Karel Karták

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KAREL KARTÁK, Praha

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0. In what follows, $\mathcal{N}$ denotes the set of all natural numbers $\{1, 2, \ldots\}$; for any $m \in \mathcal{N}$, $\mathcal{R}^m$ stands for the set of all real m-tuples $x = [x_1, \ldots, x_m]$ equipped with the distance $d(x, y) = \max\{|x_k - y_k|;\ k = 1, \ldots, m\}$. All measurability notions refer to the Lebesgue measure on $\mathcal{R}^m$. Instead of $\mathcal{R}^1$ we write merely $\mathcal{R}$; we put $\overline{\mathcal{R}} = \mathcal{R} \cup \cup \{\infty, -\infty\}$, with usual algebraic and order properties. A mapping f defined on A will be sometimes denoted by $f \mid A$ or $x \to f(x)$, $x \in A$; for $\emptyset \neq B \subset A$, $f \mid B$ denotes the reduction of f to B. A function f on a set $A \neq \emptyset$ is a mapping of A into $\overline{\mathcal{R}}$; if $A \subset \mathcal{R}^m$, then $\bar{f}$ always denotes the function such that $\bar{f} \mid A = f$, $\bar{f} \mid \mathcal{R}^m - A = 0$. If $f(x) = c \in \mathcal{R}$ for each $x \in A$, we write also $f \mid A = \hat{c}$.

Let $A \subset \mathcal{R}^m$; the symbols $\bar{A}$, A^0, $|A|$, diam (A) denote the closure of A, the interior of A, the outer Lebesgue measure of A and the diameter of A, respectively. If $x \in \mathcal{R}^m$, then $d(x, A) = \inf\{d(x, y);\ y \in A\}$; if $\varepsilon > 0$, then $O(A, \varepsilon)$ denotes the ε-neighbourhood of A in $\mathcal{R}^m$.

A set K of the form $i_1 \times \ldots \times i_m$, where $i_k = \langle a_k, b_k \rangle$, $a_k < b_k$, $k = 1, \ldots, m$, will be called an m-dimensional interval; we have thus $|K| = \Pi(b_k - a_k)$. The set of all m-dimensional intervals will be denoted by $\mathbf{J}_m$; the set of all m-dimensional intervals $I \subset K$ will be denoted by $\mathbf{J}_m(K)$. Further we put $\mathbf{J} = \bigcup_{m=1}^{\infty} \mathbf{J}_m$. We say that a sequence of intervals $\{I_n\}$, $n \in \mathcal{N}$, converges to $x \in \mathcal{R}^m$ in $K \in \mathbf{J}_m$ and write $I_n \to x|K$ iff $I_n \in \in \mathbf{J}_m(K)$, $x \in I_n$, $n \in \mathcal{N}$, and $\lim \operatorname{diam}(I_n) = 0$. Further, we write $I_n \dot{\to} x|K$ iff $I_n \to \to x|K$ and $d(x, K - I_n) > 0$, $n \in \mathcal{N}$. Let $I, I_1, I_2 \in \mathbf{J}_m$; we write $I = I_1 \dotplus I_2$, iff $I = I_1 \cup I_2$ and $(I_1 \cap I_2)^0 = \emptyset$.

Let $K \in \mathbf{J}_m$. We say that F is a function of interval on K iff F is a mapping from the set $\mathbf{J}_m(K)$ into $\mathcal{R}$. The set of all functions of interval on $K \in \mathbf{J}_m$ will be denoted by $\mathsf{U}(K)$. We say that $F \in \mathsf{U}(K)$ is superadditive on K iff $F(I_1 \dotplus I_2) \geqq F(I_1) + F(I_2)$, whenever $I_1, I_2 \in \mathbf{J}_m$, $I_1 \dotplus I_2 \subset K$. Writing $\leqq$ or $=$ instead of $\geqq$, we get the definition of a subadditive or additive function of interval. We say that $F \in \mathsf{U}(K)$ is continuous on K iff, given $\varepsilon > 0$, there exists $\delta > 0$ such that $I \in \mathbf{J}_m(K)$, $|I| < \delta \Rightarrow \Rightarrow |F(I)| < \varepsilon$.

1. In this section we give an axiomatic definition of integration (see also [3] for the 1-dimensional case).

For each measurable $A \subset \mathscr{R}^m$, $\mathscr{S}(A)$ denotes the set of all measurable functions $f : A \to \overline{\mathscr{R}}$, and $\mathscr{L}(A)$ is the set of all $f \in \mathscr{S}(A)$ such that the Lebesgue integral $(L) \int_A f$ converges; however, we shall also write merely $\int_A f$ in this case.

(1,1) Definition. Let $m \in \mathscr{N}$. An m-dimensional $\circ$-integration is a mapping $(\mathscr{F}, \iota)$ assigning to each $K \in \mathbf{J}_m$ a set $\mathscr{F}(K) \subset \mathscr{S}(K)$ and a finite function $f \to (\iota) \int_K f$, $f \in \mathscr{F}(K)$ so that the following is satisfied:

For each $K \in \mathbf{J}_m$

(I) $$\hat{1}|K \in \mathscr{F}(K) \quad \text{and} \quad (\iota)\int_K \hat{1} = |K|$$

(II) $$f_1 \in \mathscr{F}(K), \quad f_2 \in \mathscr{F}(K) \Rightarrow f_1 + f_2 \in \mathscr{F}(K),$$

$$(\iota)\int_K (f_1 + f_2) = (\iota)\int_K f_1 + (\iota)\int_K f_2$$

(here, $f_1(t) + f_2(t)$ of the form e.g. $\infty - \infty$ may be defined in an arbitrary way)

(III) $$f \in \mathscr{F}(K), \quad k \in \mathscr{R} \Rightarrow kf \in \mathscr{F}(K), \quad \text{and} \quad (\iota)\int_K kf = k(\iota)\int_K f$$

(IV) $$f \,|\, I_1 \in \mathscr{F}(I_1), \quad f \,|\, I_2 \in \mathscr{F}(I_2), \quad I_1 \dotplus I_2 = K \Rightarrow f \,|\, K \in \mathscr{F}(K)$$

and

$$(\iota)\int_K f = (\iota)\int_{I_1} f + (\iota)\int_{I_2} f.$$

The set of all m-dimensional $\circ$-integrations will be denoted by $\mathfrak{F}_m^\circ$.

Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$, $K \in \mathbf{J}_m$. If $f \in \mathscr{F}(K)$, then we say that f is ι-integrable over K, and the number $(\iota) \int_K f$ is called the ι-integral of f over K.

Let $(\mathscr{F}, \iota), (\mathscr{F}_1, \iota_1) \in \mathfrak{F}_m^\circ$; we write $(\mathscr{F}, \iota) \subset (\mathscr{F}_1, \iota_1)$ iff, for each $K \in \mathbf{J}_m$, $\mathscr{F}(K) \subset \mathscr{F}_1(K)$ and $(\iota) \int_K = (\iota_1) \int_K \,|\, \mathscr{F}(K)$. The relation $\subset$ clearly orders the set $\mathfrak{F}_m^\circ$; instead of $\mathfrak{F}_m^\circ$, we shall also write $(\mathfrak{F}_m^\circ, \subset)$.

(1,2) Theorem. *Let* $(\mathscr{F}, \iota) \in (\mathfrak{F}_m^\circ, \subset)$ *be given. Then there exists a maximal element* $(\mathscr{F}_{\max}, \iota_{\max}) \in (\mathfrak{F}_m^\circ, \subset)$ *such that* $(\mathscr{F}, \iota) \subset (\mathscr{F}_{\max}, \iota_{\max})$.

Proof. If $\{\mathscr{F}_\alpha, \iota_\alpha\}$ is a linearly ordered set of m-dimensional $\circ$-integrations, then $\bigcup_\alpha (\mathscr{F}_\alpha, \iota_\alpha) \in (\mathfrak{F}_m^\circ, \subset)$ may be defined in an obvious way. The result now follows from Zorn's lemma.

(1,3) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ be given. We say that $(\mathscr{F}, \iota)$ is saturated iff, for each $K \in \mathbf{J}_m$ and each nonnegative $f \in \mathscr{S}(K)$, $f \in \mathscr{F}(K)$ if and only if $f \in \mathscr{L}(K)$, and $(\iota) \int_K f = \int_K f$.

In theorems (1,4) to (1,9) below we suppose that $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ is saturated; as usually, K denotes an m-dimensional interval.

(1,4) Theorem. *$f \in \mathscr{F}(K) \Rightarrow |f| < \infty$ a.e. on K.*

Proof. $f \in \mathscr{F}(K) \Rightarrow (-f) \in \mathscr{F}(K)$, hence $f + (-f) \in \mathscr{F}(K)$; then $0 = (\iota) \int_K f + (\iota) \int_K (-f) = (\iota) \int_K [f + (-f)]$. When the sum is of the form e.g. $\infty - \infty$, we put $f(t) - f(t) = 1$. Then $f + (-f) \geqq 0$, lies in $\mathscr{F}(K)$, hence in $\mathscr{L}(K)$; thus, $f + (-f) = 0$ a.e. on K.

(1,5) Theorem. *$f \in \mathscr{L}(K) \Rightarrow f \in \mathscr{F}(K)$, and $\int_K f = (\iota) \int_K f$. On the other hand, $f \in \mathscr{F}(K)$, $|f| \in \mathscr{F}(K) \Rightarrow f \in \mathscr{L}(K)$.*

Proof. Easy.

(1,6) Theorem. *$f \in \mathscr{F}(K)$, $f = g$ a.e. on $K \Rightarrow g \in \mathscr{F}(K)$, and $(\iota) \int_K g = (\iota) \int_K f$.*

Proof. This is a direct consequence of (1,5).

Remark. We see that a function $f \in \mathscr{F}(K)$ may be defined only a.e. on K.

(1,7) Theorem. *$f, g \in \mathscr{F}(K)$, $f \leqq g$ a.e. on $K \Rightarrow (\iota) \int_K f \leqq (\iota) \int_K g$.*

Proof. $(\iota) \int_K g - (\iota) \int_K f = \int_K (g - f) \geqq 0$.

(1,8) Theorem. *Let*

1° $g, h \in \mathscr{F}(K)$,

2° $f \in \mathscr{S}(K)$,

3° $g \leqq f \leqq h$ *a.e. on* K.

Then $f \in \mathscr{F}(K)$.

Proof. We have $0 \leqq f - g \leqq h - g$ a.e. on K, $h - g \in \mathscr{L}(K)$, $f - g \in \mathscr{S}(K)$. Hence $f - g \in \mathscr{L}(K)$, so that $f = g + (f - g) \in \mathscr{F}(K)$.

Instead of "*f_n converge to f asymptotically*", we shall write limas $f_n = f$. We prove the following generalization of the Lebesgue convergence theorem.

(1,9) Theorem. *Let*

1° $g_n, h_n, g, h \in \mathscr{F}(K)$, $n \in \mathscr{N}$,

2° $g_n \leqq f_n \leqq h_n$ *a.e. on* K, $n \in \mathscr{N}$,

3° lim as $g_n = g$, lim as $f_n = f$, lim as $h_n = h$,

4° $\lim (\iota) \int_K g_n = (\iota) \int_K g$, $\lim (\iota) \int_K h_n = (\iota) \int_K h$,

5° $f_n \in \mathscr{S}(K)$, $n \in \mathscr{N}$.

Then $f_n, f \in \mathscr{F}(K)$, $n \in \mathscr{N}$, and $\lim (\iota) \int_K f_n = (\iota) \int_K f$.

Proof. According to (1,8), $f_n \in \mathscr{F}(K)$ for each $n \in \mathscr{N}$. Further, it is elementary that $g \leqq f \leqq h$ a.e. on K; hence $f \in \mathscr{F}(K)$. We prove that $\liminf (\iota) \int_K f_n \geqq (\iota) \int_K f$. Suppose on the contrary that $\liminf (\iota) \int_K f_n < (\iota) \int_K f$. Then there exist $n_1, n_2, \ldots$ such that $f_{n_k} \to f$, $g_{n_k} \to g$ a.e. on K and $\lim (\iota) \int_K f_{n_k} < (\iota) \int_K f$. Using Fatou's lemma we get $\int_K (f - g) = \int_K \lim (f_{n_k} - g_{n_k}) \leqq \liminf ((\iota) \int_K f_{n_k} - (\iota) \int_K g_{n_k}) = \liminf (\iota) \int_K f_{n_k} - (\iota) \int_K g$; hence $(\iota) \int_K f \leqq \liminf (\iota) \int_K f_{n_k}$. This is a contradiction. Passing to opposite functions, we obtain $(\iota) \int_K f \geqq \limsup (\iota) \int_K f_n$.

(1,10) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ be given. We say that $(\mathscr{F}, \iota)$ is hereditary iff, for each $K \in \boldsymbol{J}_m$ and each $f \in \mathscr{F}(K)$, $f \mid I \in \mathscr{F}(I)$ for each $I \in \boldsymbol{J}_m(K)$.

(1,11) Theorem. *Let a hereditary $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ be given, and let $K \in \boldsymbol{J}_m$. For each $I \in \boldsymbol{J}_m(K)$, put*

$$F(I) = (\iota) \int_I f\,. \tag{1.11.1}$$

Then $F \in \mathsf{U}(K)$ is additive on K.

Proof. Clear.

(1,12) Definition. Let a hereditary $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ be given. We say that $(\mathscr{F}, \iota)$ is continuous iff, for each $K \in \boldsymbol{J}_m$ and each $f \in \mathscr{F}(K)$, the function F defined by (1.11.1) is continuous on K.

We say that $(\mathscr{F}, \iota) \in \mathfrak{F}_m^\circ$ is an m-dimensional integration, iff it is saturated, hereditary and continuous. The set of all m-dimensional integrations will be denoted by $\mathfrak{F}_m$.

We join some usual definition relevant to the 1-dimensional case. Let a hereditary $(\mathscr{F}, \iota) \in \mathfrak{F}_1^\circ$ be given. If $K = \langle a, b \rangle$, $f \in \mathscr{F}(K)$, we put $(\iota) \int_K f = (\iota) \int_a^b f = -(\iota) \int_b^a f$, $(\iota) \int_a^a f = 0$. Given $c \in \langle a, b \rangle$, the function $t \to F(t) = (\iota) \int_c^t f$, $t \in K$, will be called a ι-antiderivative of f. If $(\mathscr{F}, \iota)$ is moreover continuous, then F is evidently continuous on K.

(1,13) Examples. Let us take $m = 1$ for simplicity. For each $K = \langle a, b \rangle$, let $\mathscr{R}(K)$ resp $\mathscr{A}(K)$ resp. $\mathscr{P}_{ap}(K)$ denote the set of all functions on K which are integrable over K in the sense of Riemann, resp. in the sense of the A-integral (see e.g. [10]), resp. in the sense defined by Burkill in [1], and let $(R) \int_K f$ resp. $(A) \int_K f$ resp. $(P_{ap}) \int_K f$ denote the corresponding integrals. Then $(\mathscr{R}, R) \subset (\mathscr{L}, L) \subset (\mathscr{A}, A)$, $(\mathscr{L}, L) \subset (\mathscr{P}_{ap}, P_{ap})$; further, $(\mathscr{R}, R)$ is not saturated, $(\mathscr{A}, A)$ is not hereditary (see [10]), $(\mathscr{P}_{ap}, P_{ap})$ is not continuous (see [1]). In [3], it is shown that it may happen that $(\mathscr{F}, \iota)$, $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}_1$ are such that $\mathscr{F}(K) = \mathscr{F}_1(K)$ for each $K \in \boldsymbol{J}_1$ whilst $(\iota) \int_K f \neq (\iota_1) \int_K f$ for some $f \in \mathscr{F}(K)$.

(1,14) Definition. We say that a mapping $(\mathscr{F}, \iota)$ defined on $\boldsymbol{J}$ is an (euclidean) integration, iff $(\mathscr{F}, \iota) \mid \boldsymbol{J}_m \in \mathfrak{F}_m$ for each $m \in \mathscr{N}$.

The set of all euclidean integrations will be denoted by $\mathfrak{F}$.

(1,15) Let $\emptyset \neq A \subset K \in \mathbf{J}_m$, and let $f \in \mathscr{S}(A)$ be given. We say that $a \in \bar{A}$ is an L-singular point of f, iff $f \mid A \cap O(a, \varepsilon) \notin \mathscr{L}(A \cap O(a, \varepsilon))$, for each $\varepsilon > 0$. The (evidently closed) set of all L-singular points of f will be denoted by $\sigma(f)$.

Let further $(\mathscr{F}, \iota) \in \mathfrak{F}_m$ be given. We write $f \in \mathscr{F}(A)$ iff $\bar{f} \mid K \in \mathscr{F}(K)$. We put then $(\iota) \int_A f = (\iota) \int_K \bar{f}$; this definition is clearly unambiguous.

2. In what follows we shall need some results on a kind of Perron integration in $\mathscr{R}^m$, $m \in \mathscr{N}$, introduced in [6]. First we stress that for $m = 1$ we get the classical Perron integration (see [6], p. 131).

Let $K \in \mathbf{J}_m$ and let $F \in \mathsf{U}(K)$. Let $x \in K$; the number $\bar{F}(x) = \sup \{\lim F(I_n)(I_n)^{-1}; I_n \to x \mid K\}$ is called the upper derivative of F at x. Similarly we introduce the notion of the lower derivative $\underline{F}(x) = \inf \{\ldots\}$.

Let f be a function on K. We say that $M \in \mathsf{U}(K)$ is a majorant of f on K iff

1° M is superadditive on K,
2° $-\infty \neq \underline{M}(x) \geqq f(x)$ for each $x \in K$.

We say that $m \in \mathsf{U}(K)$ is a minorant of f on K iff $-m$ is a majorant of $-f$ on K. Now, the upper Perron integral $\int_K^- f$ of f over K equals to $\inf \{M(K); M$ is a majorant of f on $K\}$, and similarly for the lower Perron integral $\int_{-K} f$. We say that f is Perron integrable over K and write $f \in \mathscr{P}(K)$ iff $\int_K^- f = \int_{-K} f \in \mathscr{R}$. For each $f \in \mathscr{P}(K)$, the Perron integral of f over K, denoted by $(P) \int_K f$, equals to $\int_K^- f$.

For each $K \in \mathbf{J}$, let $\mathscr{T}(K) = \{f \in \mathscr{S}(K); \sigma(f)$ is finite$\}$.

(2,1) Theorem. $(\mathscr{P}, P) \in \mathfrak{F}$.

Proof. The continuity of $(\mathscr{P}, P)$ is proved (for $m = 2$) in [2]; other results needed are contained in [6].

Let us recall some other results on Perron integration.

(2,2) Theorem. *Let K_1 resp. K_2 be an m_1-dimensional resp. m_2-dimensional interval. Let $[x_1, x_2] \to f(x_1, x_2)$ be a function on $K_1 \times K_2$, and let $f \in \mathscr{P}(K_1 \times K_2)$. Then*

$$(P) \int_{K_1 \times K_2} f = (P) \int_{K_2} \left(\int_{K_1}^- f(x_1, x_2) \right) = (P) \int_{K_2} \left(\int_{-K_1} f(x_1, x_2) \right).$$

Proof. See [6], p. 127.

(2,3) Theorem. *Let $K \in \mathbf{J}_m$, $a \in K$, $f : K \to \bar{\mathscr{R}}$ be given. Suppose that*

1° $f \in \mathscr{P}(K - I)$, *whenever* $I \in \mathbf{J}_m(K)$, $\operatorname{dist}(a, K - I) > 0$,
2° $\lim (P) \int_{K - I_n} f$ *exists, whenever* $I_n \dot{\to} a \mid K$.

Then $f \in \mathscr{P}(K)$, *and* $(P) \int_K f = \lim \int_{K - I_n} f$.

Proof. For $m = 1$, see [6], p. 133; for $m = 2$, see [2], p. 408.

For each $K \in J$, put $\dot{\mathscr{P}}(K) = \mathscr{P}(K) \cap \mathscr{T}(K)$. It is clear that $(\dot{\mathscr{P}}, P) \in \mathfrak{F}$.

(2,4) Theorem. *Let $K = \langle a_1, b_1 \rangle \times \ldots \times \langle a_m, b_m \rangle$, $m \in \mathscr{N}$, let $f \in \dot{\mathscr{P}}(K)$ and let φ be of bounded variation on $\langle a_1, b_1 \rangle$. For each $x = [x_1, \ldots, x_m] \in K$, put $\tilde{\varphi}(x) = \varphi(x_1)$. Then $f\tilde{\varphi} \in \dot{\mathscr{P}}(K)$.*

Proof. For $m = 2$, see [2], p. 410.

To show the generality of the Perron integration, let us note the following example (see [2], p. 403).

(2,5) *Let $K = \langle 0, 1 \rangle \times \langle 0, 1 \rangle$ and let $\Delta = \{[x_1, x_2] \in K,\ x_1 \geqq x_2\}$. There exists $f \in \mathscr{P}(K)$ such that $(L) \int_\Delta f = \infty$.*

This example shows that for Perron integration in $\mathscr{R}^m$, $m \geqq 2$, we cannot expect any transformation theorem, with the exception of translations. On the other hand, there are non-absolutely integrable functions invariant under isometries with respect to Perron integrability; see [2], p. 411. This example shows that there might even exist non-absolutely integrable functions invariant with respect to regular transformations, similarly to the Lebesgue case. This was proved, for $m = 2$, in an unpublished paper of the author [4], using mainly the theorem of Banach on the integral representation of variation of a continuous function. In this paper we prove this result in a different way.

3. Let $A \subset \mathscr{R}^m$, $m \geqq 2$, be a bounded measurable set. We say that $A \in \mathfrak{A}$ iff $\|A\| = \sup \{\int_A \operatorname{div} v;\ v = [v_1, \ldots, v_m]$, *$v_k$ polynomials in $x_1, \ldots, x_m$ such that $\sum_{i=1}^{m} (v_i(x))^2 \leqq 1$ for each $x \in A\} < \infty$; see* [7].

If $K \in J_m$, then $\|K\|$ equals to the elementary geometric surface of K ([7], p. 536). Further, $\max(\|A \cup B\|, \|A \cap B\|, \|A - B\|) \leqq \|A\| + \|B\|$ ([7], p. 547).

For $C, D \subset \mathscr{R}$ we write $C \sim D$ iff $|(C - D) \cup (D - C)| = 0$. Let $y = [y_1, \ldots, y_{m-1}] \in \mathscr{R}^{m-1}$ and let $k \in \{1, \ldots, m\}$. Then $A_y^k = \{t \in \mathscr{R};\ [y_1, \ldots, y_{k-1}, t, y_k, \ldots, y_{m-1}] \in A\}$.

(3,1) Theorem. *Let $A \in \mathfrak{A}$ and let an index $k \in \{1, \ldots, m\}$ be given.*

Then there exists a Borel subset $\tilde{A}(k, A) \subset \mathscr{R}^{m-1}$ with the following properties:

1° $|\mathscr{R}^{m-1} - \tilde{A}(k, A)| = 0$,

2° *for each $y \in \tilde{A}(k, A)$ there exist a nonnegative integer $r = r_A^k(y)$ and real numbers a_i, b_i, $i = 1, \ldots, r$ such that $a_1 < b_1 < \ldots < a_r < b_r$ and that $A_y^k \sim \bigcup_{i=1}^{r} (a_i, b_i)$,*

3° $2 \int_{\mathscr{R}^{m-1}} r_A^k \leqq \|A\|$,

4° *if F is a bounded Borel function on the boundary of A such that $|F| \leqq \varkappa$ and if* *we put* $\Theta_k(F, A, y) = \sum_{i=1}^{r} (F(y_1, \ldots, y_{k-1}, b_i, y_k, \ldots, y_{m-1}) - F(\ldots, a_i, \ldots))$ *for each* $y = [y_1, \ldots, y_{m-1}] \in \tilde{A}(k, A)$, *then* Θ_k *is measurable and* $\int_{\mathscr{R}^{m-1}} \Theta_k \leqq 2\varkappa \|A\|$.

Proof. See [7], p. 535, p. 545.

(3,2) Theorem. *Let* $A_n \in \mathfrak{A}$, $n \in \mathscr{N}$, *and let* $\lim \|A_n\| = 0$. *Then* $\lim |A_n| = 0$.

Proof. See [8], p. 263.

In what follows, $\Phi \mid G$ denotes always a bijective regular mapping of an open set $\emptyset \neq G \subset \mathscr{R}^m$ into $\mathscr{R}^m$, $H = \Phi(G)$, $\Psi = \Phi^{-1}$, D_Ψ = the functional determinant of Ψ. If $A \subset G$, $f : A \to \overline{\mathscr{R}}$ is given, then $f \square \Phi$ is defined as follows: $f \square \Phi(t) = f(\Psi(t)) |D_\Psi(t)|$, $t \in \Phi(A)$.

(3,3) Theorem. *Let* $\Phi \mid G$ *be given as above. Let* A *be compact,* $A \subset G$. *Then there exists* $c \in \mathscr{R}$ *such that for each measurable set* $B \subset A$ *the relation* $\|\Phi(B)\| \leqq c\|B\|$ *holds.*

Proof. See [5], p. 255.

(3,4) Theorem. *Let* $\Phi \mid G$ *be given as above. Let* $A \subset G$ *be compact and let* $f \in \mathscr{S}(A)$ *be given. Then* $\Phi(\sigma(f)) = \sigma(f \square \Phi)$.

Proof. This is a simple consequence of the transformation theorem for Lebesgue integrals.

4. In this section two euclidean integrations, denoted here $(\mathscr{H}, \omega)$, $(\mathscr{Z}, \omega)$, will be defined.

For $m = 1$ we put $(\mathscr{H}, \omega) \mid J_1 = (\dot{\mathscr{P}}, P) \mid J_1$. Let $m \geqq 2$, $K \in J_m$, and let E_n, $n \in \mathscr{N}$, be measurable subsets of $\mathscr{R}^m$. We write $E_n \dot{\to} a \mid K$ iff

1° $E_n \subset K$, $n \in \mathscr{N}$,

2° $\lim \|E_n\| = 0$, $\lim \operatorname{diam}(E_n) = 0$,

3° $d(a, K - E_n) > 0$, $n \in \mathscr{N}$.

It is clear that if especially E_n are m-dimensional intervals, then $E_n \dot{\to} a \mid K$ has the meaning introduced in section 1.

(4,1) Definition. Let $K \in J_m$, $m \geqq 2$, and let $f \in \mathscr{T}(K)$ be given. We say that f is ω-integrable over K iff (4.1.1) either $f \in \mathscr{L}(K)$; in this case we put $(\omega) \int_K f = \int_K f$ (4.1.2) or $\sigma(f) = \{a^{(1)}, \ldots, a^{(r)}\} \neq \emptyset$, and a finite limit

(4.1.3) $$\lim \int_{K - \bigcup_{i=1}^{r} E_n^{(i)}} f$$

exists, whenever $E_n^{(i)} \dot{\to} a^{(i)} \mid K$, $i = 1, \ldots, r$; in this case we put $(\omega) \int_K f =$ the limit in (4.1.3).

The set of all ω-integrable functions on K will be denoted by $\mathscr{H}(K)$.

(4,2) Lemma. *Let $K \in \mathbf{J}_m$, $m \geqq 2$, $f \in \mathscr{T}(K)$, and let $\sigma(f) = \{a^{(1)}, \ldots, a^{(r)}\} \neq \emptyset$. Then $f \in \mathscr{H}(K)$ iff $E_{n,j}^{(i)} \dot{\to} a^{(i)} \mid K$, $i = 1, \ldots, r$, $j = 1, 2$, implies*

$$\lim \left(\int_{K - \bigcup_{i=1}^{r} E_{n,1}^{(i)}} f - \int_{K - \bigcup_{i=1}^{r} E_{n,2}^{(i)}} f \right) = 0 \, .$$

Proof. Clear.

(4,3) Lemma. *Let $K \in \mathbf{J}_m$, $m \geqq 2$, and let $f \in \mathscr{S}(K)$ be given. Let $K = I_1 \dotplus I_2$. Then $f \in \mathscr{H}(K)$ iff $f \mid I_j \in \mathscr{H}(I_j)$, $j = 1, 2$; moreover,*

$$(\omega) \int_K f = (\omega) \int_{I_1} f + (\omega) \int_{I_2} f$$

holds in this case.

Proof. Let $f \in \mathscr{H}(K)$. Suppose for simplicity that $\sigma(f) \cap I_1 \cap I_2 = \emptyset$. Let e.g. $\sigma(f \mid I_1) = \{a^{(1)}, \ldots, a^{(r)}\}$, $\sigma(f \mid I_2) = \{a^{(r+1)}, \ldots, a^{(s)}\}$. Let $E_{n,j}^{(i)} \dot{\to} a^{(i)} \mid I_1$, $i = 1, \ldots, r$, $j = 1, 2$, and let $E_{n,1}^{(i)} = E_{n,2}^{(i)} = E_n^{(i)} \dot{\to} a^{(i)} \mid I_2$, $i = r + 1, \ldots, s$. Then, according to (4,2),

$$\lim \left(\int_{K - \bigcup_{i=1}^{s} E_{n,1}^{(i)}} f - \int_{K - \bigcup_{i=1}^{s} E_{n,2}^{(i)}} f \right) = \lim \left(\int_{I_1 - \bigcup_{i=1}^{r} E_{n,1}^{(i)}} f - \int_{I_1 - \bigcup_{i=1}^{r} E_{n,2}^{(i)}} f \right) = 0 \, ;$$

hence $f \in \mathscr{H}(I_1)$. The proof for other cases is similar.

If, on the other hand, $f \mid I_j \in \mathscr{H}(I_j)$, $j = 1, 2$, then (4,2) gives immediately that $f \in \mathscr{H}(K)$.

(4,4) Corollary. *Let Δ be a division of $K \in \mathbf{J}_m$ (= the cartesian product of divisions of 1-dimensional factors of K; see [6], p. 38 for a precise definition). Let $\Delta = \{I_1, \ldots, I_p\}$. Then $(\omega) \int_K f = \sum_{j=1}^{p} (\omega) \int_{I_j} f$ iff one side has a meaning.*

(4,5) Theorem. *For each $K \in \mathbf{J}$, $\mathscr{H}(K) \subset \dot{\mathscr{P}}(K)$; for each $f \in \mathscr{H}(K)$, $(\omega) \int_K f = (P) \int_K f$.*

Proof. This follows from (2,3).

(4,6) Theorem. $(\mathscr{H}, \omega) \in \mathfrak{F}$, $(\mathscr{H}, \omega) \subset (\dot{\mathscr{P}}, P)$.

Proof. (II) Using (4,4), it is sufficient to consider the case when $\sigma(f_1) \cup \sigma(f_2)$ has at most one point on K; but then it is obvious.

(IV) This follows from (4,3).

Hereditarity of $(\mathscr{H}, \omega)$ may be proved similarly to (4,3). Continuity of $(\mathscr{H}, \omega)$ is a consequence of (4,5) and (2,1).

(4,7) We introduce the integration $(\mathscr{Z}, \omega)$.

For $m = 1$, put $(\mathscr{Z}, \omega) \mid J_1 = (\dot{\mathscr{P}}, P) \mid J_1$. Let $m \geqq 2$, $m \in \mathscr{N}$, $K \in J_m$. We say that $f \in \mathscr{W}(K)$ iff

1° $f \in \mathscr{T}(K)$,

2° for each $a = [a_1, \ldots, a_m] \in K$ and each $k \in \{1, \ldots, m\}$, there exists a relative (with respect to K) neighbourhood Ω of a such that the function $x \to F_k(x)$, $x \in \Omega$ defined by

$$F_k(x) = (P) \int_{a_k}^{x_k} f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_m) \, dt$$

is bounded and Borel measurable on Ω.

Put further $\mathscr{Z}(K) = \mathscr{L}(K) \oplus \mathscr{W}(K) = \{f; f = g + h, \, g \in \mathscr{L}(K), \, h \in \mathscr{W}(K)\}$.

(4,8) Theorem. *For each $K \in J$, $\mathscr{Z}(K) \subset \mathscr{H}(K)$.*

Proof. Let $f \in \mathscr{Z}(K)$. To prove the theorem, it is sufficient to suppose that $m \geqq 2$, $\sigma(f) = \{a\}$. We may also suppose that $f \in \mathscr{W}(K)$. Let $E_{n,j} \dot{\to} a \mid K$, $j = 1, 2$. Let Ω be a relative neighbourhood of a such that $F(x) = \int_{a_1}^{x_1} f(t, x_2, \ldots, x_m) \, dt$ is in absolute value $\leqq \varkappa$ on Ω.

We have

$$\left| \int_{K - E_{n,1}} f - \int_{K - E_{n,2}} f \right| \leqq \left| \int_{E_{n,1} - E_{n,2}} f \right| + \left| \int_{E_{n,2} - E_{n,1}} f \right| ;$$

hence it is sufficient to prove that $\lim \int_{E_{n,1} - E_{n,2}} f = 0$. Put $E_{n,1} - E_{n,2} = A_n$ for short; suppose further that $A_n \subset \Omega$, $n \in \mathscr{N}$. Then, using (3,1),

$$\left| \int_{A_n} f \right| = \left| \int_{\mathscr{R}^{m-1}} \left(\int_{(A_n)_y{}^1} f(t, y) \, dt \right) dy \right| = \left| \int_{\mathscr{R}^{m-1}} \sum_{j=1}^{r(y)} F(b_i, y) - F(a_i, y) \right| \leqq$$

$$\leqq 2\varkappa \int_{\mathscr{R}^{m-1}} r \leqq \varkappa \|A_n\|$$

which proves the theorem.

(4,9) Theorem. $(\mathscr{Z}, \omega) \in \mathfrak{F}$.

Proof. Simple.

5. In this section we introduce some properties of integrations, which are fulfilled for Lebesgue integration.

(5,1) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}$ be given. We say that $(\mathscr{F}, \iota)$ has the property $(\textbf{\textit{Fub}})$ iff there exists an $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}$ such that, for each $m \in \mathscr{N}$, $m \geqq 2$, the following is satisfied: if $m = r + s$, $r, s \in \mathscr{N}$, $K \in \boldsymbol{J}_m$, $K_1 \in J_r$, $K_2 \in J_s$, $K = K_1 \times K_2$, $f \in \mathscr{F}(K)$, then

1° $y \to f(y, z) \in \mathscr{F}_1(K_1)$ for almost all $z \in K_2$,
2° $(\iota) \int_K f = (\iota_1) \int_{K_2}((\iota_1) \int_{K_1} f)$.
We write then $(\mathscr{F}, \iota)\,(\textbf{\textit{Fub}})\,(\mathscr{F}_1, \iota_1)$.

Remark 1. As it is known, $(\mathscr{L}, L)\,(\textbf{\textit{Fub}})\,(\mathscr{L}, L)$.

(5,2) Theorem. $(\dot{\mathscr{P}}, P)\,(\textbf{\textit{Fub}})\,(\dot{\mathscr{P}}, P)$.

Proof. This is a simple consequence of (2,2).

(5,3) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m$, $m \in \mathscr{N}$, be given. We say that $(\mathscr{F}, \iota)$ has the property $(\textbf{\textit{Tr}})$ iff there exists an $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}_m$ such that whenever $K \in \boldsymbol{J}_m$, $f \in \mathscr{F}(K)$, $\Phi \mid G$ is a bijective regular mapping of an open set $G \supset K$, then

1° $f \square \Phi \in \mathscr{F}_1(\Phi(K))$,
2° $(\iota) \int_K f = (\iota_1) \int_{\Phi(K)} f \square \Phi$.
We write then $(\mathscr{F}, \iota)\,(\textbf{\textit{Tr}})\,(\mathscr{F}_1, \iota_1)$.

Remark 2. As it is known, $(\mathscr{L}, L)\,(\textbf{\textit{Tr}})\,(\mathscr{L}, L)$, for each $m \in \mathscr{N}$.

(5,4) Theorem. $(\mathscr{H}, \omega)\,(\textbf{\textit{Tr}})\,(\mathscr{H}, \omega)$, *for each* $m \in \mathscr{N}$.

Proof. Let $K \in \boldsymbol{J}_m$, $m \geqq 2$, $f \in \mathscr{H}(K)$, $\sigma(f) = \{a^{(1)}, \ldots, a^{(r)}\}$. Then $\sigma(f \square \Phi) = \{\Phi(a^{(1)}), \ldots, \Phi(a^{(r)})\}$. Let $K_1 \in \boldsymbol{J}_m$ be such that $K_1 \supset \Phi(K)$. Using a suitable division of K_1, we may construct a finite set $\mathfrak{K}$ of intervals $I \in \boldsymbol{J}_m$ such that

1° $I_1, I_2 \in \mathfrak{K}$, $I_1 \neq I_2 \Rightarrow (I_1 \cap I_2)^0 = \emptyset$,
2° $\Phi(K) \subset \bigcup \mathfrak{K} \subset \Phi(G)$,
3° for each $I \in \mathfrak{K}$, $\sigma(f \square \Phi) \cap I$ has at most one point, lying then in I^0.

To prove the theorem, it clearly suffices to prove that, for each $I \in \mathfrak{K}$,

$$\overline{f \square \Phi} \mid I \in \mathscr{H}(I)\,. \tag{5.4.1}$$

This is true provided $\sigma(f \square \Phi) \cap I = \emptyset$. Let $\sigma(f \square \Phi) = \Phi(a^{(i)})$, and let $E_n \dot{\to} \Phi(a^{(i)}) \mid I$, $n \in \mathscr{N}$. Then there exists an index $n_0 \in \mathscr{N}$ and $K_2 \in \boldsymbol{J}_m$ such that

4° $K_2 \subset \Psi(I)$,
5° $\Psi(E_n) \dot{\to} a^{(i)} \mid K_2$, $n \geqq n_0$, $n \in \mathscr{N}$

as it follows from (3,3). As $\int_{I - E_n} \overline{f \square \Phi} = \int_{\Psi(I) - \Psi(E_n)} \bar{f}$ and $\bar{f} \mid K_2 \in \mathscr{H}(K_2)$, we see that (5.4.1) holds.

(5,5) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m$, $m \in \mathscr{N}$, be given. We say that $(\mathscr{F}, \iota)$ has the property **(Four)** iff there exists an $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}_m$ such that whenever $K = \langle a_1, b_1 \rangle \times \times \ldots \times \langle a_m, b_m \rangle \in \boldsymbol{J}_m$, $f \in \mathscr{F}(K)$, and $g_i \mid \langle a_i, b_i \rangle \to \mathscr{R}$, $i = 1, \ldots, m$ are of bounded variation, then $f g_1 \ldots g_m \in \mathscr{F}_1(K)$ (here, the product is defined similarly to (2,4)).
We write then $(\mathscr{F}, \iota)$ **(Four)** $(\mathscr{F}_1, \iota_1)$.

Remark 3. $(\mathscr{L}, L)$ **(Four)** $(\mathscr{L}, L)$, for each $m \in \mathscr{N}$.

(5,6) Theorem. $(\dot{\mathscr{P}}, P)$ **(Four)** $(\dot{\mathscr{P}}, P)$.

Proof. This is a simple consequence of (2,4).

Let $K \in \boldsymbol{J}$, $N \in \mathscr{N} \cup \{0\}$. We write $\varphi \in \mathscr{C}^N(K)$ iff there exists an open set $G \supset K$ such that φ has continuous N^{th}-order derivatives on G. We put $\|\varphi\|_N = \max \{|\varphi(x)|, |D\,\varphi(x)|, \ldots, |D^N \varphi(x)|; x \in K\}$, D^j denoting a differentiation operator of the j-th order, $0 \leqq j \leqq N$.

(5,7) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m$, $m \in \mathscr{N}$, be given. Let $N \in \mathscr{N} \cup \{0\}$. We say that $(\mathscr{F}, \iota)$ has the property **(Pr** N**)** iff there exists an $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}_m$ such that whenever $f \in \mathscr{F}(K)$, $\varphi \in \mathscr{C}^N(K)$, then $f\varphi \in \mathscr{F}_1(K)$.
We write then $(\mathscr{F}, \iota)$ **(Pr** N**)** $(\mathscr{F}_1, \iota_1)$.

Remark 4. $(\mathscr{L}, L)$ **(Pr** 0**)** $(\mathscr{L}, L)$.

(5,8) Theorem. $(\mathscr{Z}, \omega)$ **(Pr** 1**)** $(\mathscr{Z}, \omega)$.

Proof. Let $f \in \mathscr{Z}(K)$, $K \in \boldsymbol{J}_m$, $m \geqq 2$, $a \in K$. It is evidently sufficient to suppose that $f \in \mathscr{W}(K)$. Let $\varphi \in \mathscr{C}^1(K)$ and put $F(x) = (P) \int_{a_1}^{x_1} f(t, y)\,dt$, $x = [x_1, y] \in K$. Then $(P) \int_{a_1}^{x_1} f(t, y)\,\varphi(t, y)\,dt = F(x)\,\varphi(x) - \int_{a_1}^{x_1} F(t, y)\,(\partial\varphi/\partial t)\,(t, y)\,dt$; the right-hand side shows immediately that $f\varphi \in \mathscr{W}(K)$. This proves the theorem.

(5,9) Definition. Let $(\mathscr{F}, \iota) \in \mathfrak{F}_m$, $m \in \mathscr{N}$, be given. Let $N \in \mathscr{N} \cup \{0\}$. We say that $(\mathscr{F}, \iota)$ has the property **(Distr** N**)** iff it has the property **(Pr** N**)**, i.e. $(\mathscr{F}, \iota)$ **(Pr** N**)** $(\mathscr{F}_1, \iota_1)$ for some $(\mathscr{F}_1, \iota_1) \in \mathfrak{F}_m$, and if $\varphi_n \in \mathscr{C}^N(K)$, $\lim \|\varphi_n\|_N = 0 \Rightarrow \lim (\iota_1) \int_K f\varphi_n = 0$.
We write then $(\mathscr{F}, \iota) \in$ **(Distr** N**)**.

Remark 5. As it is known, $(\mathscr{L}, L) \in$ **(Distr** 0**)**, for each $m \in \mathscr{N}$.

(5,10) Theorem. $(\mathscr{Z}, \omega) \in$ **(Distr** 1**)**.

Proof. Let $\varepsilon > 0$ be given. Let $f \in \mathscr{Z}(K)$, $K \in \boldsymbol{J}_m$, $m \geqq 2$, $\varphi_j \in \mathscr{C}^1(K)$, $\lim \|\varphi_j\|_1 = 0$. We may suppose that $\sigma(f) = a \in K$. Let $F(x) = (P) \int_{a_1}^{x_1} f(t, y)\,dt$, $\varkappa \geqq 0$, and suppose that $\|\varphi_j\|_1 \leqq \varkappa$, $j \in \mathscr{N}$, $|F| \leqq \varkappa$ on a relative neighbourhood Ω of a. Let $E_n \dot{\to} a \mid K$, $E_n \subset \Omega$; then

$$\left| \int_K f\varphi_j \right| \leqq \left| \int_{K - E_n} f\varphi_j \right| + \left| \int_{E_n} f\varphi_j \right|$$

for each $j, n \in \mathcal{N}$. Using (3,1) we have immediately that

$$\left|\int_{E_n} f\varphi_j\right| = \left|\int_{\mathcal{R}^{m-1}} \left(\int_{(E_n)_y{}^1} f\varphi_j \,\mathrm{d}t\right) \mathrm{d}y\right| =$$

$$= \left|\int_{\mathcal{R}^{m-1}} \left(\sum_{i=1}^{r(y)} \int_{a_i}^{b_i} f(t, y)\, \varphi_j(t, y)\, \mathrm{d}t\right) \mathrm{d}y\right| =$$

$$= \left|\int_{\mathcal{R}^{m-1}} \sum_{i=1}^{r(y)} \left[F(b_i, y)\, \varphi_j(b_i, y) - F(a_i, y)\, \varphi_j(a_i, y) - \int_{a_i}^{b_i} F(t, y) \frac{\partial \varphi_j}{\partial t}(t, y)\, \mathrm{d}t\right] \mathrm{d}y\right| \leqq$$

$$\leqq \int_{\mathcal{R}^{m-1}} (2\varkappa^2\, r(y) + \varkappa^2 |(E_n)_y^1|)\, \mathrm{d}y \leqq \varkappa^2(\|E_n\| + |E_n|),$$

for each $j, n \in \mathcal{N}$.

Choose $n_0 \in \mathcal{N}$ such that $\varkappa^2(\|E_{n_0}\| + |E_{n_0}|) < \varepsilon/2$; now it is sufficient to find $j_0 \in \mathcal{N}$ such that $j \geqq j_0 \Rightarrow |\int_{K - E_{n_0}} f\varphi_j| < \varepsilon/2$. This proves the theorem.

6. We introduce the following concept.

(6,1) Definition. Let $(\mathcal{F}, \iota) \in \mathfrak{F}$ be given. We say that $(\mathcal{F}, \iota)$ is a quasi-Lebesgue integration iff there exists an $(\mathcal{F}_1, \iota_1) \in \mathfrak{F}$ such that

(6.1.1) $$(\mathcal{F}, \iota)\, (\mathbf{Fub})\, (\mathcal{F}_1, \iota_1)$$

and for each $m \in \mathcal{N}$

(6.1.2) $$(\mathcal{F}, \iota) \mid \boldsymbol{J}_m (\mathbf{Tr})\, (\mathcal{F}_1, \iota_1) \mid \boldsymbol{J}_m$$

(6.1.3) $$(\mathcal{F}, \iota) \mid \boldsymbol{J}_m\, (\mathbf{Four})\, (\mathcal{F}_1, \iota_1) \mid \boldsymbol{J}_m$$

and for some $N \in \mathcal{N} \cup \{0\}$

(6.1.4) $$(\mathcal{F}, \iota) \mid \boldsymbol{J}_m (\mathbf{Pr}\, N)\, (\mathcal{F}_1, \iota_1) \mid \boldsymbol{J}_m$$

(6.1.5) $$(\mathcal{F}, \iota) \in (\mathbf{Distr}\, N)$$

(6,2) Theorem. $(\mathcal{L}, \omega)$ is a quasi-Lebesgue integration.

Proof. This is a consequence of the preceding theorems.

From Remarks 1 to 5 of section 5 we see that if $(\mathcal{F}, \iota) = (\mathcal{L}, L)$, then $(\mathcal{F}_1, \iota_1)$ may be chosen equal to $(\mathcal{F}, \iota)$.

(6,3) Problem. Does there exist any other euclidean integration possessing the above property?

7. Let us still mention another example of integration, which was studied in [5].

For each $K \in \boldsymbol{J}_1$, put $\mathcal{B}(K) = \{f \in \mathcal{P}(K);\ \sigma(f)$ is countable$\}$; see also [9]. For each $f \in \mathcal{B}(K)$, put $(\beta) \int_K f = (P) \int_K f$.

If $m \geqq 2$, $K \in \mathbf{J}_m$, let $\mathscr{B}(K) = \{f \in \mathscr{S}(K); (\beta) \int_K f$ defined in [5] exists$\}$.

(7,1) Theorem. $(\mathscr{B}, \beta) \in \mathfrak{F}$.

Proof. The only point here is to prove continuity for $m \geqq 2$. To this end, we use the following lemma (for notions mentioned below, see [5]).

(7,2) Lemma. *Let φ be an additive function defined on a ring of sets* Dom φ. *Then φ is continuous with respect to the convergence $\rightarrow$ iff given $\varepsilon > 0$, $C > 0$, $B \in \text{Dom}\,\varphi$, there exists a $\delta > 0$ such that*

$$(7.2.1) \qquad |A| < \delta\,, \quad \|A\| < C\,, \quad A \subset B\,, \quad A \in \text{Dom}\,\varphi \Rightarrow |\varphi(A)| < \varepsilon\,.$$

Proof. 1° Let φ be continuous with respect to $\rightarrow$. Suppose on the contrary that there exist $\varepsilon > 0$, $C > 0$, $B \in \text{Dom}\,\varphi$ such that for each $n \in \mathscr{N}$, there exist $A_n \in \in \text{Dom}\,\varphi$ such that $|A_n| < n^{-1}$, $\|A_n\| < C$, $A_n \subset B$, $|\varphi(A_n)| \geqq \varepsilon$. Then $B - A_n \in \in \text{Dom}\,\varphi$, $B - A_n \rightarrow B$, and $\lim \varphi(B - A_n) = \lim (\varphi(B) - \varphi(A_n)) \neq \varphi(B)$. This is a contradiction.

2° Let the conditions of the lemma be fulfilled, and suppose that $B_n \rightarrow B$, B_n, $B \in \in \text{Dom}\,\varphi$. There exists $C > 0$ such that $\|B - B_n\| < C$. Let $\varepsilon > 0$. Let $\delta > 0$ be such that (7.2.1) is fulfilled for these B, C. As $|B - B_n| \rightarrow 0$, there exists n_0 such that $n > n_0 \Rightarrow |B - B_n| < \delta$. Then $n > n_0 \Rightarrow |\varphi(B) - \varphi(B_n)| = |\varphi(B - B_n)| < \varepsilon$. Hence $\lim \varphi(B_n) = \varphi(B)$, which proves the lemma.

To prove the theorem, put $B = K \in \mathbf{J}_m$, $\varphi = (\beta) \int$. Let $C = \|K\|$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that (7.2.1) is fulfilled. If $I \in \mathbf{J}_m(K)$, then evidently $\|I\| \leqq C$, $I \in \text{Dom}\,\varphi$. Hence, according to (7,2), $|I| < \delta \Rightarrow |\varphi(I)| < \varepsilon$, which proves the continuity of $(\beta) \int$.

(7,3) Let us still mention that in view of [5], Theorem 11, p. 255, $(\mathscr{B}, \beta)$ **(Tr)** $(\mathscr{B}, \beta)$ for each $m \geqq 2$; for $m = 1$ this is well-known.

Properties **(Fub)**, **(Four)**, **(Pr)**, **(Distr)** have not been investigated for this type of integration.

(7,4) Let us also note that for $m = 1$, it may occur that $|\sigma(f)| > 0$ for e.g. $f \in \in \mathscr{P}(K)$, with properties **(Four)**, **(Tr)**, ... still holding true. I do not know any m-dimensional ($m \geqq 2$) integration with this property.

8. We show that $\mathscr{W}(K) - \mathscr{L}(K)$, $K \in \mathbf{J}_2$, is nonempty. Instead of $[x_1, x_2]$, we write $[x, y]$.

(8,1) Theorem. *Let $\varrho = \sqrt{(x^2 + y^2)}$. Define f as follows: $f(0, 0) = 0$, $f(x, y) = = \varrho^{-2} \sin \varrho^{-3}$, $\varrho \neq 0$. Then*

1° *f is continuous on $\mathscr{R}^2 - [0, 0]$,*

2° *$F(x, y) = (P) \int_0^x f(t, y)\,\mathrm{d}t$ is continuous on $\mathscr{R}^2$,*

3° *$\sigma(f) = [0, 0]$.*

Proof. 1° is clear.

2° We show that $(P)\int_0^x f(t, y)\,dt$ exists. It suffices to consider the case $y = 0$, $x > 0$. Let $0 < \varepsilon < x$. Then $\int_\varepsilon^x f(t, 0)\,dt = \int_\varepsilon^x t^{-2} \sin t^{-3}\,dt = \frac{1}{3}\int_{x^{-3}}^{\varepsilon^{-3}} z^{-2/3} \sin z\,dz$ so that existence of $(P)\int_0^x f(t, 0)\,dt$ follows.

We show that F is continuous at $[0, 0]$. Let first $x > 0$, $y > 0$, $\sqrt{(x^2 + y^2)} = r < 1$. Then $\int_0^x f(t, y)\,dt = \int_0^x (y^2 + t^2)^{-1} \sin (y^2 + t^2)^{-3/2}\,dt = \int_y^r \varrho^{-1}(\varrho^2 - y^2)^{-1/2}$ . . $\sin \varrho^{-3}\,d\varrho$, as we get using $t = \sqrt{(\varrho^2 - y^2)}$. Let $y_1 = \min(y + y^4, r)$. Then $\int_y^r \ldots = \int_y^{y_1} \ldots + \int_{y_1}^r \ldots$ We estimate the first integral. It holds

$$\left|\int_y^{y_1} \varrho^{-1}(\varrho^2 - y^2)^{-1/2} \sin \varrho^{-3}\,d\varrho\right| \leqq \int_y^{y_1} \varrho^{-1}(\varrho + y)^{-1/2}(\varrho - y)^{-1/2}\,d\varrho \leqq$$

$$\leqq \int_y^{y_1} y^{-3/2}(\varrho - y)^{-1/2}\,d\varrho = 2y^{-3/2}(y_1 - y)^{1/2} \leqq 2\sqrt{y} \leqq 2\sqrt{r}\,.$$

To estimate the second one, we suppose that $y_1 < r$. We have $\int_{y_1}^r \ldots = \int_{y_1}^r \varrho^3(\varrho^2 - y^2)^{-1/2}\,\varrho^{-4} \sin \varrho^{-3}\,d\varrho$. The derivative of the function $\lambda(\varrho) = \varrho^3(\varrho^2 - y^2)^{-1/2}$, $\varrho \in \langle y_1, r\rangle$ equals to $\varrho^2(2\varrho^2 - 3y^2)(\varrho^2 - y^2)^{-3/2}$, and is therefore negative for $\varrho < y\sqrt{\frac{3}{2}}$, positive for $\varrho > y\sqrt{\frac{3}{2}}$. Supposing $r \leqq y\sqrt{\frac{3}{2}}$, the function λ attains its maximum for $\varrho = y_1$, and $\lambda(y_1) = \lambda(y + y^4) = \sqrt{(y)}\,(1 + y^3)^3\,(2 + y^3)^{-1/2}$; as $y < 1$, we have $\lambda(y_1) < 4\sqrt{(2y)}$. If $r > y\sqrt{\frac{3}{2}}$, then λ attains its maximum on the boundary of $\langle y_1, r\rangle$. It holds $\lambda(r) = r^3x^{-1}$; as $2r^2 > 3y^2$, we have $3x^2 > r^2$ so that $\lambda(r) < r^2\sqrt{3}$. Hence $0 < \lambda(\varrho) < 4\sqrt{(2r)}$, $\varrho \in \langle y_1, r\rangle$, in each case.

Now put $\psi(\varrho) = \varrho^{-4} \sin \varrho^{-3}$ and estimate $\int_{y_1}^r \lambda(\varrho)\,\psi(\varrho)\,d\varrho$. It is immediate that, for each $k_1, k_2 > 0$, $\left|\int_{k_1}^{k_2} \psi\right| \leqq \frac{2}{3}$. The interval $\langle y_1, r\rangle$ may eventually be divided into two subintervals on each of which λ is monotone. Using there the second mean-value theorem, we get $\left|\int_{y_1}^r \lambda\psi\right| \leqq 4.\ 4\sqrt{(2r)}\,.\,\frac{2}{3} < 16\sqrt{r}$. Hence $\left|\int_y^r \ldots\right| \leqq 16\sqrt{(r)} + 2\sqrt{(r)} = 18\sqrt{r}$. As $F(-x, y) = -F(x, y)$, $F(x, -y) = F(x, y)$, $F(0, 0) = 0$, we have $|F(x, y)| < 18\sqrt{r}$ for each $[x, y]$ such that $x \neq 0$, $x^2 + y^2 < 1$. As $F(0, 0) = 0$ and $F(x, 0)$ is continuous on $\mathscr{R}$, the continuity of F at $[0, 0]$ follows at once.

Further, F is continuous at each $[x, y]$ such that $y \neq 0$. Let $x_0 > 0$; we show that F is continuous at $[x_0, 0]$. Let $\varepsilon > 0$; let $\delta_1 > 0$ be such that $|x| \leqq \delta_1$, $|y| \leqq \delta_1 \Rightarrow |F(x, y)| < \varepsilon/3$. The function $G(x, y) = \int_{\delta_1}^x f(t, y)\,dt$ is clearly continuous at $[x_0, 0]$; further we have $F(x, y) = \int_0^x \ldots = \int_0^{\delta_1} \ldots + \int_{\delta_1}^x \ldots = F(\delta_1, y) + G(x, y)$. Choose a neighbourhood Ω of $[x_0, 0]$ such that $[x, y] \in \Omega \Rightarrow |G(x, y) - G(x_0, 0)| < \varepsilon/3$. Then $[x, y] \in \Omega$, $|y| \leqq \delta_1 \Rightarrow |F(x, y) - F(x_0, 0)| = |F(\delta_1, y)| + |F(\delta_1, 0)| + |G(x, y) - G(x_0, 0)| < \varepsilon$. This proves 2°.

3° Suppose on the contrary that $(L)\int_C f$, $C = \{[x, y];\ x^2 + y^2 \leqq 1\}$, exists. Then also $(L)\int_{C^*} r^{-1} \sin r^{-3}\,dr\,d\varphi$, $C^* = \{[r, \varphi];\ 0 \leqq r \leqq 1,\ 0 \leqq \varphi < 2\pi\}$, exists, and using Fubini's theorem we get that $(L)\int_0^1 r^{-1} \sin r^{-3}\,dr = (L)\,\frac{1}{3}\int_1^\infty z^{-1} \sin z\,dz$ exists, which is a contradiction. This proves 3°.

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Author's address: Technická 1905, Praha 6 - Dejvice, (Vysoká škola chemicko-technologická).