t} j - \mathscr{T}^{\cdot}\right)^{-(p+1)}$$

for every $t \in R^+$ and $p = 1, 2, \ldots, p > t\omega$.

4° $\mathscr{V}_p(t)(x) \to \mathscr{T}(t)(x)$ for every $x \in E$ and $t \in R^+$ as $p \to \infty$, $p > t\omega$.

Proof. By the use of 1,2 in [3], 2° and 1,19 in [3].

5° $\mathscr{V}_p(t)^*(x) \to \mathscr{T}(t)^*(x)$ for every $x \in E$ and $t \in R^+$ as $p \to \infty$, $p > t\omega$.

Proof. By 3,4 $\mathscr{T}^*$ is regular semigroup and $\mathscr{T}^{*\cdot} = \mathscr{T}^{\cdot *}$. Further by 3,1 $((\lambda j - \mathscr{T}^{\cdot})^{-1})^* = (\lambda j - \mathscr{T}^{*\cdot})^{-1}$ for every $\lambda > \omega$ and hence easily

$$\mathscr{V}_p(t)^* = \left(\frac{p}{t}\right)^{p+1} \left(\frac{p}{t} j - \mathscr{T}^{*\cdot}\right)^{-(p+1)}$$

for every $t \in R^+$ and $p > t\omega$.

On the other hand since $\mathscr{T}^*$ is regular semigroup as stated above, we may use 4° (one replaces $\mathscr{T}$ by $\mathscr{T}^*$) and obtain $\mathscr{V}_p(t)^*(x) \to \mathscr{T}^*(t)(x) = \mathscr{T}(t)^*(x)$ and this completes the proof of 5°.

6° $(\lambda j - \mathscr{T}^{\cdot})^{-1}$ is normal for every $\lambda > \omega$.

Proof use 2° and 3,6.

7° $\mathscr{V}_p(t)$ is normal for every $t \in R^+$ and $p > t\omega$.

Proof use 1,17 and 6°.

8° Final part: If we use 4°, 5°, 7° and 1,19 we obtain that $\mathscr{T}(t)$ is normal for every $t \in R^+$.

3,10. Fundamental theorem. *If $\mathscr{T}$ is a regular semigroup, then $\mathscr{T}(t)$ is normal for every $t \in R^+$ if and only if $\mathscr{T}^{\cdot}$ is normal.*

Proof. Consequence of 3,8 and 3,9.

3,11. Theorem. *If $\mathscr{T}$ is a regular semigroup of selfadjoint operators, then $\mathscr{T}^{\cdot}$ is selfadjoint.*

Proof. On the base of 1,20 and 3,8 it suffices to prove that $\mathscr{T}^{\cdot}$ is symmetric. But this is easy.

3,12. *If $\mathscr{T}$ is a regular semigroup such that $\mathscr{T}^{\cdot}$ is selfadjoint, then $\mathscr{T}(t)$ is selfadjoint for every $t \in R^+$.*

Proof. We proceed analogously as in the proof of 3,9 with some simplifications.

3,13. Fundamental theorem. *If $\mathscr{T}$ is a regular semigroup, then $\mathscr{T}(t)$ is selfadjoint for every $t \in R^+$ if and only if $\mathscr{T}^{\cdot}$ is selfadjoint.*

Proof. Use 3,11 and 3,12.

3,14. Theorem. *If $\mathscr{C}$ is a regular cosine function of normal operators, then $\mathscr{C}^{:}$ is a normal operator.*

Proof. 1° There exist two constants $M \geqq 0$, $\omega \geqq 0$ such that $\|\mathscr{C}(t)\| \leqq M \operatorname{coh}(\omega t)$ for every $t \in R^+$.

Proof by 2,5 in [3].

2° $\lambda^2 j - \mathscr{C}^{:}$ is one-to-one, $(\lambda^2 j - \mathscr{C}^{:})^{-1} \in \mathfrak{L}(E)$ and $\lambda(\lambda^2 j - \mathscr{C}^{:})^{-1}(x) = \int_0^\infty e^{-\lambda\tau} \mathscr{C}(\tau)(x) \, d\tau$ for $\lambda > \omega$ and $x \in E$.

See the proof of 3,1 in [3].

3° Let us define a sequence of operator functions $\mathscr{Q}_p$ defined for $\lambda > \omega$ with values in $\mathfrak{L}(E)$ as follows: $\mathscr{Q}_p(\lambda)(x) = \sum_{k=1}^{p^2} e^{-\lambda k/p^2} \mathscr{C}(k/p^2)(x)$ for every $x \in E$, $\lambda > \omega$ and $p = 1, 2, \ldots$

4° $\mathscr{Q}_p(\lambda)(x) \to \lambda(\lambda^2 j - \mathscr{C}^{:})^{-1}(x)$ for $x \in E$ and $\lambda > \omega$ if $p \to \infty$.

Proof is easy by the use of 2° and 2,7 in [3].

5° $\mathscr{Q}_p(\lambda)^*(x) \to \lambda((\lambda^2 j - \mathscr{C}^{:})^{-1})^*(x)$ for every $x \in E$ and $\lambda > \omega$.

Proof. By 3,1 $((\lambda^2 j - \mathscr{C}^{:})^{-1})^* = (\lambda^2 j - \mathscr{C}^{:*})^{-1}$ for $\lambda > \omega$. Further by 3,4 $\mathscr{C}^*$ is regular cosine function and $\mathscr{C}^{*:} = \mathscr{C}^{:*}$. Since evidently $\mathscr{Q}_p(\lambda)^*(x) = \sum_{k=1}^{p^2} e^{-\lambda k/p^2} \mathscr{C}^*(k/p)(x)$, we may use 4° and obtain our result.

6° $\mathscr{Q}_p(\lambda)$ is normal for every $\lambda > \omega$ and $p = 1, 2, \ldots$

Proof use 1,17.

7° $(\lambda^2 j - \mathscr{C}^{\cdot})^{-1}$ is normal for $\lambda > \omega$.

Proof by the use of 2°, 5° and 1,19.

8° Final part: Since $\mathscr{C}^{\cdot}$ is densely defined by 2,20 in [3] we obtain by 1,14 and 7° that $\lambda^2 j - \mathscr{C}^{\cdot}$ is normal for $\lambda > \omega$. Now 1,15 proves our theorem.

3,15. Theorem. *If $\mathscr{C}$ is a regular cosine function such that $\mathscr{C}^{\cdot}$ is a normal operator, then $\mathscr{C}(t)$ is normal for every $t \in R^+$.*

Proof. 1° and 2° as in the proof of 3,14.

3° Let us define a sequence of operator functions $\mathscr{V}_p$ defined on R^+ with values in $\mathfrak{L}(E)$ as follows:

$$\mathscr{V}_p(t) = \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{k+1} \left[\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} (\lambda(\lambda^2 j - \mathscr{C}^{\cdot})^{-1})\right]_{\lambda = p/t}$$

for every $t \in R^+$ and $p = 1, 2, \ldots, p > t\omega$.

4° $\mathscr{V}_p(t)(x) \to \mathscr{C}(t)(x)$ for every $x \in E$ and $t \in R^+$ as $p \to \infty$, $p > t\omega$.

Proof. By the use of 1,2 in [3], 2° and 1,19 in [3].

5° $\mathscr{V}_p(t)^*(x) \to \mathscr{C}(t)^*(x)$ for every $x \in E$ and $t \in R^+$ as $p \to \infty$, $p > t\omega$.

Proof. By 3,5 $\mathscr{C}^*$ is a regular cosine function and $\mathscr{C}^{*\cdot} = \mathscr{C}^{\cdot *}$. Further by 3,1 $((\lambda^2 j - \mathscr{C}^{\cdot})^{-1})^* = (\lambda^2 j - \mathscr{C}^{*\cdot})^{-1}$ for every $\lambda > \omega$ and hence

$$\mathscr{V}_p(t)^* = \frac{(-1)^k}{p!} \left(\frac{p}{t}\right)^{p+1} \left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} (\lambda(\lambda^2 j - \mathscr{C}^{*\cdot})^{-1}\right]_{\lambda = p/t}.$$

On the other hand since $\mathscr{C}^*$ is regular cosine function as stated above, we may use 4° (on replaces $\mathscr{C}$ by $\mathscr{C}^*$) and obtain $\mathscr{V}_p(t)^*(x) = \mathscr{C}^*(t)(x) = \mathscr{C}(t)^*(x)$ and this completes the proof of 5°.

6° $(\lambda^2 j - \mathscr{C}^{\cdot})^{-1}$ is normal for every $\lambda > \omega$.

Proof by the use of 2° and 3,6.

7° $\mathscr{V}_p(t)$ is normal for every $t \in R^+$ and $p > t\omega$.

Proof by 1,17 and 6°.

8° Final part: If we use 4°, 5°, 7° and 1,19 we obtain that $\mathscr{C}(t)$ is normal for every $t \in R^+$.

3,16. Fundamental theorem. *If $\mathscr{C}$ is a regular cosine function, then $\mathscr{C}(t)$ is normal for every $t \in R^+$ if and only if $\mathscr{C}^{\cdot}$ is normal.*

Proof. Consequence of 3,14 and 3,15.

3.17. Theorem. *If $\mathscr{C}$ is a regular cosine function of selfadjoint operators, then $\mathscr{C}^{\cdot\cdot}$ is selfadjoint.*

Proof. With respect to 1,20 and 3,14 it suffices to prove that $\mathscr{C}^{\cdot\cdot}$ is symmetric. But this is easy.

3,18. Theorem. *If $\mathscr{C}$ is a regular cosine function such that $\mathscr{C}^{\cdot\cdot}$ is selfadjoint, then $\mathscr{C}(t)$ is selfadjoint for every $t \in R^+$.*

Proof. We proceed analogously as in the proof of 3,15 with some simplification.

3,19. Fundamental theorem. *If $\mathscr{C}$ is a regular cosine function, then $\mathscr{C}(t)$ is selfadjoint for every $t \in R^+$ if and only if $\mathscr{C}^{\cdot\cdot}$ is selfadjoint.*

Proof. Consequence of 3,17 and 3,18.

4. GENERATION OF SEMIGROUPS AND COSINE FUNCTIONS OF NORMAL OPERATORS

4,0. Orientation. In this section, the generation problem is discussed. In 4,3 we, prove that an operator generates a regular semigroup of normal operators if and only if it is normal and bounded above. Analogous result is not true for cosine functions. In this case, we are able to give certain sufficient conditions (4,10) and to prove only that an operator generates a regular cosine function of selfadjoint operators if and only if it is selfadjoint and bounded above (cf. 4,12). This last assertion is well known. At the end we give an example of a normal operator bounded above which generates no regular cosine function (4,13).

4,1. Lemma. *If $A \in \mathfrak{L}^+(E)$ is a normal operator and ω a constant such that $\langle A(x), x\rangle \leqq \omega\langle x, x\rangle$, then*

(1) *$\lambda I - A$ is one-to-one and $(\lambda I - A)^{-1} \in \mathfrak{L}(E)$ for every $\lambda > \omega$,*

(2) *$\|(\lambda I - A)^{-1}\| \leqq 1/(\lambda - \omega)$ for $\lambda > \omega$.*

Proof. 1° $\|\lambda x - A(x)\| \geqq (\lambda - \omega)\,\|x\|$ for every $x \in \mathfrak{D}(A)$ and $\lambda > \omega$.

Proof. We may write: $\|\lambda x - A(x)\| \geqq \langle \lambda x - A(x), x/\|x\|\rangle = 1/\|x\|\,(\lambda\langle x, x\rangle - \langle A(x), x\rangle) \geqq (1/\|x\|)\,(\lambda - \omega)\,\langle x, x\rangle = (\lambda - \omega)\,\|x\|$ if $x \in \mathfrak{D}(A)$, $x \neq 0$ and $\lambda > \omega$. If $x = 0$, the inequality is trivial.

2° $\lambda j - A$ is one-to-one for every $\lambda > \omega$.

Proof by the use of 1°.

3° $\{\lambda x - A(x) : x \in \mathfrak{D}(A)\}$ is dense in E for every $\lambda > \omega$.

Proof. Assume contrary for some fixed $\lambda > \omega$. There exists a $y \in E$, $y \neq 0$ so that $\langle \lambda x - A(x), y \rangle = 0$ for every $x \in \mathfrak{D}(A)$. Hence $y \in \mathfrak{D}(\lambda j - A^*)$ and $(\lambda j - A^*)$ . . $(y) = 0$. Since A is normal, $\lambda j - A$ is also normal by 1,15. Therefore $y \in \mathfrak{D}(\lambda j - A)$ and $\|(\lambda j - A)(y)\| = \|(\lambda j - A^*)(y)\| = 0$, i.e. $(\lambda j - A) = 0$. But this is impossible since $\|(\lambda j - A)(y)\| \geqq (\lambda - \omega)\|y\| > 0$ by 1°.

4° $(\lambda j - A)^{-1} \in \mathfrak{L}(E)$ for every $\lambda \leqq \omega$.

Proof. Since A is closed by 1,11, $(\lambda j - A)^{-1}$ is also closed. But by 3° $(\lambda j - A)^{-1}$ is densely defined and hence belongs to $\mathfrak{L}(E)$.

5° $\|(\lambda j - A)^{-1}\| \leqq 1/(\lambda - \omega)$ for every $\lambda > \omega$.

Proof follows from 1°.

6° Final part. Our lemma is prowed by 2°, 4°, 5°.

4,2. Theorem. *If $A \in \mathfrak{L}^+(E)$ is a normal operator and ω a constant such that $\langle A(x)\, x \rangle \leqq \omega \langle x, x \rangle$, then there exists a regular semigroup $\mathscr{T}$ such that*

(1) $\mathscr{T}^{\boldsymbol{\cdot}} = A$,

(2) $\|\mathscr{T}(t)\| \leqq e^{\omega t}$ *for every* $t \in R^+$.

Proof follows immediately from 4,1 and 4,7 of [3].

4.3. Fundamental theorem. *If $A \in \mathfrak{L}^+(E)$, then there exists a regular semigroup $\mathscr{T}$ of normal operators such that $\mathscr{T}^{\boldsymbol{\cdot}} = A$ if and only if A is normal and bounded above.*

Proof. Easy consequence of 4,2, 3,8 and 2,1.

4,4. Definition. Let E be a Banach space. We shall denote elements of $E \times E$ by $\mathfrak{z} = \begin{pmatrix} x \\ y \end{pmatrix}$. Norm in $E \times E$ is given by $\|\mathfrak{z}\| = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{(\|x\|^2 + \|y\|^2)}$. Subsequently, $E \times E$ is a Hilbert space if E.

If $A \in \mathfrak{L}^+(E)$, then operator $\mathfrak{A}$ is defined on $\mathfrak{D}(A) \times E$ and $\mathfrak{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ A(x) \end{pmatrix}$ for every $x \in \mathfrak{D}(A)$, $y \in E$.

4,5. Lemma. *If $A \in \mathfrak{L}^+(E)$ and λ is a constant so that*[1]) *$\lambda^2 I - A$ is one-to-one and $(\lambda^2 I - A)^{-1} \in \mathfrak{L}(E)$, then $\lambda I - \mathfrak{A}$ is one-to-one, $(\lambda I - \mathfrak{A})^{-1} \in \mathfrak{L}(E \times E)$ and for every $x, y \in E$*

[*] $$(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\lambda^2 I - A)^{-1}(\lambda x + y) \\ \lambda(\lambda^2 I - A)^{-1}(\lambda x + y) - x \end{pmatrix}.$$

Proof by direct verification of [*].

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[1]) In 4,5—4,10 one writes I instead of j.

4,6. Lemma. *If $A \in \mathfrak{L}^+(E)$ is normal and α a nonnegative constant such that*

(I) $\langle A(x), x\rangle \leqq 0$ *for every* $x \in \mathfrak{D}(A)$,

(II) *there exists a constant* $\vartheta \geqq 0$ *so that* $\frac{1}{2}[\langle A(x), y\rangle - \langle A(y), x\rangle] \leqq \alpha[\vartheta\langle x, x\rangle + \langle y, y\rangle - \langle A(x), x\rangle] - \vartheta\langle x, y\rangle$ *for every* $x, y \in \mathfrak{D}(A)$, *then*

(1) $\mu I - A$ *is one-to-one and* $(\mu I - A)^{-1} \in \mathfrak{L}(E)$ *for* $\mu > \sqrt{\alpha}$,

(2) $\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 I - A)^{-1} \right\| \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}$

for $\lambda > \alpha$ *and* $n = 0, 1, 2, \ldots$

Proof. 1° Let us define on $\mathfrak{D}(A) \times E$:

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \vartheta\langle x_1, x_2\rangle + \langle y_1, y_2\rangle - \tfrac{1}{2}[\langle A(x_1), x_2\rangle + \langle A(x_2), x_1\rangle]$$

and

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \sqrt{(\vartheta\langle x, x\rangle + \langle y, y\rangle - \langle A(x), x\rangle)}\,.$$

2° $\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle$ is a symmetric, bilinear function,

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| \geqq 0 \quad \text{and} \quad \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left| \begin{pmatrix} x \\ y \end{pmatrix} \right|^2.$$

3°

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle \leqq \left| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right|.$$

Proof. Let $\mathfrak{z}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathfrak{z}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. We can suppose first $\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle > 0$. Then, for every ξ, there is

$$0 \leqq \langle \mathfrak{z}_1 + \xi\mathfrak{z}_2, \mathfrak{z}_1 + \xi\mathfrak{z}_2\rangle =$$

$$= \langle \mathfrak{z}_1, \mathfrak{z}_1\rangle + 2\xi\langle \mathfrak{z}_1, \mathfrak{z}_2\rangle + \xi^2\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle \quad \text{and for} \quad \xi = -\frac{\langle \mathfrak{z}_1, \mathfrak{z}_2\rangle}{\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle}$$

we hawe $\langle \mathfrak{z}_1, \mathfrak{z}_1\rangle\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle - \langle \mathfrak{z}_1, \mathfrak{z}_2\rangle^2 \geqq 0$ i. e. 3° is valid under assumption $\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle > 0$.

If $\langle \mathfrak{z}_2, \mathfrak{z}_2\rangle = 0$, one sees from 1° that $y_2 = 0$. Let $y_2^{(n)} \neq 0$, $y_2^{(n)} \to 0$ $(n \to \infty)$ and $\mathfrak{z}_2^{(n)} = \begin{pmatrix} x_2 \\ y_2^{(n)} \end{pmatrix}$. Then easily $\langle \mathfrak{z}_1, \mathfrak{z}_2^{(n)}\rangle \leqq |\mathfrak{z}_1| \cdot |\mathfrak{z}_2^{(n)}|$, $\langle \mathfrak{z}_1, \mathfrak{z}_2^{(n)}\rangle \to \langle \mathfrak{z}_1, \mathfrak{z}_2\rangle$ and $|\mathfrak{z}_2^{(n)}| \to |\mathfrak{z}_2|$. This completes the proof.

4° If $\mathfrak{A}$ is operator defined in 4,4 then

$$(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}x\\y\end{pmatrix} \in \mathfrak{D}(A) \times \mathfrak{D}(A) \quad \text{for} \quad x \in \mathfrak{D}(A)\,, \quad y \in E\,.$$

Proof. This follows immediately from 4,5.

5° If $x, y \in \mathfrak{D}(A)$, then $\left\langle \mathfrak{A}\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right\rangle \leqq \alpha \left|\begin{pmatrix}x\\y\end{pmatrix}\right|^2$.

Proof. It is

$$\left\langle \mathfrak{A}\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}y\\A(x)\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right\rangle = \vartheta\langle x, y\rangle + \langle A(x), y\rangle -$$
$$- \tfrac{1}{2}[\langle A(x), y\rangle + \langle A(y), x\rangle] = \vartheta\langle x, y\rangle + \tfrac{1}{2}[\langle A(x), y\rangle - \langle A(y), x\rangle] \leqq$$
$$\leqq \alpha[\vartheta\langle x, x\rangle + \langle y, y\rangle - \langle A(x), x\rangle] = \alpha \left|\begin{pmatrix}x\\y\end{pmatrix}\right|^2.$$

6° $\left|(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right| \leqq \dfrac{1}{\lambda - \alpha}\left|\begin{pmatrix}x\\y\end{pmatrix}\right|$ for every $x \in \mathfrak{D}(A)$, $y \in E$ and $\lambda > \alpha$.

Proof. Let $x \in \mathfrak{D}(A)$, $y \in E$. Then by 4° $(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}x\\y\end{pmatrix} \in \mathfrak{D}(A) \times \mathfrak{D}(A)$. Define now $\begin{pmatrix}u\\v\end{pmatrix} = (\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}x\\y\end{pmatrix}$. Then we have to prove $\left|(\lambda I - \mathfrak{A})\begin{pmatrix}u\\v\end{pmatrix}\right| \geqq (\lambda - \alpha)\,.\,\left|\begin{pmatrix}u\\v\end{pmatrix}\right|$.

This is evident if $\left|\begin{pmatrix}u\\v\end{pmatrix}\right| = 0$. Let now $\left|\begin{pmatrix}u\\v\end{pmatrix}\right| > 0$. We obtain by 3° and 5°

$$\left|(\lambda I - \mathfrak{A})\begin{pmatrix}u\\v\end{pmatrix}\right| \left|\begin{pmatrix}u\\v\end{pmatrix}\right| \geqq \left\langle (\lambda I - \mathfrak{A})\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}u\\v\end{pmatrix}\right\rangle =$$
$$= \lambda\left|\begin{pmatrix}u\\v\end{pmatrix}\right|^2 - \left\langle \mathfrak{A}\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}u\\v\end{pmatrix}\right\rangle \geqq (\lambda - \alpha)\left|\begin{pmatrix}u\\v\end{pmatrix}\right|^2.$$

7°

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right| \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}\left|\begin{pmatrix}x\\y\end{pmatrix}\right|$$

for every $x \in \mathfrak{D}(A)$, $y \in E$, $\lambda > \alpha$.

Proof. As known, $(\mathrm{d}^n/\mathrm{d}\lambda^n)(\lambda I - \mathfrak{A})^{-1} = (-1)^n\, n!\,(\lambda I - \mathfrak{A})^{-n+1}$. Now we obtain by induction from 6° and 4°:

$$\left|(\lambda I - \mathfrak{A})^{-q}\begin{pmatrix}x\\y\end{pmatrix}\right| \leqq \frac{1}{(\lambda - \alpha)^q}\left|\begin{pmatrix}x\\y\end{pmatrix}\right| \quad \text{for} \quad x \in \mathfrak{D}(A)\,, \quad y \in E, \quad q = 1, 2, \ldots$$

8° Final part: By lemma 4,1 and 4,5 we have for $x \in E$ and $\lambda > 0$:

$$(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}0\\x\end{pmatrix} = \begin{pmatrix}(\lambda^2 I - A)^{-1}(x)\\ \lambda(\lambda^2 I - A)^{-1}(x)\end{pmatrix}$$

and consequently

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda I - \mathfrak{A})^{-1}\begin{pmatrix}0\\x\end{pmatrix} = \begin{pmatrix}\dfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^2 I - A)^{-1}(x)\\ \dfrac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda(\lambda^2 I - A)^{-1}(x)\end{pmatrix}$$

for $n = 0, 1, 2, \ldots$

Now 7° implies for $x \in E$, $\lambda > \alpha$ and $p = 0, 1, 2, \ldots$

$$\Bigg(\vartheta\left\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^2 I - A)^{-1}(x)\right\|^2 + \left\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda(\lambda^2 I - A)^{-1}(x)\right\|^2 -$$

$$- \left\langle A\left(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^2 I - A)^{-1}(x)\right), \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda^2 I - A)^{-1}(x)\right\rangle\Bigg)^{1/2} \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}\|x\|$$

$\left(\text{note } \left\|\begin{pmatrix}0\\x\end{pmatrix}\right\| = \|x\|\right)$. This proves that

$$\left\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda(\lambda^2 I - A)^{-1}(x)\right\| \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}\|x\|$$

and completes the proof.

4,7. Lemma. *If $A \in \mathfrak{L}^+(E)$ and α is a nonnegative constant such that*

(I) *$\mu I - A$ is one-to-one and $(\mu I - A)^{-1} \in \mathfrak{L}(E)$ for every $\mu > \sqrt{\alpha}$*

(II) $$\left\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda(\lambda^2 I - A)^{-1}\right\| \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}$$

for every $\lambda > \alpha$ and $n = 0, 1, 2, \ldots$,

then for every $\omega \geqq 0$:

(1) *$\lambda^2 I - \omega^2 I - A$ is one-to-one and $(\lambda^2 I - \omega^2 I - A)^{-1} \in \mathfrak{L}(E)$ for $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$.*

(2) $$\left\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\lambda(\lambda^2 I - \omega^2 I - A)^{-1}\right\| \leqq (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2}\right)$$

for every $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$ and $n = 0, 1, 2, \ldots$

Proof. 1° If $\lambda > \alpha$, $n = 0, 1, 2, \ldots$, then

$$\left\| \frac{d^n}{d\lambda^n} (\lambda^2 I - A)^{-1} \right\| \leqq (-1)^n \frac{d^n}{d\lambda^n} \left(\frac{1}{\lambda(\lambda - \alpha)} \right).$$

Proof.

$$\left\| \frac{d^n}{d\lambda^n} (\lambda^2 I - A)^{-1} \right\| = \left\| \frac{d^n}{d\lambda^n} \left(\frac{1}{\lambda} \lambda(\lambda^2 I + A)^{-1} \right) \right\| =$$

$$= \left\| \sum_{i=0}^{n} \binom{n}{i} \frac{d^i}{d\lambda^i} \left(\frac{1}{\lambda} \right) \frac{d^{n-i}}{d\lambda^{n-i}} \lambda(\lambda^2 I - A)^{-1} \right\| \leqq (-1)^n \frac{d^n}{d\lambda^n} \left(\frac{1}{\lambda(\lambda - \alpha)} \right).$$

2° If $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$, then $\lambda > \alpha$, $\lambda^2 - \omega^2 > \alpha^2$ and $\omega^2/(\lambda(\lambda - \alpha)) < 1$.

Proof is easy.

3° If $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$, then $(\lambda^2 I - \omega^2 I - A)^{-1} = (\lambda^2 I - A)^{-1} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1}$.

Proof. It follows from 1° and 2° that $\|\omega^2(\lambda^2 I - A)^{-1}\| \leqq \omega^2/(\lambda(\lambda - \alpha)) < 1$. Thus it remains only to verify 3° by direct computation.

4° If $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$, then

$$\frac{d}{d\lambda} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} = \omega^2 (I - \omega^2(\lambda^2 I - A)^{-1})^{-2} \frac{d}{d\lambda} (\lambda^2 I - A)^{-1}.$$

Proof. $(1/h) [(I - \omega^2((\lambda + h^2) I - A)^{-1})^{-1} - (I - \omega^2(\lambda^2 I - A)^{-1})^{-1}] = \omega^2(I - \omega^2((\lambda + h)^2 I - A)^{-1})^{-1} (I - \omega^2(\lambda^2 I - A)^{-1} (1/h) [((\lambda + h)^2 I - A)^{-1} - (\lambda^2 I - A)^{-1}]_{\overrightarrow{h \to 0}} \omega^2(I - \omega^2(\lambda^2 I - A)^{-1})^{-2} (d/d\lambda) (\lambda^2 I - A)^{-1}$.

5° For every $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$, $p = 0, 1, 2, \ldots$

$$\left\| \frac{d^n}{d\lambda^n} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} \right\| \leqq (-1)^n \frac{d^n}{d\lambda^n} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right).$$

Proof by induction. Because

$$\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} = \left(1 - \frac{\omega^2}{\lambda(\lambda - \alpha)} \right)^{-1}$$

we verify easy the case $n = 0$.

For $n + 1$ we obtain following 4°:

$$\left\| \frac{d^{n+1}}{d\lambda^{n+1}} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} \right\| =$$

$$= \omega^2 \left\| \frac{d^n}{d\lambda^n} \left[(I - \omega^2(\lambda^2 I - A)^{-1})^{-2} \frac{d}{d\lambda} (\lambda^2 I - A)^{-1} \right] \right\| =$$

$$= \omega^2 \left\| \sum_{i=0}^{n} \binom{n}{i} \left[\sum_{j=0}^{i} \binom{i}{j} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} \,. \right. \right.$$

$$\left. \left. \cdot \frac{\mathrm{d}^{i-j}}{\mathrm{d}\lambda^{i-j}} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} \right] \frac{\mathrm{d}^{n-i+1}}{\mathrm{d}\lambda^{n-i+1}} (\lambda^2 I - A)^{-1} \right\| \leqq$$

$$\leqq \omega^2 \sum_{i=0}^{n} \binom{n}{i} \left[\sum_{j=0}^{i} \binom{i}{j} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right) (-1)^{i-j} \,. \right.$$

$$\left. \cdot \frac{\mathrm{d}^{i-j}}{\mathrm{d}\lambda^{i-j}} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right) \right] (-1)^{n-i} \frac{\mathrm{d}^{n-i+1}}{\mathrm{d}\lambda^{n-i+1}} \left(\frac{1}{\lambda(\lambda - \alpha)} \right) =$$

$$= (-1)^{n+1} \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right)^2 \frac{\mathrm{d}^{n-i+1}}{\mathrm{d}\lambda^{n-i+1}} \left(\frac{\omega^2}{\lambda(\lambda - \alpha)} \right) =$$

$$= (-1)^{n+1} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[\left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right)^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\omega^2}{\lambda(\lambda - \alpha)} \right) \right] =$$

$$= (-1)^{n+1} \frac{\mathrm{d}^{n+1}}{\mathrm{d}\lambda^{n+1}} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right),$$

because

$$\left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right)^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\omega^2}{\lambda(\lambda - \alpha)} \right) = \left(1 - \frac{\omega^2}{\lambda(\lambda - \alpha)} \right)^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\omega^2}{\lambda(\lambda - \alpha)} \right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(1 - \frac{\omega^2}{\lambda(\lambda - \alpha)} \right)^{-1} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right).$$

6° Final part: It follows immediately from 3° and 5° for $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$, $n = 0, 1, 2, \ldots$:

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 I - \omega^2 I - A)^{-1} \right\| =$$

$$= \left\| \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} \lambda(\lambda^2 I - A)^{-1} \frac{\mathrm{d}^{n-i}}{\mathrm{d}\lambda^{n-i}} (I - \omega^2(\lambda^2 I - A)^{-1})^{-1} \right\| \leqq$$

$$\leqq \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}\lambda^i} \left(\frac{1}{\lambda - \alpha} \right) (-1)^{n-i} \frac{\mathrm{d}^{n-i}}{\mathrm{d}\lambda^{n-i}} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right) =$$

$$= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\frac{1}{\lambda - \alpha} \frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\frac{\lambda}{\lambda^2 - \alpha\lambda - \omega^2} \right).$$

4,8. Lemma. *For every* $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$

$$\frac{\lambda}{\lambda^2 - \alpha\lambda - \omega^2} = \int_0^\infty e^{-\lambda\tau} e^{\alpha\tau/2} \operatorname{coh}[\tau \sqrt{(\tfrac{1}{4}\alpha^2 + \omega^2)}] \,\mathrm{d}\tau +$$

$$+ \tfrac{1}{2}\alpha \int_0^\infty e^{-\lambda\tau} e^{\alpha\tau/2} \int_0^\tau \operatorname{coh}[\sigma \sqrt{(\tfrac{1}{4}\alpha^2 + \omega^2)}] \,\mathrm{d}\sigma \,\mathrm{d}\tau \,.$$

Proof. We may write:

$$\frac{\lambda}{\lambda^2 - \alpha\lambda - \omega^2} = \frac{\lambda - \frac{1}{2}\alpha}{(\lambda - \frac{1}{2}\alpha)^2 - (\frac{1}{4}\alpha^2 + \omega^2)} + \frac{\alpha}{2} \frac{1}{(\lambda - \frac{1}{2}\alpha)^2 - (\frac{1}{4}\alpha^2 + \omega^2)}.$$

On the other hand, it is known that $\mu/(\mu^2 - \omega^2) = \int_0^\infty e^{-\mu\tau} \operatorname{coh}(\omega\tau)\, d\tau$ $(\mu > \omega)$ a this gives immediately our lemma.

4,9. Lemma. *There is*

$$(-1)^n \frac{d^n}{d\lambda^n}\left(\frac{\lambda}{\lambda^2 - \alpha\lambda - \omega^2}\right) \leqq n!\left(\frac{1}{[\lambda + \alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}} + \right.$$
$$\left. + \frac{1}{[\lambda - \alpha - \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}}\right)$$

for every $\lambda > \alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$ *and* $n = 0, 1, 2, \ldots$

Proof. By 4,8

$$(-1)^n \frac{d^n}{d\lambda^n}\left(\frac{\lambda}{\lambda^2 - \alpha\lambda - \omega^2}\right) = \int_0^\infty e^{-\lambda\tau}\tau^n e^{\alpha\tau/2}\left(1 + \frac{\alpha}{2}\tau\right)\operatorname{coh}[\tau\sqrt{(\tfrac{1}{4}\alpha^2 + \omega^2)}] \leqq$$
$$\leqq \int_0^\infty e^{-\lambda\tau}\tau^n e^{\alpha\tau} \exp(\tau\sqrt{(\tfrac{1}{4}\alpha^2 + \omega^2)}) \leqq 2\int_0^\infty e^{-\lambda\tau}\tau^n \operatorname{coh}\{\tau[\alpha + \sqrt{(\tfrac{1}{4}\alpha^2 + \omega^2)}]\} =$$
$$= n!\left(\frac{1}{[\lambda + \alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}} + \frac{1}{[\lambda - \alpha - \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}}\right).$$

4,10. Fundamental theorem. *If* $A \in \mathfrak{L}^+(E)$ *is normal and* α, ω *are two nonnegative constants such that*

(I) $\langle A(x), x\rangle \leqq \omega^2\langle x, x\rangle$ *for* $x \in \mathfrak{D}(A)$,

(II) *there exists a constant* $\vartheta \geqq 0$ *so that*

$$\tfrac{1}{2}[\langle A(x)\, y\rangle - \langle A(y), x\rangle] \leqq$$
$$\leqq \alpha[\vartheta\langle x, x\rangle + \langle y, y\rangle + \omega^2\langle x, x\rangle - \langle Ax, x\rangle] - \vartheta\langle x, y\rangle$$

for every $x, y \in \mathfrak{D}(A)$

then there exists a regular cosine function $\mathscr{C}$ *so that*

(1) $\mathscr{C}^{\cdot} = A$,

(2) $\|\mathscr{C}(t)\| \leqq e^{\alpha t/2} \operatorname{coh}[t\sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}] + \frac{1}{2}\alpha\, e^{\alpha t/2} \int_0^t \operatorname{coh}[\sigma\sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]\, d\sigma$ *for every* $t \in R^+$.

Proof. Let us define $A_0 = A - \omega^2 I$. Then A_0 fullfils all assumptions of the lemma 4,6 and therefore $\mu I - A_0$ is one-to-one, $(\mu I - A)^{-1} \in \mathfrak{L}(E)$ for $\mu > \sqrt{\alpha}$ and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 I - A_0)^{-1} \right\| \leqq \frac{n!}{(\lambda - \alpha)^{n+1}}$$

for $\lambda > \alpha$ and $n = 0, 1, 2, \ldots$ This implies that all assumptions of 4,7 are also fulfilled and consequently $\lambda^2 I - \omega^2 I - A$ is one-to-one, $(\lambda^2 I - \omega^2 I - A)^{-1} \in \mathfrak{L}(E)$ and

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 I - \omega^2 I - A)^{-1} \right\| \leqq (-1)^p \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left(\frac{\lambda(\lambda - \alpha)}{\lambda^2 - \alpha\lambda - \omega^2} \right)$$

for $\lambda > \frac{1}{2}\alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}$ and $n = 0, 1, 2, \ldots$

But by 4,9

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 I - \omega^2 I - A)^{-1} \right\| \leqq$$

$$\leqq n! \left(\frac{1}{[\lambda + \alpha + \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}} + \frac{1}{[(\lambda - \alpha - \sqrt{(\frac{1}{4}\alpha^2 + \omega^2)}]^{n+1}} \right)$$

and by 3,2 [3] there exists a regular cosine function $\mathscr{C}$ such that $\mathscr{C}^{\cdot} = A$. Finally, 1,19 [3] with 4,8 establishes the second assertion of our theorem.

Remark. Condition (II) describes a "subordination" of antisymetric part of A to symetric part.

4,11. Corollary. *If A is a selfadjoint operator and ω a nonnegative constant such that $\langle A(x), x\rangle \leqq \omega^2 \langle x, x\rangle$ for $x \in \mathfrak{D}(A)$, then there exists a regular cosine function $\mathscr{C}$ such that*

(1) $\mathscr{C}^{\cdot} = A$,

(2) $\|\mathscr{C}(t)\| \geqq \operatorname{coh}(\omega t)$ *for every* $t \in R^+$.

Proof. This is an immediate consequence of 4,10 if we take $\alpha = \vartheta = 0$.

Remark. Another proof of this corollary with the use of spectral resolution can also be given and is well known.

4,12. Fundamental theorem. *If $A \in \mathfrak{L}^+(E)$, then there exists a regular cosine function $\mathscr{C}$ of selfadjoint operators such that $\mathscr{C}^{\cdot} = A$ if and only if A is normal and bounded above.*

Proof. Easy consequence of 4,11, 3,17 and 2,3.

4,13. Theorem. *In every infinitedimensional Hilbert space E there exists a normal (antisymmetric) operator A bounded above such that for no regular cosine function $\mathscr{C}$ in E we have $\mathscr{C}^{\cdot} = A$.*

Proof. 1° As known in every infinitedimensional Hilbert space E there exists an infinitedimensional separable subspace D. Denote Π the symmetric projection of E onto D. Further we choose a fixed ortonormal basis $e_0, e_1, e_2, \ldots$ in D.

Now for arbitrary $x \in E$ we shall denote x_i the i-th coordinate of $\Pi(x)$ in D with respect to the basis $e_0, e_1, e_2, \ldots$, i.e. $x_i = \langle \Pi(x), e_i \rangle$.

2° Let us define an operator $A \in \mathfrak{L}^+(E)$ as follows: $x \in \mathfrak{D}(A)$ if and only if $\sum_{k=0}^{\infty} k(x_{2k}^2 + x_{2k+1}^2)$ exists; if $x \in \mathfrak{D}(A)$, then we take $A(x) = \sum_{k=0}^{\infty} k(x_{2k+1}e_{2k} - x_{2k}e_{2k+1})$.

3° A is antisymmetric.

Proof. If $x, y \in \mathfrak{D}(A)$ then $\langle A(x), y \rangle = \langle \sum_{k=0}^{\infty} k(x_{2k+1}e_{2k} - x_{2k}e_{2k+1}), \sum_{k=0}^{\infty} (y_{2k}e_{2k} + y_{2k+1}e_{2k+1}) \rangle = \sum_{k=0}^{\infty} k(x_{2+1}y_{2k} - x_{2k}y_{2k+1}) = -\sum_{k=0}^{\infty} k(y_{2k+1}x_{2k} - y_{2k}x_{2k+1}) = -\langle \sum_{k=0}^{\infty} k(y_{2k+1}e_{2k} - y_{2k}e_{2k+1}), \sum_{k=0}^{\infty} (x_{2k}e_{2k} + x_{2k+1}e_{2k+1}) \rangle = -\langle A(y), x \rangle$.

4° $\mathfrak{D}(A) \subseteq \mathfrak{D}(A^*)$.

Proof. Follows immediately from 3°.

5° $\mathfrak{D}(A^*) \subseteq \mathfrak{D}(A)$.

Proof. Let $y \in \mathfrak{D}(A^*)$, i.e. $\langle A(x), y \rangle$ is continuous in $x \in \mathfrak{D}(A)$ and hence we can find a constant $c \geqq 0$ such that $|\langle A(x), y \rangle| \leqq c\|x\|$ for every $x \in \mathfrak{D}(A)$.

Let us now define a sequence $x^{(p)} \in E$ $(p = 0, 1, 2, \ldots)$ as follows: $x^{(p)} = \sum_{k=0}^{p} (-y_{2k+1}e_{2k} + y_{2k}e_{2k+1})$. Evidently $x^{(p)} \in \mathfrak{D}(A)$ and $\langle A(x^{(p)}), y \rangle = \sum_{k=0}^{p} k(y_{2k}^2 + y_{2k+1}^2)$. But by our assumption $|\langle A(x^{(p)}), y \rangle| \leqq c\|x^{(p)}\|$. Since evidently $\|x^{(p)}\| \leqq \|y\|$, we obtain for all $p = 0, 1, 2, \ldots$ that $\sum_{k=0}^{p} k(y_{2k}^2 + y_{2k+1}^2) \leqq c\|y\|$. Therefore by 2° $y \in \mathfrak{D}(A)$ and hence $\mathfrak{D}(A^*) \subseteq \mathfrak{D}(A)$.

6° $A^* = -A$.

Proof. This is an immediate consequence of 3°, 4°, 5°.

7° A is normal and $\langle A(x), x \rangle = 0$ for every $x \in \mathfrak{D}(A)$.

Proof. This follows from 6°.

8° There does not exist a regular cosine function $\mathscr{C}$ such that $\mathscr{C}^{\cdot\cdot} = A$.

Proof. Assume contrary. Then by 2,5 in [3] there exist two nonnegative constants M, ω such that $\|\mathscr{C}(t)\| \leqq M \operatorname{coh}(\omega t)$. Now by the use of 3,1 in [3] we obtain that $\lambda^2 j - \mathscr{C}^{\cdot}$ is one-to-one, $(\lambda^2 j - \mathscr{C}^{\cdot})^{-1} \in \mathfrak{L}(E)$ and

$$[*] \qquad \left\| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 j - \mathscr{C}^{\cdot})^{-1} \right\| \leqq \frac{Mn!}{2} \left(\frac{1}{(\lambda + \omega)^{n+1}} + \frac{1}{(\lambda - \omega)^{n+1}} \right)$$

for every $\lambda > \omega$ and $n = 0, 1, 2, \ldots$

Now a simple computation shows that

$$(\mu j - \mathscr{C}^{\cdot})^{-1} (\alpha_1 e_{2k} + \alpha_2 e_{2k+1}) =$$
$$= \left(\frac{\mu}{\mu^2 + k^2} \alpha_1 + \frac{k}{\mu^2 + k^2} \alpha_2 \right) e_{2k} + \left(\frac{\mu}{\mu^2 + k^2} \alpha_1 - \frac{k}{\mu^2 + k^2} \alpha_2 \right) e_{2k+1}$$

for every $\mu > \omega^2$, $\alpha_1, \alpha_2 \in R$ and $k = 0, 1, 2, \ldots$

This implies $\langle (\mu j - \mathscr{C}^{\cdot})^{-1} (e_{2k}), e_{2k} \rangle = \mu/(\mu^2 + k^2)$ and $\langle \lambda(\lambda^2 j - \mathscr{C}^{\cdot})^{-1} (e_{2k}), e_{2k} \rangle = \lambda^3/(\lambda^4 + k^2)$ for every $\mu > \omega^2$, $\lambda > \omega$ and $k = 0, 1, 2, \ldots$

Since from [*]

$$\left| \left\langle \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \lambda(\lambda^2 j - \mathscr{C}^{\cdot})^{-1} (e_{2k}), e_{2k} \right\rangle \right| \leqq \frac{Mn!}{2} \left(\frac{1}{(\lambda + \omega)^{n+1}} + \frac{1}{(\lambda - \omega)^{n+1}} \right)$$

we obtain that

$$\left| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \frac{\lambda^3}{\lambda^4 + k^2} \right| \leqq \frac{Mn!}{2} \left(\frac{1}{(\lambda + \omega)^{n+1}} + \frac{1}{(\lambda - \omega)^{n+1}} \right).$$

But

$$\frac{\lambda^3}{\lambda^4 + k^2} = \int_0^\infty e^{-\lambda\tau} \cos \left[\sqrt{\left(\frac{k}{2} \right)} \tau \right] \operatorname{coh} \left[\sqrt{\left(\frac{k}{2} \right)} \tau \right] \mathrm{d}\tau$$

for every $\lambda > 0$ (cf. the proof of 4,11 in [3]). If we proceed analogously as in the proof of 4,11 in [3] we obtain that $\cos [\sqrt{(k/2)}\, t] \operatorname{coh} [\sqrt{(k/2)}]\, t \leqq M \operatorname{coh} (\omega t)$ for every $t \in R^+$ and $k = 0, 1, 2, \ldots$ But this is evidently impossible for sufficiently large k. We have obtained a contradiction and hence proved 8°.

9° Final part: 7° and 8° establish our theorem.

References

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