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GENERALIZED INTERPRETABILITY IN TERMS OF MODELS (NOTE TO A PAPER OF R. MONTAGUE)

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In [2], MONTAGUE considers three relations between two sets of sentences Φ, Ψ, namely:

(1) all members of Ψ are derivable from Φ;

(2) the theory axiomatized by Ψ is interpretable in the theory axiomatized by Φ;

(3) the theory axiomatized Ψ is relatively interpretable in the theory axiomatized by Φ.

He gives semantic definitions of the relations (2) and (3), and proves that these new definitions are equivalent to the original syntactic definitions, which he states to have an accidental character.

The function f from the definition of (relative) interpretability, which associates to every standard atomic formula of the language of Ψ (and to a new unary predicate) a formula of the language of Φ, can be called either an interpretation (of Ψ in Φ) or a syntactic model (of Ψ in Φ), [1]. If a syntactic model of Ψ in Φ is given, i.e. actually constructed, then the relative consistency of Ψ with respect to Φ is (effectively) demonstrated. The need of effectivity (consequently, the need of finite metamathematics without set-theoretical means) seems to be adequate, if we (as mathematical logicians) inquire, what can the matematicians do (prove, decide) and what cannot they do? I believe that, in this case, the syntactic definitions of the relations (1)–(3) are not entirely accidental, and indeed that they are the only possible ones. The metamathematical framework sketched by Montague seems to correspond to another question of the logician, namely, what are relations between the languages of the matematicians and of the external "world"? In this case, indeed, semantic definitions of the relations (1)–(3) are more interesting than the syntactic ones.

In order to answer the first metamathematical question in particular cases, a generalized notion of interpretability, the so-called notion of a parametrical syntactic model (see below), was used (see e.g. [4]) and explicitely formulated (in [1]). We also have the fourth (actually used) relation between two systems of sentences:

(4) The theory axiomatized by Ψ has a parametrical syntactic model in the theory axiomatized by Φ (one may say that the former theory is parametrically interpretable in the latter one).

A semantic definition of this relation can be found, and proved to be equivalent to the syntactic definition by modifying the proof from [2]. This is carried out in the present paper.

The framework sketched in [2] will be employed here. The only difference is that we consider the logical calculus without preferred equality predicate (and, consequently, without operation symbols and constants). Obviously, it is possible that a theory contain an equality predicate; the condition for a predicate to be an equality predicate in a theory are well-known. Then logical operations and constants can be introduced as (metamathematical) abbreviations. This conceptions enables us to interpret the predicate declared to be the equality predicate not necessarily as the equality predicate of the theory in which it is interpreted (cf. footnote 17) in [3]). This fact can at least simplify constructions of syntactic models (see e.g. [5]). However, it seems that the modification of our consideration so as to apply to the metamathematics given in [2] does not present any problems.

Definition 1. A triple ϑ, χ, f is called a parametrical translation of a language Γ_1 into a language Γ_2 with n parameters, n a positive integer, iff (i) ϑ is a formula of Γ_2 such that the free variables of ϑ are among $v_1, v_3, \ldots, v_{2n-1}$ (n variables);

(ii) χ is a formula of Γ_2 such that the free variables of χ are among $v_0, v_1, v_3, \ldots, \ldots, v_{2n-1}$ ($n + 1$ variables);

(iii) f is a function whose domain is the set of all standard atomic formulas φ of Γ_1; for every such formula $P(v_0, \ldots, v_q)$, $f(\varphi)$ is a formula such that its free variables are among $v_1, v_3, \ldots, v_{2n-1}, v_0, v_2, \ldots, v_{2q}$ (and none of these variables are bound in $f(\varphi)$).

Definition 2. Let $t = \langle \vartheta, \chi, f \rangle$ be a parametrical translation of Γ_1 into Γ_2 (with n parameters). With every formula φ of Γ_1 one associates a formula φ_t of Γ_2 in the following way:

(a) if φ is atomic, say $P(v_{k_0}, \ldots, v_{kq})$, and $f(P(v_0, \ldots, v_q))$ is $\psi(v_1, v_3, \ldots, v_{2n-1}, v_0, v_2, \ldots, v_{2q})$, then φ_t is $\psi(v_1, v_3, \ldots, v_{2n-1}, v_{2k_0}, v_{2k_1}, \ldots, v_{2kq})$;

(b) if φ is $\varphi_1 \wedge \varphi_2$ (or $\varphi_1 \vee \varphi_2$ or $\neg\varphi_1$, etc.) then φ_t is $(\varphi_1)_t \wedge (\varphi_2)_t$ (or $(\varphi_1)_t \vee \vee (\varphi_2)_t$, or $\neg(\varphi_1)_t$ respectively);

(c) if φ is $\bigwedge v_k \psi$ or $\bigvee v_k \psi$, then φ_t is $\bigwedge v_{2k}(\chi(v_{2k}, v_1, \ldots, v_{2n-1}) \rightarrow \psi_t)$ or $\bigvee_{2k}(\chi(v_{2k}, v_1, \ldots, v_{2n-1}) \wedge \psi_t)$ respectively.

Definition 3. (i) Let t, Γ_1, Γ_2 be as in Definition 2, let Γ_2 be the language of a theory Φ, φ a formula of Γ_1. Then φ is said to hold in the translation t iff the formula $\bigwedge v_1, \ldots, v_{2n-1}(\vartheta(v_1, \ldots, v_{2n-1}) \rightarrow \varphi_t)$ belongs to Φ.

(ii) Under the same assumption, let Γ_1 be the language of a theory ψ axiomatized by a set of formulas Ψ_0. The translation t is said to be a parametrical syntactic model of Ψ in Φ iff the formula

$$(1)\qquad \bigvee v_1, \ldots, v_{2n-1} \vartheta(v_1, \ldots, v_{2n-1}) \wedge \bigwedge v_1, \ldots, v_{2n-1}[\vartheta(v_1, \ldots, v_{2n-1}) \rightarrow \rightarrow \bigvee v_0 \chi(v_0, v_1, \ldots, v_{2n-1})]$$

belongs to Φ and, for every $\psi \in \Psi_0$, ψ holds in t.

If $\Psi_0 = \{\psi\}$ is one-element-set (and ψ be closed), then the conjunction of (1) with the formula

$$\bigwedge v_1, \ldots, v_{2n-1}[\vartheta(v_1, \ldots, v_{2n-1}) \rightarrow \psi_t]$$

is denoted by Mod_t. (Mod_t is a closed formula of the language Γ_2.)

Definition 4. Let Φ, Ψ be theories. Then Ψ is said to be parametrically interpretable in Φ iff, for some positive integer n, there is a parametrical syntactic model with n parameters of Ψ in Φ.

Lemma. *Let Φ be a theory, Ψ a theory axiomatized by Ψ_0, t a parametrical syntactic model of Ψ in Φ. Then, for every $\psi \in \Psi$, ψ holds in t. Further more, if Φ is consistent, then Ψ is also consistent.* (See [1].)

Definition 5. A set F is called a family of semantic models (of a theory Φ, with n parameters) iff F is a function such that the domain of F is a non-empty n-ary relation and the range of F consists of some semantic models of Φ. Write $F(y) = \langle A_y, g_y \rangle$ for every y in the domain of F.

With the family of models F one associates a triple $\langle P_F, Q_F, g_F \rangle$ in the following manner: (i) P_F is the domain of F;

(ii) Q_F is the $(n + 1)$-ary relation such that $\langle x_0, \ldots, x_n \rangle \in Q_F$ if and only if $\langle x_0, \ldots, x_{n-1} \rangle \in P_F$ and $x_n \in A_{\langle x_0, \ldots, x_{n-1} \rangle}$;

(iii) g_F is a function, the domain of g_F constists of all standard atomic formulas of the language of Φ and, for every k-ary φ in the domain of g_F, $g_F(\varphi)$ is the $(n + k)$-ary relation such that $\langle x_0, \ldots, x_{n-1}, x_n, \ldots, x_{n+k-1} \rangle \in g_F'\varphi)$ if and only if $\langle x_0, \ldots, x_{n-1} \rangle \in P_F$ and $\langle x_n, \ldots, x_{n+k-1} \rangle \in g_{\langle x_0, \ldots, x_{n-1} \rangle}(\varphi)$.

Definition 6. A family of models F is said to be definable in a model A iff the relations P_F, Q_F and all relations in the range of g_F are such.

Theorem. *If Φ is a theory and Ψ is a finitely axiomatizable theory, then Ψ is parametrically interpretable in Φ if and only if, for each model A of Φ, there is a family of models of Ψ which is definable in A.*

Proof. We modify the proof of Theorem 1 in [2]. Assume the hypothesis. The implication from left to right is obvious. Assume that for every model A of Φ there is a family of models of F which is definable in A. Let G be the set of all parametrical translations of the language of Ψ into the language of Φ; let ψ_0 be the conjunction of all members of a finite axiom system of Ψ, let $\Psi_0 = \{\psi_0\}$. It is easy to see that, for every model A of Φ, there is a family of models of Ψ definable in A if and only if the sentence Mod_t is true in A for some $t \in G$. It follows from the Compactness Theorem that there is a finite subset D of G such that, for every model A of Φ, there exists a t in D for which Mod_t is true in A. Let $D = \{t_1, \ldots, t_n\}$, let $t_i = \langle \vartheta_i, \chi_i, f_i \rangle$ for every $1 \leqq i \leqq n$. The disjunction $\mathrm{Mod}_{t_1} \vee \ldots \vee \mathrm{Mod}_{t_n}$ is true in every model A of Φ, and consequently, $\Phi \vdash \mathrm{Mod}_{t_1} \vee \ldots \vee \mathrm{Mod}_{t_n}$. Define a translation $t_0 = \langle \vartheta_0, \chi_0, f_0 \rangle$ as follows:

$$\vartheta_0 \text{ is the formula } (\mathrm{Mod}_{t_1} \wedge \vartheta_1) \vee (\neg\, \mathrm{Mod}_{t_1} \wedge \mathrm{Mod}_{t_2} \wedge \vartheta_2) \vee \ldots$$
$$\ldots \vee (\neg\, \mathrm{Mod}_{t_1} \wedge \neg\, \mathrm{Mod}_{t_2} \wedge \ldots \wedge \neg\, \mathrm{Mod}_{t_{n-1}} \wedge \mathrm{Mod}_{t_n} \wedge \vartheta_n);$$
$$\chi_0 \text{ is the formula } (\mathrm{Mod}_{t_1} \wedge \chi_1) \vee (\neg\, \mathrm{Mod}_{t_1} \wedge \mathrm{Mod}_{t_2} \wedge \chi_2) \vee \ldots$$
$$\ldots \vee (\neg\, \mathrm{Mod}_{t_1} \wedge \neg\, \mathrm{Mod}_{t_2} \wedge \ldots \wedge \neg\, \mathrm{Mod}_{t_{n-1}} \wedge \mathrm{Mod}_{t_n} \wedge \chi_n);$$

for every standard atomic $\varphi, f_0(\varphi)$ is the formula

$$(\mathrm{Mod}_{t_1} \wedge f_1(\varphi)) \vee \ldots \vee (\neg\, \mathrm{Mod}_{t_1} \wedge \ldots \wedge \neg\, \mathrm{Mod}_{t_{n-1}} \wedge \mathrm{Mod}_{t_n} \wedge f_n(\varphi)).$$

D being finite, there is a positive integer n_0 such that t_0 is a parametrical translation with n_0 parameters. In analogy with Montague's procedure one proves $\Phi \vdash \mathrm{Mod}_{t_0}$, and this suffices to show that Ψ is parametrically interpretable in Φ.

Corollary. *Let Φ, Ψ be theories, let Ψ be finitely axiomatizable. Then Ψ is parametrically interpretable in Φ if and only if Ψ is parametrically interpretable in every complete extension of Φ.*

Appendix. It is obvious that every theory Ψ which is relatively interpretable in Φ is parametrically interpretable in Φ. The notion of parametrical syntactic models is at least useful as a means to simplify some syntactic constructions (consistency proofs). In the case of the Bernays-Gödel set theory Σ, the following holds: Every theory which is parametrically interpretable in Σ by means of a normal model (see [1]) is (nonparametrically) relatively interpretable in Σ. (A weaker statement is proved in [1], Theorem 7; the present assertion was proved by I. Korec.) Next there is exhibited a simple example of theories Φ, Ψ such that Ψ is parametrically interpretable but not relatively interpretable in Φ. Let the language of both Φ and Ψ consist of one unary predicate and one binary predicate $=$, let the axioms of Φ be

(1) $\bigvee v_0\, P(v_0) \wedge \bigvee v_0 \bigvee v_1 v_0 \neq v_1,$

(2) $\bigwedge v_0 \bigwedge v_1 [(P(v_0) \wedge v_0 = v_1) \rightarrow P(v_1)],$

(3) reflexivity, transitivity and symmetry of $=$,

let the axioms of Ψ be (1), (2), (3)and

(4) $\bigvee v_0 \neg P(v_0)$.

In order to prove that Ψ is parametrically interpretable in it suffices to take $\vartheta(v_1) \equiv \equiv P(v_1)$, $\chi(v_1, v_0) \equiv v_0 = v_0$, $P_t(v_1, v_0) \equiv v_0 = v_1$, $=_t (v_1, v_0, v_2) \equiv v_0 = v_2$.

Now proceed to prove that Ψ is not relatively interpretable in Φ. Let A be any set consisting of at least two elements, let $g(P(v_0)) = A$, $g(v_0 = v_1) = \{\langle x, x\rangle; x \in A\}$, $A = \langle A, g\rangle$. If Ψ were relatively interpretable in Φ, then, by Theorem 2 in [2], two disjoint non-empty sets would be definable in A. But the only sets definable in A by menas of the language $\{P, =\}$ are the empty set $\emptyset$ and A. This can be shown by proving (by induction) the following assertion: If φ is a formula of the language $\{P, =\}$, π is a permutation of the set A, $\{a_n\}_\omega$ is a countable sequence of elements of A and $\{a_n\}_\omega$ fulfils φ in A, then also the sequence $\{\pi(a_n)\}_\omega$ fulfils φ in A.

Finally, let the axioms of Ψ_1 be (1), (2), (3) and

(5) $$\bigwedge v_0 \bigwedge v_1 [(P(v_0) \wedge P(v_1)) \rightarrow v_0 = v_1] .$$

The theory Ψ_1 is interpretable in Φ (put $P_t(v_0) \equiv P(v_0)$, $v_0 =_t v_1 \equiv ((P(v_0) \wedge \wedge P(v_1)) \vee v_0 = v_1))$; further more, Ψ_1 is parametrically interpretable in Φ in such a manner that the equality predicate is interpreted absolutely (take ϑ, χ, P_t, $=_t$ from the preceeding example); however, Ψ_1 is not relatively interpretable in Φ if the equality predicate is considered as absolute (consider e.g. the set of all positive integers with the equality relation and with the subset of all odd numbers).

References

[1] *P. Hájek:* Syntactic models of axiomatic theories, Bull. Acad. Polon. Sci. *XIII* (1965), 273—278.

[2] *R. Montague:* Interpretability in terms of models, Indag. Math. *XXVII* (1965), 467—476.

[3] *A. Tarski, A. Mostowski, R. N. Robinson:* Undecidable Theories, Amsterdam 1953.

[4] *P. Vopěnka:* Postroenie modélei teorii množestv metodom ul'traproizvedenia, Zeitschr. für Math. Log. *8* (1962), 281—292.

[5] *P. Hájek:* Die durch die schwach inneren relationen gegebenen Modelle der Mengenlehre, ibid. *10* (1964), 151—157.

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ZOBECNĚNÁ INTERPRETOVATELNOST V TERMINOLOGII MODELŮ (POZNÁMKA K PRÁCI R. MONTAGUEHO)

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Montague podává v práci [4] sémantické definice syntaktických pojmů interpretovatelnosti a relativní interpretovatelnosti libovolné axiomatické teorie v konečně axiomatizovatelné teorii. Podávám analogickou sémantickou charakterizaci obecnějšího pojmu parametrické interpretovatelnosti axiomatické teorie.

Věta. *Buďte Φ, Ψ axiomatické teorie, budiž Φ konečně axiomatizovatelná. Ψ je parametricky interpretovatelná ve Φ (tj. Ψ má parametrický syntaktický model ve Φ) právě tehdy, když ke každému sémantickému modelu* A *teorie Φ existuje rodina* F *sémantických modelů teorie Ψ definovatelná v* A. *(Pojem rodiny sémantických modelů a její definovatelnosti je zaveden jistým přirozeným způsobem.)*

Je podán příklad axiomatizovatelných teorií Φ, Ψ takových, že Ψ je parametricky interpretovatelná ve Φ, ale není relativně interpretovatelná ve Φ.

Резюме

ОБОБЩЕННАЯ ИНТЕРПРЕТИРУЕМОСТЬ В ПОНЯТИЯХ МОДЕЛЕЙ (ЗАМЕТКА К РАБОТЕ Р. МОНТАГЮ)

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Монтагю ввел семантические определения синтаксических понятий интерпретируемости и относительной интерпретируемости про извольной аксиоматической теории в конечно-аксиоматизируемой теории. В предлагаемой работе дается аналогичное семантическое определение более общего понятия параметрической интерпретируемости теории в конечно-аксиоматизируемой теории.

Теорема. Пусть Φ, Ψ — аксиоматические теории, пусть Φ — конечно-аксиоматизируема. Ψ параметрически интерпретируема в Φ тогда и только тогда, когда для всякой семантической модели A теории Φ существует семейство F семантических моделей теории Ψ, определимое в A. (Понятия семейства моделей и его определимости вводятся естественным образом.)

Дается пример конечно-аксиоматизируемых теорий Φ, Ψ таких, что Ψ параметрически интерпретируема в Φ, но не является относительно интерпретируемой в Φ.