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CHARACTERIZATION OF FUNCTIONS WITH ZERO TRACES BY INTEGRALS WITH WEIGHT FUNCTIONS I

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INTRODUCTION

Let Ω be a bounded domain of the Euclidean N-space E_N and let $\varrho(X)$ be the distance between the point $X \in \Omega$ and the boundary $\partial\Omega$ of the domain Ω. We assume the boundary $\partial\Omega$ fulfils locally the Lipschitz condition only.

We define the space $L_{p,\alpha}(\Omega)$ for $p \geqq 1$ and α real as the set of functions defined almost everywhere in Ω such that the norm

$$\|u\|_{L_{p,\alpha}(\Omega)} = \left[\int_\Omega |u(X)|^p \varrho^\alpha(X)\, \mathrm{d}X\right]^{1/p} \tag{0.1}$$

is finite.

If k is a natural number then $W^{(k)}_{p,\alpha}(\Omega)$ denotes the Banach space of all functions u (defined almost everywhere in Ω) with the generalized derivatives

$$D^i u = \frac{\partial^{|i|}}{\partial x_1^{i_1} \dots \partial x_N^{i_N}}$$

$(i = (i_1, i_2, \dots, i_N)$, $|i| = \sum_{s=1}^N i_s$, i_s – non- negative integers) of the order $|i|$ less or equal to k such, that $D^i u \in L_{p,\alpha}(\Omega)$ for i, $0 \leqq |i| \leqq k$. We take the norm

$$\|u\|_{W^{(k)}{}_{p,\alpha}(\Omega)} = [\sum_{|i|=0}^k \|D^i u\|^p_{L_{p,\alpha}(\Omega)}]^{1/p} \tag{0.2}$$

in $W^{(k)}_{p,\alpha}(\Omega)$.

From the well-known inequality of Hardy it follows

$$\|D^j u\|_{L_{p,\alpha-(k-|j|)p}(\Omega)} \leqq c\|u\|_{W^{(k)}{}_{p,\alpha}(\Omega)} \quad (0 \leqq |j| \leqq k) \tag{0.3}$$

under certain conditions on u, Ω and α. Here c is a positive constant independent on u. Let us denote by V the space of functions with the finite norm

$$(0.4) \qquad \|u\|_V = \Big[\sum_{|j|=0}^{k} \|D^j u\|^p_{L_{p,\alpha-(k-|j|)p}(\Omega)}\Big]^{1/p};$$

then it follows from (0.3), that $\|u\|_V \leqq c\|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$, i.e. $V \supset W^{(k)}_{p,\alpha}(\Omega)$.

Put ${}^0W = {}^0W^{(k)}_{p,\alpha}(\Omega)$ where ${}^0W^{(k)}_{p,\alpha}(\Omega)$ is the closure of the set $\mathscr{D}(\Omega)$ in the norm (0.2). Here $\mathscr{D}(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω.

In this paper, we show the other characterization of the space ${}^0W$ than this latter one (i.e. as the closure of $\mathscr{D}(\Omega)$ or as the "space of functions from $W^{(k)}_{p,\alpha}(\Omega)$ with zero traces of functions and its derivatives up to the order $k-1$ on the boundary $\partial\Omega$")[1]. To characterize ${}^0W$ it is sufficient to introduce an equivalent norm of the type (0.4) (except for some singular values of α – see the part II of this paper). Then we can show that the set of functions for which this new norm is finite is equal to the closure of $\mathscr{D}(\Omega)$, i.e. $V = {}^0W$ with equivalent norms.

As a special case of our results follows e.g. this theorem: For the function u to be in ${}^0W^{(1)}_2(\Omega)$ is necessary and sufficient that

$$\frac{\partial u}{\partial x_i} \in L_2(\Omega) \quad \text{and} \quad \frac{u}{\varrho} \in L_2(\Omega).$$

This fact has its applications in the theory of differential equations.

1. BASIC NOTIONS AND AUXILIARY ASSERTIONS

1.1 In the following, we shall consider bounded domains of the type $\mathfrak{N}^{(0),1}$, i.e. domains, for which the following conditions hold:

1. There are m coordinate systems $[x'_r, x_{rN}]$ (with $x'_r = (x_{r1}, x_{r2}, \ldots, x_{r,N-1})$) and m functions $a_r = a_r(x'_r)$ $(r = 1, 2, \ldots, m)$ defined on $(N-1)$-dimensional cubes $\Delta_r = \{x'_r \mid |x_{ri}| < \gamma$ for $i = 1, 2, \ldots, N-1\}$ such that for every point $X \in \partial\Omega$ there exists r such that $X = [x'_r, x_{rN}]$ and $x_{rN} = a_r(x'_r)$.

2. Functions $a_r(x'_r)$ fulfil the Lipschitz condition for $x'_r \in \Delta_r$.

3. There exists $\beta > 0$ such that the "cylinders"

$$B_r = \{[x'_r, x_{rN}] \mid x'_r \in \Delta_r;\ a_r(x'_r) - \beta < x_{rN} < a_r(x'_r)\}$$

lie in Ω and "cylinders"

$$C_r = \{[x'_r, x_{rN}] \mid x'_r \in \Delta_r;\ a_r(x'_r) - \beta < x_{rN} < a_r(x'_r) + \beta\}$$

cover the boundary $\partial\Omega$ and intersection of Ω and C_r is equal to B_r. We assume $\beta < 1$.

[1]) Let us remark that in some cases traces in the usual sense do not exist.

1.2 Function $a_r(x_r')$ fulfils the Lipschitz condition and so we have for $X \in B_r$ that the distance $\varrho(X)$ of X to $\partial\Omega$ is equivalent to the distance of X to $\partial\Omega$ in the direction of the coordinate axis x_{rN}, i.e.

$$c_1 \varrho(x_r', x_{rN}) \leqq a_r(x_r') - x_{rN} \leqq c_2 \varrho(x_r', x_{rN}) \tag{1.1}$$

(see for example [3]). So we can in the local coordinate system $[x_r', x_{rN}]$ write $[a_r(x_r') - x_{rN}]^\alpha$ instead of the original weight function $\varrho^\alpha(X)$.

1.3 Let us remember the inequality of Hardy

$$\int_0^\infty |f(t)|^p t^{\beta-p} \, dt \leqq \left(\frac{p}{|\beta - p + 1|}\right)^p \int_0^\infty |f'(t)|^p t^\beta \, dt \tag{1.2}$$

which takes place for $\beta > p - 1$, if $\lim_{t\to\infty} f(t) = 0$, and for $\beta < p - 1$ if $\lim_{t\to 0} f(t) = 0$. Inequality (1.2) follows from Theorem 330 in [1]. The value $\beta = p - 1$ is singular and in this case we shall give some modification of (1.2) in part II of this paper.

1.4 We shall use function $F_h(X)$ $(h > 0)$ with the following properties:

A. $F_h(X) \in \mathcal{D}(\Omega)$.

B. $F_h(X) \equiv 1$ for $\varrho(X) \geqq h$ (so we have $D^i F_h(X) \equiv 0$ for $|i| > 0$ and $\varrho(X) > h$).

C. $0 \leqq F_h(X) \leqq 1$.

D. $|D^i F_h(X)| \leqq c(i)\, h^{-|i|}$ for $|i| > 0$ $(h < 1)$.

The existence of such a function F_h has been shown in [2].

1.5 For the bounded domain Ω there is a constant c_3 such that $\varrho^{\alpha_1}(X) \leqq c_3 \varrho^{\alpha_2}(X)$ for $\alpha_1 \geqq \alpha_2$. By (0.1) we have

$$\|u\|_{L_{p,\alpha_1}} \leqq c_4 \|u\|_{L_{p,\alpha_2}} \quad \text{i.e.} \quad L_{p,\alpha_1} \subset L_{p,\alpha_2} \quad \text{for} \quad \alpha_2 \geqq \alpha_1 . \tag{1.3}$$

2. EQUIVALENCE OF $^\circ W$ AND V

The main result of this paper is the following

Theorem 1. *Let $\Omega \in \mathfrak{N}^{(0),1}$, $\varrho(X) = \operatorname{dist}(X, \partial\Omega)$ and $^\circ W$ be the closure of $\mathcal{D}(\Omega)$ in the norm (0.2). Let*

$$V = \{u \mid D^j u \in L_{p,\alpha-(k-|j|)p}(\Omega) \quad \text{for} \quad 0 \leqq |j| \leqq k\}$$

with the norm (0.4). Then

$$^\circ W = V$$

for every $\alpha \neq ip - 1$, where $i = 1, 2, \ldots, k$. The norms (0.2) and (0.4) are equivalent.

Proof. I) We prove that $\|u\|_V \leqq c\|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$ for $u \in {}^0W$, i.e. that ${}^0W \subset V$. From the density of $\mathscr{D}(\Omega)$ in ${}^0W$ it follows, that we can assume $u \in \mathscr{D}(\Omega)$.

The cylinders C_r from 1.1 form a covering of $\partial\Omega$. Let us denote by C_{m+1} such an open set that $\overline{C_{m+1}} \subset \Omega$ and $\Omega = \sum_{r=1}^{m} B_r + C_{m+1}$. Let functions $\varphi_i \in \mathscr{D}(C_i)$ ($i = 1, 2, \ldots, m+1$) form the corresponding decomposition of unit on $\overline{\Omega}$.

Consider one fixed coordinate system $[x'_i, x_{iN}]$ and the corresponding function φ_i. Then $\varphi_i \in \mathscr{D}(C_i)$, $u \in \mathscr{D}(\Omega)$ and so $u_i = u\varphi_i \in \mathscr{D}(B_i)$. We have $\|u_i\|_{W^{(k)}_{p,\alpha}(C_i)} \leqq C\|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$.

Further we omit the index i.

Let $|j| \leqq k$ and put $v = D^j u$.

a) If $|j| = k$, we obviously have $\|v\|_{L_{p,\alpha}(\Omega)} \leqq \|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$.

b) Let $|j| = k - s$, where $k \geqq s \geqq 1$. We have to prove

$$\|v\|_{L_{p,\alpha-sp}(\Omega)} \leqq c\|u\|_{W^{(k)}_{p,\alpha}(\Omega)},$$

where we take by (1.1)

$$\|v\|^p_{L_{p,\alpha-sp}} = \int_B |v(x', x_N)|^p \, [a(x') - x_N]^{\alpha-sp} \, \mathrm{d}X = \tag{2.1}$$

$$= \int_\Delta \mathrm{d}x' \int_{a(x')-\beta}^{a(x')} |v(x', x_N)|^p \, [a(x') - x_N]^{\alpha-sp} \, \mathrm{d}x_N \, .$$

The inner integral, after substitution $t = a(x') - x_N$, we can write as

$$\mathscr{I}(x') = \int_0^\infty |v(x', a(x') - t)|^p \, t^{\alpha-sp} \, \mathrm{d}t \, ,$$

where we use $u \in \mathscr{D}(B)$ and so $v = D^j u \in \mathscr{D}(B)$, i.e. $v(x', a(x') - t) = 0$ for t sufficiently close to zero and for $t \geqq \beta$. The integral $\mathscr{I}(x')$ we can estimate, using the inequality of Hardy (1.2) where we put $f(t) = v(x', a(x') - t)$ and $\beta - p = \alpha - sp$. From the properties of function $v(x', a(x') - t)$ it follows, by (1.2),

$$\mathscr{I}(x') \leqq \left(\frac{p}{|\alpha - sp + 1|}\right)^p \int_0^\infty \left|\frac{\partial v}{\partial x_N}(x', a(x') - t)\right|^p t^{\alpha-(s-1)p} \, \mathrm{d}t$$

for $\beta \neq p - 1$, i.e. $\alpha \neq sp - 1$. To estimate the latter integral, we substitute $f(t) = (\partial v/\partial x_N)(x', a(x') - t)$ and $\beta - p = \alpha - (s-1)p$ in (1.2). This function $f(t)$ again vanishes in the neighbourhood of 0 and ∞ and $\alpha \neq (s-1)p - 1$; we obtain by (1.2)

$$\mathscr{I}(x') \leqq \left(\frac{p}{|\alpha - sp + 1|}\right)^p \left(\frac{p}{|\alpha - (s-1)p + 1|}\right)^p \int_0^\infty \left|\frac{\partial^2 v}{\partial x_N^2}(x', a(x') - t)\right|^p t^{\alpha-(s-2)p} \, \mathrm{d}t \, .$$

The integral on the right hand side we estimate in the same manner and finally we obtain (repeating this procedure s-times)

$$\mathscr{I}(x') \leqq \prod_{i=0}^{s-1} \left(\frac{p}{|\alpha - (s - i) p + 1|} \right)^p \int_0^{\infty} \left| \frac{\partial^s v}{\partial x_N^s} (\alpha', a(x') - t) \right|^p t^\alpha \, dt =$$

$$= c_1 \int_{a(x')-\beta}^{a(x')} \left| \frac{\partial^s v}{\partial x_N^s} (x', x_N) \right|^p [a(x') - x_N]^\alpha \, dx_N .$$

Because $\alpha \neq ip - 1$ for $i = 1, 2, \ldots, k$.

Now we have $\partial^s v / \partial x_N^s = D^h u$ where $|h| = k$ and, integrating the latter inequality by x', we obtain by (2.1)

$$(2.2) \qquad \|D^j u\|^p_{L_{p,\alpha-(k-|j|)p}(B)} \leqq c_1 \int_B |D^h u|^p [a(x') - x_N]^\alpha \, dX \leqq c_2 \|u\|^p_{W^{(k)}_{p,\alpha}(\Omega)}$$

$$(0 \leqq |j| \leqq k) .$$

The inequalities (2.2) take place for $u = u_i$, $i = 1, 2, \ldots, m$. If we put $u = u\varphi_{m+1} = u_{m+1}$ we obtain (2.2) again because φ_{m+1} has its support in $\overline{C_{m+1}} \subset \Omega$ and the weight function is bounded, continuous and sharp positive on $\overline{C_{m+1}}$. From (2.2) finally follows the inequality

$$\|u\varphi_i\|_V \leqq C_3 \|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$$

where by (1.1) we take the weight function $\varrho(x)$ instead of $[a_r(x') - x_{rN}]$ on every C_r. Further, $u = \sum_{i=1}^{m+1} u\varphi_i$, an so we have

$$\|u\|_V \leqq C_4 \|u\|_{W^{(k)}_{p,\alpha}(\Omega)}$$

for every $u \in \mathscr{D}(\Omega)$. From the density of $\mathscr{D}(\Omega)$ in ${}^0W$ it follows that the latter inequality holds also for $u \in {}^0W$, and so ${}^0W \subset V$.

II) We shall prove $V \subset {}^0W$.

Let $u \in V$ and put $u_h(X) = u(X) F_h(X)$, where $F_h(X)$ is the function from the section 1.4. The function u_h has a compact support in Ω (see property A) and from the property B it follows $u(X) - u_h(X) = u(X)(1 - F_h(X)) = 0$ for such $X \in \Omega$ that $\varrho(X) \geqq h$.

We have $u \in V$ and so $D^j u \in L_{p,\alpha-(k-|j|)p}(\Omega)$. By (1.3) is $D^j u \in L_{p,\alpha}(\Omega)$ for $0 \leqq |j| \leqq k$, it means that $u \in W^{(k)}_{p,\alpha}(\Omega)$, i.e.

$$(2.3) \qquad V \subset W^{(k)}_{p,\alpha}(\Omega) .$$

We have obviously also $u_h \in W^{(k)}_{p,\alpha}(\Omega)$.

We shall prove $\|u - u_h\|_{W^{(k)}_{p,\alpha}(\Omega)} \to 0$ as $h \to 0$. Put $P_h = \{X \in \Omega \mid \varrho(X) < h\}$; we can integrate $u - u_h$ only over P_h, because $u_h(X) = u(X)$ as $X \notin P_h$.

Let $|i| \leqq k$. Then

$$D^i(u - u_h) = D^i[u(1 - F_h)] = (1 - F_h)\, D^i u + \sum_{\substack{m+n=i \\ |n| \geqq 1}} c_{mn} D^m u D^n F_h$$

and so

$$\|D^i(u - u_h)\|_{L_{p,\alpha}(\Omega)} \leqq \|(1 - F_h)\, D^i u\|_{L_{p,\alpha}(\Omega)} + \sum c_{mn} \|D^m u D^n F_h\|_{L_{p,\alpha}(\Omega)}\,. \tag{2.4}$$

From $0 \leqq F_h(X) \leqq 1$ it follows $|1 - F_h(X)| \leqq 1$ and so

$$\|D^i u(1 - F_h)\|^p_{L_{p,\alpha}(\Omega)} \leqq \int_{P_h} |D^i u|^p \varrho^\alpha(X)\, \mathrm{d}X \to 0 \tag{2.5}$$

as $h \to 0$, because $u \in W^{(k)}_{p,\alpha}(\Omega)$ and so $D^i u \in L_{p,\alpha}(\Omega)$ and further mes $(P_h) \to 0$ as $h \to 0$.

For $m + n = i$, $|n| \geqq 1$ we put

$$\mathscr{I}_{mn}(h) = \|D^m u D^n F_h\|^p_{L_{p,\alpha}(\Omega)} = \int_{P_h} |D^m u|^p\, |D^n F_h|^p\, \varrho^\alpha(X)\, \mathrm{d}X\,.$$

With respect to the property D (see section 1.4) we have $|D^n F_h|^p \leqq c^p(n)\, h^{-|n|p}$ and further $X \in P_h$, i.e. $\varrho(X) \leqq h$ and $h^{-|n|p} \leqq \varrho^{-|n|p}(X)$. We have $n = i - m$, $|i| \leqq k$, and so $|n| = |i| - |m| \leqq k - |m|$ and (see section 1.5) $\varrho^{-|n|p}(X) \leqq c_5 \varrho^{-(k-|m|)p}(X)$. So we finally obtain the inequality

$$|D^n F_h|^p\, \varrho^\alpha(X) \leqq c_6 \varrho^{\alpha-(k-|m|)p}(X)$$

and the following estimation

$$\mathscr{I}_{mn}(h) \leqq c_6 \int_{P_h} |D^m u|^p\, \varrho^{\alpha-(k-|m|)p}(X)\, \mathrm{d}X \to 0$$

as $h \to 0$ because $u \in V$ and so $D^m u \in L_{p,\alpha-(k-|m|)p}(\Omega)$ and further mes $(P_h) \to 0$ for $h \to 0$. If we consider moreover (2.5) and (2.4) we obtain

$$\lim_{h\to 0} \|u - u_h\|_{W_{p,\alpha}(k)} = \lim_{h \to 0} \Big[\sum_{|i|=0}^{k} \|D^i(u - u_h)\|^p_{L_{p,\alpha}} \Big]^{1/p} = 0\,.$$

To given $\varepsilon > 0$ we can take a fixed $h > 0$ so that $\|u - u_h\|_{W_{p,\alpha}(k)} < \varepsilon/2$. Let us denote by $(u_h)_\delta$ the regularisation of the function u_h (see e.g. [4]). The function u_h has a compact support in Ω and so $(u_h)_\delta \in \mathscr{D}(\Omega)$ and for sufficiently small δ is also $u_h - (u_h)_\delta$ small in the norm $W^{(k)}_p(\Omega)$ (for $\alpha = 0$!). Now u_h and $(u_h)_\delta$ vanish in the neighbourhood of $\partial\Omega$ and the weight function has no influence on functions with the same compact support in Ω. So we can choose δ so small, that $\|u_h - (u_h)_\delta\|_{W^{(k)}_{p,\alpha}(\Omega)} < \varepsilon/2$.

To an arbitrary function $u \in V$ and given $\varepsilon > 0$ we found a function $v = (u_h)_\delta \in \mathscr{D}(\Omega)$ such that $\|u - v\|_{W^{(k)}_{p,\alpha}(\Omega)} < \varepsilon$. It means $u \in {}^0W$ an so $V \subset {}^0W$. By part I of this proof we have $V = {}^0W$ and so the theorem is proved.

3.1 Theorem 1 takes place for every value of the parameter α except $\alpha_i = ip - 1$ ($i = 1, 2, \ldots, k$). It is obvious $\alpha_i \geqq 0$ for $i = 1, 2, \ldots, k$ and so Theorem 1 holds for $\alpha < 0$.

Now, let $\alpha \leqq -1$ and let $f(t)$ be defined in $(0, 1)$ and such that

$$\int_0^\infty |f(t)|^p t^\alpha \, dt < \infty \quad \text{and} \quad \int_0^\infty |f'(t)|^p t^\alpha \, dt < \infty .$$

If $p > 1$ then

$$|f(t + h) - f(t)| = \left| \int_t^{t+h} f'(s) \, ds \right| = \left| \int_t^{t+h} f'(s) \, . \, s^{\alpha/p} \, . \, s^{-\alpha/p} \, ds \right| \leqq$$

$$\leqq \left\{ \int_t^{t+h} |f'(s)|^p s^\alpha \, ds \right\}^{1/p} . \left\{ \int_t^{t+h} s^{-\alpha/(p-1)} \, ds \right\}^{p/(p-1)} .$$

The right hand side of the latter inequality converges to zero as $h \to 0$ uniformely with respect to t. So $f(t)$ is uniformely continuous in $(0, 1)$ and there exists the limit

$$\lim_{t \to 0+} f(t) = a .$$

On the other hand the integral $\int_0^1 |f(t)|^p t^\alpha \, dt$ is finite and for $\alpha \leqq -1$ is $\int_0^1 t^\alpha \, dt = \infty$. So we necessarily have $a = 0$.

So $f(t)$ has in the point $t = 0$ a trace which vanishes, i.e. a zero trace.

Now, let $u \in W_{p,\alpha}^{(k)}(\Omega)$, $\alpha \leqq -1$. In the local coordinate system (see section 1.1) the function $v(x_i', t) = u(x_i', a_i(x_i') - t)$ is an element of the space $W_{p,\alpha}^{(k)}(\Delta_i \times (0, \beta))$ with the weight function t^α.

Put $f(t) = D^j v(x_i', t)$ for $|j| = 0, 1, \ldots, k - 1$. From our considerations this result follows: The function $D^j v$ for $0 \leqq |j| \leqq k - 1$ has for almost all $x_i' \in \Delta_i$ limits on the hyperplane $t = 0$ equal to zero; so we obtain for the function u this assertion:

The function $D^j u$ for $0 \leqq |j| \leqq k - 1$ has zero traces on $\partial\Omega$.

In this case we can automatically use Hardy's inequality, repeat the considerations of the first part of the proof of Theorem 1 and then we obtain

$$W_{p,\alpha}^{(k)}(\Omega) \subset V . \tag{2.6}$$

By Theorem 1 we have $V \subset W_{p,\alpha}^{(k)}(\Omega)$ – see (2.3) – and so

$$W_{p,\alpha}^{(k)}(\Omega) = V \; (= {}^0W_{p,\alpha}^{(k)}(\Omega)) \quad \text{for} \quad \alpha \leqq -1 . \tag{2.7}$$

3.2 The same result we obtain also for $\alpha > kp - 1$: In part I of the proof of Theorem 1 we take $u_i = u\varphi_i$ for $u \in W_{p,\alpha}^{(k)}(\Omega)$, where φ_i is the decomposition of unit (in local coordinates). The function $u_i(x_i', x_{iN})$ vanishes for $x_{iN} \leqq a_i(x_i') - \beta$ and then $f(t) = u_i(x_i, a_i(x_i') - t) = 0$ for $t \geqq \beta$. Now we can again use Hardy's inequality (namely its second eventuality: $\lim_{t \to \infty} f(t) = 0$) for $\alpha > kp - 1$ and obtain (by the same way as by part I of the proof of Theorem 1) the inclusion $W_{p,\alpha}^{(k)}(\Omega) \subset V$. So we can write

Theorem 2. *Let $\Omega \in \mathfrak{N}^{(0),1}$ and $\alpha \notin (-1, kp - 1\rangle$. Then*

$$W_{p,\alpha}^{(k)}(\Omega) = {}^0W_{p,\alpha}^{(k)}(\Omega) = V \tag{2.8}$$

where $V = \{u \mid D^i u \in L_{p,\alpha-(k-|i|)p}(\Omega);\ 0 \leqq |i| \leqq k\}$. Both norms (0.2) and (0.3) are equivalent on $W_{p,\alpha}^{(k)}(\Omega)$.

Proof follows from Theorem 1 and from the considerations of sections 3.1 and 3.2.

3.3 The assertion of Theorem 2 is in general not true for $\alpha \in (-1, kp - 1\rangle$. It is possible that the function $u \in W_{p,\alpha}^{(k)}(\Omega)$ or some of its derivatives have traces which are not equal to zero identically (see [3]).

References

[1] *G. H. Hardy, J. E. Littlewood, G. Pólya:* Inequalities, Cambridge 1934.

[2] *J. Kadlec:* On domain of the type $\mathfrak{P}$, Czech. Math. Journal *16* (86), (1966), 247–259.

[3] *J. Nečas:* Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle, Ann. Scuola Norm. Sup. Pisa, ser. 3, *16*, 4 (1962), 305–326.

[4] *С. Л. Соболев:* Некоторые применения функционального анализа в математической физике, Ленинград 1950.

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Výtah

CHARAKTERIZACE FUNKCÍ S NULOVÝMI STOPAMI POMOCÍ INTEGRÁLŮ S VAHOU

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Buď Ω omezená oblast v E_N, jejíž hranice $\partial\Omega$ splňuje lokálně Lipschitzovu podmínku, a buď $W_{p,\alpha}^{(k)}(\Omega)$ Sobolevův prostor s vahou $\varrho^\alpha(X)$, kde $\varrho(X)$ je vzdálenost bodu $X \in \Omega$ od hranice $\partial\Omega$; norma ve $W_{p,\alpha}^{(k)}(\Omega)$ je dána vzorcem (0.2). Buď dále $\mathscr{D}(\Omega)$ množina funkcí nekonečně diferencovatelných v E_N s kompaktním nosičem v Ω.

V práci je ukázáno, že funkce z prostoru ${}^0W = {}^0W^{(k)}_{p,\alpha}(\Omega)$, definovaného jako uzávěr množiny $\mathscr{D}(\Omega)$ v normě (0.2), lze charakterizovat též pomocí jiného váhového prostoru V:

$$V = \{u \mid D^j u \,.\, \varrho^{\alpha/p-(k-|j|)}(X) \in L_p(\Omega)\ ;\ 0 \leqq |j| \leqq k\}$$

s normou (0.4).

Ve větě 1 je dokázáno, že pro $\alpha \neq ip - 1$, $i = 1, 2, \ldots, k$, je ${}^0W = V$, přičemž normy (0.2) a (0.4) jsou na ${}^0W$ ekvivalentní.

Pomocí této věty je ve větě 2 ukázáno, že pro $\alpha \notin (-1, kp - 1\rangle$ je dokonce ${}^0W^{(k)}_{p,\alpha}(\Omega) \equiv W^{(k)}_{p,\alpha}(\Omega)$.

Резюме

ХАРАКТЕРИСТИКА ФУНКЦИЙ С НУЛЕВЫМИ СЛЕДАМИ ПРИ ПОМОЩИ ИНТЕГРАЛОВ С ВЕСОМ

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Пусть Ω – ограниченная область в E_N, граница $\partial\Omega$ которой удовлетворяет локально условию Липшица, и пусть $W^{(k)}_{p,\alpha}(\Omega)$ – пространство Соболева с весом $\varrho^\alpha(X)$, где $\varrho(X)$ – расстояние точки $X \in \Omega$ от границы $\partial\Omega$; норма в пространстве $W^{(k)}_{p,\alpha}(\Omega)$ определена формулой (0.2). Пусть, далее, $\mathscr{D}(\Omega)$ – множество финитных в Ω функций.

В работе показано, что функции из пространства $W \doteq W^{(k)}_{p,\alpha}(\Omega)$, определенного как замыкание множества $\mathscr{D}(\Omega)$ в норме (0.2), можно охарактеризовать с помощью пространства

$$V = \{u \mid D^j u \,.\, \varrho^{\alpha/p-(k-|j|)}(x) \in L_p(\Omega);\ 0 \leqq |j| \leqq k\}$$

с нормой (0.4).

В теореме 1 доказывается, что для $\alpha \neq ip - 1$, $i = 1, 2, \ldots, k$ имеет место тождество ${}^\circ W = V$ и что нормы (0.2) и (0.4) равносильны для $u \in {}^\circ W$.

С помощью этой теоремы показано в теореме 2, что для $\alpha \notin (-1, kp - 1\rangle$ даже имеет место тождество ${}^\circ W^{(k)}_{p,\alpha}(\Omega) = W^{(k)}_{p,\alpha}(\Omega)$.