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A CONTRIBUTION TO RELATIONS BETWEEN GÖDELIAN AND ZERMELIAN SET THEORIES

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The main purpose of this paper is to study the set theory Σ_{∞}^{-}. In a certain sense, this theory lies between Gödel-Bernays' set theory Σ and Zermelo-Fraenkel's set theory ZF. It contains the concept of a class but instead of the axiom C4, that stands for the whole infinite axiom-scheme of the theory ZF in Σ, that scheme is accepted.

Relations between theories Σ and ZF have been investigated by various authors – see [6], [8], [9] – who proved that both theories are consistent or inconsistent simultaneously. These proofs are either nonfinitistic or (cf. [9]) finitistic but the latter are not carried through with the use of the concept of a syntactical model.

In this paper properties of the theory Σ_{∞}^{-} are studied by finitistic means and it is proved that Σ_{∞}^{-} is actually weaker than Σ. At the same time it is shown – regardless of our being sure, on the basis of cited results, that no statement concerning sets only can be proved in Σ and can not be proved in Σ_{∞}^{-} – that it is not possible to construct a syntactic parametric model of the theory Σ in Σ_{∞}^{-}.

Concepts and notations introduced in [3] will be commonly used. The concepts of syntactic parametric model and strongly regular model can be found in [4] and [2].

1. THE SETS p_α

1.1. Definition. $x \in K_1 \equiv x \in On \,\&\, (\exists y)\,(y \in On \,\&\, x = y + 1), x \in K_2 \equiv x \in On \,\&$ $\&\, x \notin K_1$. Any ordinal number from K_2 is called a limit ordinal number.

1.2. Definition. Define a function G over V (universal class) as follows

$$\mathfrak{D}(x) \in K_2 \rightarrow G'x = \mathfrak{S}(\mathfrak{W}(x))$$
$$\mathfrak{D}(x) \in V - K_2 \rightarrow G'x = \mathfrak{P}(\mathfrak{S}(\mathfrak{W}(x))).$$

The existence of G follows by M6 in [3].

We can now define a function F over On by the following postulate $F'\alpha = G'(F \upharpoonright \alpha)$. The existence and uniqueness of F follow from Theorem 7.5 in [3]. We shall denote $F'\alpha = p_\alpha$.

The sets p_α have the following properties:

(a) $p_0 = 0$.
(b) $p_{\alpha+1} = \mathfrak{P}(p_\alpha)$.
(c) $p_\alpha = \bigcup_{\beta\in\alpha} p_\beta$ for $\alpha \in K_2$.

1.3. Lemma.
(a) $(\forall\alpha)(\alpha \neq 0 \rightarrow p_\alpha \neq \emptyset)$.
(b) $(\forall\alpha)(p_\alpha \in p_{\alpha+1})$.
(c) $(\forall\alpha, \beta)(\beta \in \alpha \rightarrow p_\beta \subset p_\alpha)$.

1.4. Lemma. $(\forall\alpha)(\mathrm{Comp}(p_\alpha))$.

1.5. Lemma. $\mathfrak{S}(p_\alpha) = p_\alpha$ *if and only if* $\alpha \in K_2$; *if* $\alpha \in K_1$ *then* $\mathfrak{S}(p_\alpha) = p_{\alpha-1}$.

1.6. Lemma. $(\forall\alpha)(\alpha \in p_{\alpha+1})$.
For proofs of these lemmas see [1].

1.7. Lemma. *Let* $\alpha_1, \alpha_2, \ldots$ *be a sequence of ordinal numbers. Then* $\bigcup_{n\in\omega} p_{\alpha_n} = p_{\bigcup_{n\in\omega}\alpha_n}$.

Proof. 1) There is a maximal element α_{n_0} among α_n $(n = 1, 2, \ldots)$. Then $\bigcup_{n\in\omega}\alpha_n = \alpha_{n_0}$. Because the number α_{n_0} is one of numbers α_n, we have $p_{\alpha_{n_0}} \subseteq \bigcup_{n\in\omega} p_{\alpha_n}$. On the other hand, if $x \in \bigcup_{n\in\omega} p_{\alpha_n}$, there is an integer m such that $\alpha_m \leqq \alpha_{n_0}$ (and then $p_{\alpha_m} \subseteq p_{\alpha_{n_0}}$) and $x \in p_{\alpha_m}$. Then $x \in p_{\bigcup_{n\in\omega}\alpha_n}$.

2) There is not a maximal element among α_n $(n = 1, 2, \ldots)$. Then $\lambda = \bigcup_{n\in\omega}\alpha_n$ is a limit ordinal number and $p_\lambda = \bigcup_{\beta\in\lambda} p_\beta$. Certainly $\bigcup_{n\in\omega} p_{\alpha_n} \subseteq \bigcup_{\beta\in\lambda} p_\beta$. On the other hand, for each $\beta \in \lambda$ there exists an integer n such that $\beta \leqq \alpha_n$ (otherwise $\lambda = \bigcup_{n\in\omega}\alpha_n$ is not true); then $p_\beta \subseteq p_{\alpha_n}$ and from this it follows $\bigcup_{\beta\in\lambda} p_\beta \subseteq \bigcup_{n\in\omega} p_{\alpha_n}$. This means that $\bigcup_{n\in\omega} p_{\alpha_n} = p_{\bigcup_{n\in\omega}\alpha_n}$.

1.8. Lemma. *Let* $\alpha_1, \alpha_2, \ldots$ *be a sequence of limit ordinal numbers. Then* $\bigcup_{n\in\omega}\alpha_n$ *is a limit ordinal number.*

The proof is easy.

1.9. Lemma. $\bigcup_{\alpha\in On} p_\alpha = V$.

For the proof see [10].

As a consequence of this lemma we can now define the type of a set x (we denote it by $\tau(x)$) as the least ordinal number α such that $x \in p_\alpha$.

1.10. Definition. $\tau(x) = \alpha \equiv . \, x \in p_\alpha \,\&\, (\forall \beta)\,(x \in p_\beta \to \alpha \leqq \beta)$

$$\bar{\tau}(X) = \bigcup_{y \in X} \tau(y)\,.$$

1.11. Lemma. $(\forall x)\,(\tau(x) = \bar{\tau}(x) + 1)$

$$\mathfrak{Pr}(X) \equiv \bar{\tau}(X) = On\,.$$

1.12. Lemma. $x \in p_\alpha \equiv x \subseteq p_\alpha \,\&\, \bar{\tau}(x) < \alpha.$

$$\bar{\tau}(\alpha) = \bar{\tau}(p_\alpha) = \alpha\,.$$

For proofs of these lemmas see [1].

1.13. Lemma. *If α is a limit ordinal number, then*

$$x \in p_\alpha \,\&\, y \subseteq x \to y \in p_\alpha\,.$$

Proof. Since α is a limit ordinal number we have $p_\alpha = \bigcup_{\beta \in \alpha} p_\beta$ and consequently, there is $\beta_0 \in \alpha$ such that $x \in p_{\beta_0}$. We have $x \subset p_{\beta_0}$ by 1.4 and consequently $y \subset p_{\beta_0} \subset \subset p_\alpha$. It is clear that $\tau(y) \leqq \beta_0 + 1$ and hence $\bar{\tau}(y) \leqq \beta_0 < \alpha$. From 1.12 it follows that $y \in p_\alpha$.

1.14. Lemma. *Let α be a limit ordinal number, let $x, y \in p_\alpha$. Then*

(a) $\{x, y\} \in p_\alpha$.
(b) $\mathfrak{S}(x) \in p_\alpha$.
(c) $\mathfrak{P}(x) \in p_\alpha$.

Proof. As α is a limit ordinal number we have $\tau(x) < \alpha$, $\tau(y) < \alpha$.

(a) $\{x, y\} \subset p_\alpha$, $\bar{\tau}(\{x, y\}) = \mathrm{Max}\,(\tau(x), \tau(y)) < \alpha$ and hence $\{x, y\} \in p_\alpha$ by 1.12.
(b) There is $\beta < \alpha$ such that $x \in p_\beta$, hence $x \subseteq p_\beta$. $\mathfrak{S}(x) \subseteq \mathfrak{S}(p_\beta) \subseteq p_\beta \subset p_\alpha$ (see 1.5), then $\tau(\mathfrak{S}(x)) \leqq \beta + 1$, $\bar{\tau}(\mathfrak{S}(x)) \leqq \beta < \alpha$ From 1.12 it follows that $\mathfrak{S}(x) \in p_\alpha$.
(c) Similarly.

2. THE THEORY Σ_∞^-

Let $\varphi(x, y, t_1, \ldots, t_r)$ be a ppf (primitive propositional formula) containing set variables only. We shall often write briefly $\varphi(x, y, \boldsymbol{t})$. In particular, $\boldsymbol{t}$ can be an empty sequence.

Let us denote $\mathfrak{A}_\varphi(t)$ a term such that

$$(1) \qquad \vdash_{\Sigma} (\forall t)(\forall z)(z \in \mathfrak{A}_\varphi(t) \equiv (\exists x, y)(z = \langle x, y\rangle \mathbin{\&} \varphi(x, y, t)))\,.$$

($\vdash_{\Sigma} \varphi$ denotes the fact that φ is provable in Σ). The formula in (1) means that, for every $t_1, \ldots, t_r$, $\mathfrak{A}_\varphi(t)$ is precisely the class of all ordered pairs $\langle x, y\rangle$ for which $\varphi(x, y, t)$ holds.

Now we write formulas

$$(2) \qquad (\forall t)(\mathrm{Un}(\mathfrak{A}_\varphi(t)) \rightarrow (\forall p)\, \mathrm{M}(\mathfrak{A}_\varphi(t)'' p))\,.$$

C3′: $(\forall x)(\exists y)(\forall U)(U \subseteq x \rightarrow U \in y)$.

C3′ is an (inessential) modification of the axiom C3 of the theory Σ.

We shall denote Σ_∞^- the theory whose language is the same as that of Σ and whose axioms are A1 – A4, B1 – B8, C1, C2, C3′, D, and (2) for each ppf φ with set variables only. Consequently, the theory Σ_∞^- is obtained from the Gödel-Bernays set theory Σ by replacing the axiom C4 by a given axiom-scheme (and we have C3′ instead of C3).

Further we denote Σ_n^- the theory that differs from Σ_∞^- only in the following item: instead of the whole scheme, Σ_n^- uses only the first n instances of the scheme for its axioms (n is a metamathematical natural number).

Now, let $\varphi(x, y, t_1, \ldots, t_r)$ be a ppf ($r \geqq 0$). We shall introduce the following abbreviations:

Un_φ – we read „the formula φ is single-valued" – is the abbreviation for the formula

$$(\forall x_1, x_2, y)(\varphi(x_1, y, t) \mathbin{\&} \varphi(x_2, y, t) \rightarrow x_1 = x_2)\,.$$

$\mathrm{Im}_\varphi(p, q)$ – we read „the set q is an image of the set p under the formula φ" – is the abbreviation for the formula

$$(\forall u)(u \in q \equiv (\exists v)(v \in p \mathbin{\&} \varphi(u, v, t)))\,.$$

2.1. Lemma. *Let $\varphi(x, y, t_1, \ldots, t_r)$ be a ppf, let $\mathfrak{A}_\varphi(t)$ be a term introduced by the formula* (1). *Then the formula* (2) *is equivalent to the formula whose abbreviation is*

$$(3) \qquad (\forall t)(\mathrm{Un}_\varphi \rightarrow (\forall p)(\exists q)(\mathrm{Im}_\varphi(p, q)))\,.$$

The proof is easy.

We shall realize the following construction[1]) in the theory Σ^*. Let n be a fixed metamathematical natural number, let $\varphi_1(x, y, t), \ldots, \varphi_n(x, y, t)$ be ppfs that appear in the axioms of the theory Σ_n^-. Let us denote Φ a formula that is a conjunction of for-

[1]) During my paper had been printed I was told that the construction presented here was used to prove the so called Reflection Principle by Montague.

mulas, abbreviations of which are (3) (where we write $\varphi_1, \ldots, \varphi_n$ instead of φ). We shall bring Φ to the prenex normal form, i.e. to the form

$$(4) \qquad (\forall x_1, \ldots, x_m)(\exists y_1, \ldots, y_l)(\forall z_1, \ldots, z_p)(\exists t_1, \ldots, t_q) \ldots$$
$$\ldots \Psi(x_1, \ldots, x_m, y_1, \ldots, y_l, z_1, \ldots, z_p, t_1, \ldots, t_q, \ldots)$$

where Ψ has no quantifiers.

We shall form a formula equivalent to (4) and containing no existential quantifier in the following way. (cf. [5]). We rewrite (4) to the form

$$(5) \qquad (\forall x_1, \ldots, x_m)(\exists y_1)\, \Psi_1 .$$

Ψ_1 is the formula

$$(\exists y_2, \ldots, y_l)(\forall z_1, \ldots, z_p)(\exists t_1, \ldots, t_q) \ldots \Psi .$$

In the set theory we can describe the set of members of the smallest type from every y_1 for which (5) holds and from this set we shall select one member (with the aid of the axiom of choice) which we denote by $\bar{y}_1$. Consequently, we can introduce a logical function $\mathfrak{S}^1_\Psi(A)$, depending on one variable for a chosen class, such that

$$\vdash_{\Sigma_*} (\forall x_1, \ldots, x_m)(\forall y_1)(\langle y_1, x_1, \ldots, x_m\rangle \in \mathfrak{S}^1_\psi(A) \equiv y_1 = \bar{y}_1) .$$

Then we have in the set theory:

$$(6) \qquad (\forall x_1, \ldots, x_m)(\exists y_1)\, \Psi_1 \equiv (\forall x_1, \ldots, x_m)\, \overline{\Psi}_1 ,$$

where $\overline{\Psi}_1$ is the formula

$$(\exists y_2, \ldots, y_l)(\forall z_1, \ldots, z_p)(\exists t_1, \ldots, t_q) \ldots$$
$$\ldots \Psi(x_1, \ldots, x_m, \mathfrak{S}^1_\Psi(A)' \langle x_1, \ldots, x_m\rangle, y_2, \ldots, y_l, z_1, \ldots, z_p, t_1, \ldots, t_q, \ldots) .$$

Since only a finite number of existential quantifiers occurs in (4), we get after a finite number of steps k logical functions such that the formula (4) is equivalent to the formula (we briefly write $\mathfrak{S}^i_\Psi$ instead of $\mathfrak{S}^i_\Psi(A)$, $i = 1, \ldots, k$)

$$(7) \qquad (\forall x_1, \ldots, x_m)(\forall z_1, \ldots, z_p) \ldots \Psi(x_1, \ldots, x_m, \mathfrak{S}^{1'}_\Psi\langle x_1, \ldots, x_m\rangle, \ldots$$
$$\ldots, \mathfrak{S}^{l'}_\Psi\langle x_1, \ldots, x_m\rangle, z_1, \ldots, z_p, \mathfrak{S}^{l+1'}_\Psi\langle x_1, \ldots, x_m, z_1, \ldots, z_p\rangle, \ldots$$
$$\ldots, \mathfrak{S}^{l+q'}_\Psi\langle x_1, \ldots, x_m, z_1, \ldots, z_p\rangle, \ldots) .$$

Let us realize the following construction (the number n and also the formula Φ are kept fixed). First, we denote $p_{\alpha_n 0} = p_{\omega+1}$. Second, we define the set $p_{\alpha_n i}$ by the following induction for each $i = 1, 2, \ldots$: α_n^i is the least limit ordinal number such that the set $p_{\alpha_n i}$ contains (as its members) all values of functions $\mathfrak{S}^1_\Psi, \ldots, \mathfrak{S}^k_\Psi$ (i.e. all $\mathfrak{S}^{1'}_\Psi\langle x_1, \ldots, x_m\rangle, \ldots, \mathfrak{S}^{l'}_\Psi\langle x_1, \ldots, x_m\rangle$; $\mathfrak{S}^{l+1'}_\Psi\langle x_1, \ldots, x_m, z_1, \ldots, z_p\rangle, \ldots$

$\ldots, \mathfrak{S}_{\Psi}^{l+q'}\langle x_1, \ldots, x_m, z_1, \ldots, z_p\rangle, \ldots)$ for each $x_1, \ldots, x_m, z_1, \ldots, z_p, \ldots$ from $p_{\alpha_n{}^{i-1}}$. Finally we put $p_{\alpha_n} = \bigcup_{i \in \omega} p_{\alpha_n{}^i}$. α_n is a limit ordinal number from 1.7 and 1.8.

2.2. Metatheorem. *There exists a strongly regular model of the theory Σ_n^- in the theory Σ^* for each metamathematical natural number n.*

Proof. Let n be any metamathematical natural number and let p_{α_n} be a constant denoting the set constructed in the way just described. (We realized our construction in the theory Σ^* so that p_{α_n} is really a set from the point of view of this theory). We define fundamental predicates Cls*, M*, $\in^*$ as follows:

$$\text{Cls}^*(X^*) \equiv X^* \subseteq p_{\alpha_n}, \quad \text{M}^*(X^*) \equiv X^* \in p_{\alpha_n}, \quad X^* \in^* Y^* \equiv X^* \in Y^*.$$

We shall denote classes of the model (i.e. subsets of p_{α_n}) by $X^*, Y^*, Z^*, \ldots$, sets of the model (i.e. members of p_{α_n}) by $x^*, y^*, \ldots$

We define $X^* =^* Y^* \equiv X^* = Y^*$.

Next, two statements hold:

$$\text{M}^*(X^*) \equiv (\exists Y^*)(X^* \in^* Y^*). \tag{8}$$

$$X^* =^* Y^* \equiv (\forall z^*)(z^* \in^* X^* \equiv z^* \in^* Y^*). \tag{9}$$

We are to prove that all axioms of the theory Σ_n^- for predicates Cls*, M*, $\in^*$ hold (i.e. that if we sign all axioms of Σ_n^- with an asterisk * we obtain formulas, provable in Σ^*. In φ is any formula of the theory Σ_n^- we obtain φ^* – the formula of the theory Σ^* belonging to φ – by relativizing quantifiers on the set p_{α_n}).

A1* follows from 1.4.
A2* follows from (8).
A3* follows from (9).
A4* follows from (a) in 1.14.
B1*: we put $A^* = A \cap p_{\alpha_n}$.
B2* – B8* are easily provable.
C1*: $\omega \in p_{\omega+1}$ from 1.6 and $p_{\omega+1} \subset p_{\alpha_n}$ from (c) in 1.3 (from the construction of p_{α_n} it follows that $\omega + 1 \in \alpha_n$). Then $\omega \in p_{\alpha_n}$.
C2* follows from (b) in 1.14.
C3* follows from 1.13 and from (c) in 1.14.

Finally, it remains to prove that if we supply the formulas (2) (where we write $\varphi_1, \ldots, \varphi_n$ instead of φ) with an asterisk then the formula that is a conjunction of these is provable in Σ^*. According to 2.1 it is sufficient to prove the formula Φ^*. As we have shown Φ is equivalent to (7) and clearly a consequence of (7) is the formula

$$(\forall x_1^*, \ldots, x_m^*)(\forall z_1^*, \ldots, z_p^*) \ldots \Psi(x_1^*, \ldots, x_m^*, \mathfrak{S}_{\Psi}^1\langle x_1^*, \ldots, x_m^*\rangle, \ldots$$
$$\ldots, \mathfrak{S}_{\Psi}^l\langle x_1^*, \ldots, x_m^*\rangle, z_1^*, \ldots, z_p^*, \mathfrak{S}_{\Psi}^{l+1}\langle x_1^*, \ldots, x_m^*, z_1^*, \ldots, z_p^*\rangle, \ldots$$
$$\ldots, \mathfrak{S}_{\Psi}^{l+q}\langle x_1^*, \ldots, x_m^*, z_1^*, \ldots, z_p^*\rangle, \ldots).$$

As it follows from the construction of p_{α_n}, we have

$$(\forall x_1^*, \ldots, x_m^*)(\forall z_1^*, \ldots, z_p^*) \ldots (\mathfrak{S}_\Psi^{1'}\langle x_1^*, \ldots, x_m^*\rangle \in p_{\alpha_n} \& \ldots$$
$$\ldots \& \mathfrak{S}_\Psi^{l}\langle x_1^*, \ldots, x_m^*\rangle \in p_{\alpha_n} \& \mathfrak{S}_\Psi^{l+1}\langle x_1^*, \ldots, x_m^*, z_1^*, \ldots, z_p^*\rangle \in p_{\alpha_n} \& \ldots$$
$$\ldots \& \mathfrak{S}_\Psi^{l+q}\langle x_1^*, \ldots, x_m^*, z_1^*, \ldots, z_p^*\rangle \in p_{\alpha_n} \& \ldots)$$

and then

$$(\forall x_1^*, \ldots, x_m^*)(\exists y_1^*, \ldots, y_l^*)(\forall z_1^*, \ldots, z_p^*)(\exists t_1^*, \ldots, t_q^*) \ldots$$
$$\ldots \Psi(x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_l^*, z_p^*, \ldots, z_p^*, t_1^*, \ldots, t_q^*, \ldots).$$

The last formula is equivalent to Φ^*. The theorem follows.

2.3. Corollary. *As there is a syntactic model of the theory Σ^* in the theory Σ (see the Δ-model in [3]) we can put the metatheorem 3.2 in the following way:*

There is a strongly regular model of the theory Σ_n^- in the theory Σ for every metamathematical natural number n.

3. THE RELATIONS BETWEEN Σ_∞^- AND Σ

3.1. Metatheorem. *There is no strongly regular model of the theory Σ in the theory Σ.*

For the proof see [11].

3.2. Metatheorem. *The axiom C4 of the theory Σ is not provable in the theory Σ_∞^-.*

Proof. Suppose there is a proof of C4 in Σ_∞^-. This proof is a finite sequence of formulas $\varphi_1, \ldots, \varphi_k$ and these are either axioms of predicate calculus or axioms of Σ_∞^- or follow from the preceding ones by the rule of modus ponens. Let $\varphi_1, \ldots, \varphi_m$ $(0 \leqq m < k)$ be those formulas among $\varphi_1, \ldots, \varphi_k$ which are axioms from the scheme, i.e. formulas of the form (2). Let n_0 be the least metamathematical natural number such that the theory $\Sigma_{n_0}^-$ has all the formulas $\varphi_1, \ldots, \varphi_m$ as its axioms. Then our proof is the proof of C4 in $\Sigma_{n_0}^-$ and hence Σ and $\Sigma_{n_0}^-$ are equivalent. By 2.3, there is a strongly regular model Σ_n^- in Σ for each metamathematical natural number, for n_0 in particular. By this we have proved that a strongly regular model of Σ in Σ exists but this is a contradiction with 3.1.

Mostowski proved in [6] that each formula of the theory Σ expressible in the theory ZF (i.e. each formula with set variables only) and provable in Σ is proavable in ZF, too. (For the same result see [8] and [9]). Clearly each formula provable in ZF is provable in Σ_∞^-, too, thus in this connection we can speak about Σ_∞^- instead of ZF. The proof of the given statement by Mostowski is based on [7] where the relative consistency of Σ with respect to ZF is not proved in a finitistic way (the same holds for [8]). A finitistic proof of this result is given in [9]. We prove a theorem concerning the possibility of a finitary finding of a proof in ZF (or in Σ_∞^-) from a given proof in Σ, which is negative is a certain sense. We show that it is impossible to transfer proofs by the method of parametric models (generalized interpretations).

3.3. Metatheorem. *There is no parametric syntactic model of the theory Σ in the theory Σ_∞^-.*

Proof. Let $\mathfrak{M}_1$ be a model of Σ in Σ_∞^-. Consequently, if we denote Φ the conjunction of axioms of the theory Σ, the formula Φ^* (that is a formula of the language of Σ_∞^- obtained by the translation of Φ through $\mathfrak{M}_1$) is proavble in Σ_∞^-. Similarly as in the proof of 3.2, since the proof of the formula Φ^* consists of a finite number of steps, it is the proof in some $\Sigma_{n_0}^-$ and thus $\mathfrak{M}_1$ is clearly a model of Σ in $\Sigma_{n_0}^-$. By 2.3 there is a strongly regular model $\mathfrak{M}_2$ of $\Sigma_{n_0}^-$ in Σ. The composition $\mathfrak{M} = \mathfrak{M}_2 * \mathfrak{M}_1$ is a strongly regular model of Σ in Σ which is a contradiction with 3.1.

References

[1] *Б. Балцар - Т. Ех:* Модели теории множеств образованные совершенным отношением. Čas. pro pěst. mat., *90* (1965), 430—440.

[2] *Bukovský L. - Hájek P.:* On the Standardness and Regularity of Normal Syntactic Models of the Set Theory. Bull. Acad. Sci. Polon., *XIV*, 3 (1966), 101—105.

[3] *Gödel K.:* The Consistency of the Axiom of Choice etc. Princeton Univ. Press, 1940.

[4] *Hájek P.:* Syntactic Models of Axiomatic Theories. Bull. Acad. Sci. Polon. *XIII*, 4 (1965), 273—278.

[5] *Mostowski A.:* An Undecidable Arithmetical Statement. Fund. Math. *XXXVI* (1949).

[6] *Mostowski A.:* Some Impredicative Definitions in the Axiomatic Set Theory. Fund. Math. *XXXVII* (1950), 111—124.

[7] *Novak L. I.:* A Construction for Consistent Systems. Fund. Math. *XXXVII* (1950), 87—110.

[8] *Rosser J. B. - Wang Hao:* Non-standard Models for Formal Logics. Journ. of Symb. Logics. *15* (1950), 113—129.

[9] *Shoenfield J. R.:* A Relative Consistency Proof. Journ. of Symb. Logic. *19* (1954), 21—28.

[10] *Вопенка П.:* Модели теории множеств. Zeitschr. f. mat. Logik, *8* (1962), 281—292.

[11] *Vopěnka P.:* A New Proof of the Gödel's Result on Non-provability of Consistency. Bull. Acad. Sci. Polon. XIV, *3* (1966), 111—116.

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