Časopis pro pěstování matematiky

Pavol Brunovský

A “bang-bang” principle in the problem of ε-stabilization of linear control systems

Časopis pro pěstování matematiky, Vol. 91 (1966), No. 3, 344--351

Persistent URL: `http://dml.cz/dmlcz/117571`

Časopis pro pěstování matematiky, roč. 91 (1966), Praha

A "BANG-BANG" PRINCIPLE IN THE PROBLEM OF ε-STABILIZATION OF LINEAR CONTROL SYSTEMS

Pavol Brunovský, Bratislava

(Received October 26, 1965)

In [1], the concept of ε-stabilizing control for two-dimensional linear control systems was introduced.

In the same way it may be introduced for systems of arbitrary dimension.

Let us have a linear control system

$$\dot{x} = Ax + Bu + \varepsilon p\,, \tag{1}$$

where x is an n-vector of state variables, u an m-vector of control, p an n-vector of perturbations, A, B — $n \times n$, and $n \times m$ constant matrices, respectively. Further, let there be given two convex compacts $P \subset E_n$, $Q \subset E_m$ (E_k being the k-dimensional Euclidean space).

By perturbation we shall denote a measurable function $p(t)$ on $\langle t_0, \infty)$, satisfying $p(t) \in P$ for a.e. $t \in \langle t_0, \infty)$. By control we shall denote a measurable function $u(x)$, defined on E_n and satisfying $u(x) \in Q$ a.e. in E_n.

Denote $\|x\|$ the Euclidean norm in E_n. Let $X \subset E_n$. Denote co X the convex hull of X, $\varrho(X, x) = \inf_{y \in X} \|y - x\|$, $S(X, \delta) = \{y \in E_n : \varrho(X, y) < \delta\}$, $f(X) = \{f(x) : x \in X\}$ for an arbitrary function f, defined on X.

Let $u(x)$ be a given control. $x(t)$ will be called a solution of (1) on an interval I, if it is absolutely continuous on I and satisfies a.e. on I the relation

$$\dot{x}(t) \in Ax(t) + BU(x(t)) + \varepsilon\, p(t)$$

where

$$U(x) = \bigcap_{\delta > 0} \bigcap_{\text{mes } N = 0} \overline{\text{co}\, u(S(x, \delta) - N)}$$

and $p(t)$ is an arbitrary perturbation, defined on I.

The reason for the generalization of the notion of solution is the fact, that as controls discontinuous functions of state variables are allowed (cf. [3], [4]). For continuous $u(x)$, the former definition is equivalent to the classical one.

In the following we shall apply the fact, that $x(t)$ is a solution of (1) if and only if it is a solution of the contingent equation

$$\dot{x} \in Ax + BU(x) + \varepsilon P \tag{2}$$

(cf. [1], [4]).

A control $u(x)$ will be called ε-stabilizing, if a compact region G containing the origin exists such that if $x(t)$ is a solution of (1) with $x(t_0) \in G$, then $x(t) \in G$ for $t \geqq t_0$; the region G will be called (u, ε)-invariant.

Clearly a product of two (u, ε)-invariant regions (with u fixed) is (u, ε)-invariant again. Hence, to every ε-stabilizing control u the smallest (u, ε)-invariant region $G(u)$ exists in the sense, that it is contained in every other (u, ε)-invariant region.

Therefore, we may estimate the quality of the ε-stabilizing controls according to their smallest (u, ε)-invariant regions.

Let $|x|$ be a given norm in E_n. Denote $|G| = \max_{x \in G} |x|$ for an arbitrary compact G.

Let u_1, u_2 be two ε-stabilizing controls. u_1 will be said better then u_2 (u_2 worse than u_1), if $|G(u_1)| < |G(u_2)|$.

For two-dimensional systems under sufficiently general assumptions for $\varepsilon > 0$ sufficiently small the best ε-stabilizing control has been proved to exist and constructed in [1].

In [2], the n-dimensional controllable systems are treated. It is shown, that for special P and $\varepsilon > 0$ sufficiently small a control $u(x)$ exists such that the origin itself is a (u, ε)-invariant region and, moreover, the system (1) is asymptotically stable under an arbitrary perturbation.

If Q contains the origin in its interior and (1) is controllable, i.e. if among the vectors $b_1, \ldots, A^{n-1}b_1, b_2, \ldots, A^{n-1}b_2, \ldots, b_m, \ldots, A^{n-1}b_m$ ($b_1, \ldots, b_m$ being the column vectors of B) are n linearly independent, then for $\varepsilon > 0$ sufficiently small an ε-stabilizing control exists. This may be demonstrated as follows:

From [5] it follows, that the unperturbed system

$$\dot{x} = Ax + Bu \tag{3}$$

may be done asymptotically stable by a linear function $u = Cx$ and, hence, there exists a positive definite quadratic form $V = \frac{1}{2}(Wx, x)$, W being symmetric, which is a Lyapunov function for (3), i.e. the form

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3)} = (Wx, (A + BC)\,x) = (W(A + BC)\,x, x)$$

is negative definite. Henceforth, it satisfies the inequality

$$(W(A + BC)\,x, x) \leqq q\|x\|^2, \quad q < 0.$$

Calculating $\mathrm{d}V/\mathrm{d}t$ according to the system (1) we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} = (W(A + BC)\,x, x) + (Wx, \varepsilon p) \leqq$$

$$\leqq q\|x\|^2 + \varepsilon\|W\| \,.\, \|P\|\,\|x\| = (q\|x\| + \varepsilon\|Q\|\,\|P\|)\,\|x\|\,.$$

From this it may be seen, that for $\varepsilon > 0$ an $\eta(\varepsilon) > 0$ exists such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} \leqq 0$ if $V(x) = \eta(\varepsilon)$. If $\varepsilon > 0$ is small enough, $2Cx \in Q$ if $V(x) = \eta(\varepsilon)$. Hence, defining a control $u(x)$ such that $u(x) = Cx$ in some neighbourhood of the surface $V(x) = \eta(\varepsilon)$ we obtain an ε-stabilizing control with a (u, ε)-invariant region $V(x) \leqq \eta(\varepsilon)$.

However, the question of the existence of a best ε-stabilizing control is in general open.

The main purpose of this paper is to prove a theorem, which enables us in the problem of choosing a best ε-stabilizing control to restrict ourselves on the so called "bang-bang" controls and which, in analogy to a theorem in the optimal control theory may be denoted as a "bang-bang" principle.

The "bang-bang" principle, according to [6] may be formulated as follows:

If an optimal control exists, then there exists an optimal control, which is bang-bang.

In [6], Q is a polyhedron and by "bang-bang" control there is meant a control which acquires as values only the vertices of Q.

The bang-bang controls for more general Q (and even more general control systems) are discussed in [7].

The ε-stabilization bang-bang principle will be given as a corollary of a theorem which we are going to prove.

According to [7] denote tend Q the least compact set the convex hull of which is Q.

Theorem. *Let u be an ε-stabilizing control with a (u, ε)-invariant region G. Then, there exists an ε-stabilizing control u_0, acquiring its values only from* tend Q *and such that $G_0 = \operatorname{co} G$ is a (u_0, ε)-invariant region.*

The proof of the theorem will be accomplished in several steps.

Let x be a boundary point of a closed convex set C. Denote M_x the set of all normals of the support planes of C at x, i.e. $M_x = \{\psi : (\psi, x) = \max_{y \in C} (\psi, y)\}$.

Lemma 1. *Let $C \subset E_n$ be a convex compact and let $x \in E_n$. Then*

1° *There exists a unique point $q(x) \in C$ such that $\|x - q(x)\| = \varrho(C, x)$;*

2° $(x - q(x), q(x)) = \sup_{y \in C} (x - q(x), y)$, *(in particular $x - q(x) \in M_{q(x)}$, if $x \bar{\in} C$);*

3° $\|q(x_1) - q(x_2)\| \leqq \|x_1 - x_2\|$ *for $x_1, x_2 \in E_n$.*

Proof. 1° For $x \in C$ we have clearly $q(x) = x$. If $x \bar{\in} C$, the existence of $q(x)$

follows from the compactness of C. If there were two distinct points $y_1 \in C$, $y_2 \in C$, satisfying $\|x - y_i\| = \varrho(C, x)$, $i = 1, 2$ then for the point $\frac{1}{2}(y_1 + y_2)$ we would have $\|x - \frac{1}{2}(y_1 + y_2)\| < \varrho(C, x)$. This is impossible, as $\frac{1}{2}(y_1 + y_2) \in C$.

2° If $x \in C$, 2° is trivial. In order to prove 2° for $x \bar{\in} C$, suppose the contrary, i.e. that a point $y_0 \in C$ exists such that

$$(3) \qquad (x - q(x), y_0) > (x - q(x), q(x))\,.$$

Denote $y(\alpha) = \alpha y_0 + (1 - \alpha)\, q(x)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \|y(\alpha) - x\|^2 = 2(y_0 - q(x),\ y(\alpha) - x)\,,$$

$y(0) = q(x)$, $y(\alpha) \in C$ for $\alpha \in \langle 0, 1\rangle$. Due to (3) we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \|y(\alpha) - x\|^2|_{\alpha=0} = 2(y_0 - q(x)\,,\quad q(x) - x) < 0$$

from which it follows, that for $\alpha > 0$ sufficiently small $\|y(\alpha) - x\| < \|q(x) - x\| = \varrho(C, x)$. This is impossible, as $y(x) \in C$ for $\alpha \in \langle 0, 1\rangle$.

3° From 2° it follows $(x_1 - q(x_1), q(x_1) - q(x_2)) \geqq 0$, $(q(x_2) - x_2, q(x_1) - q(x_2)) \geqq 0$. Adding these two inequalities, we obtain $(x_1 - q(x_1) + q(x_2) - x_2, q(x_1) - q(x_2)) \geqq 0$, i.e. $(x_1 - x_2, q(x_1) - q(x_2)) \geqq \|q(x_1) - q(x_2)\|^2$. From the last inequality it follows $\|x_1 - x_2\| \geqq \|q(x_1) - q(x_2)\|$, q.e.d.

Lemma 2. *Let x be a boundary point of G. Then, for every $\psi \in M_x$ an $u_\psi \in$ tend Q exists such that*

$$(4) \qquad (\psi, Ax + Bu_\psi + \varepsilon p) \leqq 0$$

for an arbitrary $p \in P$.

Proof. First suppose that the theorem fails to hold for a boundary point of G, say x_0. Let $\psi \in M_{x_0}$. Then, for every $u \in$ tend Q we have

$$(5) \qquad (\psi, Ax_0 + Bu + \varepsilon p_\psi) > 0\,,$$

where p_ψ is such that $(\psi, p_\psi) = \max_{p \in P} (\psi, p)$. Now, let $u \in Q$. Then, we may choose $u_i \in$ tend Q, $\lambda_i \geqq 0$, $i = 1, 2, \ldots, m + 1$, such, that $\sum_{i=1}^{m+1} \lambda_i = 1$ and $u = \sum_{i=1}^{m+1} \lambda_i u_i$ (cf. [8]). Hence

$$(\psi, Ax_0 + Bu + \varepsilon p_\psi) = \sum_{i=1}^{m+1} \lambda_i (\psi, Ax_0 + Bu_i + \varepsilon p_\psi) > 0\,,$$

i.e. (5) is valid for every $u \in Q$.

Due to the contingent equation existence theorem ([4], [9]) a solution $x(t)$ of the contingent equation

$$\dot{x} \in Ax + BU(x) + \varepsilon p_{\psi} \tag{6}$$

with $x(t_0) = x_0$ exists. This solution satisfies the relation $\text{cont}\, x(t_0) \subset Ax_0 + BU(x_0) + \varepsilon p_{\psi} \subset Ax_0 + BQ + \varepsilon p_{\psi}$. From this and (5) we obtain $(\psi, z) > 0$ for every $z \in \text{cont}\, x(t_0)$. This is possible only if $x(t)$ leaves G. But $x(t)$ being a solution of (6) is also a solution of (2) and, hence, of (1). Thus, according to the assumption, it cannot leave G. This contradiction proves the validity of the theorem for $x \in G$.

Now, let x_0 be an arbitrary boundary point of G_0. Let $\psi \in M_{x_0}$. Then, we may choose $x_i \in G$, $\lambda_i > 0$, $i = 1, 2, \ldots, r$, $r \leqq n + 1$ such that $x_0 = \sum_{i=1}^{r} \lambda_i x_i$, $\sum_{i=1}^{r} \lambda_i = 1$. It is easy to show that x_i should be boundary points of G and $\psi \in M_{x_i}$, $i = 1, 2, \ldots, r$. Hence, r points $u_i \in Q$ exist such that $(\psi, Ax_i + Bu_i + \varepsilon p) \leqq 0$ for $p \in P$, $i = 1, 2, \ldots, r$. Adding these inequalities we obtain $(\psi, Ax_0 + B \sum_{i=1}^{r} \lambda_i u_i + \varepsilon p) \leqq 0$ for $p \in P$. Due to the convexity of Q, $\sum_{i=1}^{r} \lambda_i u_i \in Q$. Applying the same argument as in the first part of the proof, we conclude from this the existence of the desired $u \in \text{tend}\, U$.

Lemma 3. *Let $u(x)$ be a given control and let $x(t)$ be a solution of* (1) *on I. Let C be a given convex compact. Then, $r(t) = q(x(t))$ is absolutely continuous on I and $(x(t) - r(t), \dot{r}(t)) = 0$ for a.e. $t \in I$.*

Proof. The absolute continuity of $r(t)$ follows from the absolute continuity of $x(t)$ and lemma 1, 3°. As $r(t)$ is absolutely continuous, it has a derivate a.e. an I. Let the derivative $\dot{r}(t)$ at t exist. Suppose $(x(t) - r(t), \dot{r}(t)) \neq 0$. If

$$(x(t) - r(t), \dot{r}(t)) > 0\,, \tag{7}$$

then we have $(x(t) - r(t), h^{-1}(r(t + h) - r(t))) > 0$ for $|h|$ sufficiently small. For $h > 0$ we have $(x(t) - r(t), r(t + h)) > (x(t) - r(t)), r(t))$. This contradicts lemma 1, 2°. If, instead of (7), the opposite inequality holds, we obtain a contradiction with lemma 1, 2° for $h < 0$.

Denote $V(x) = \{u \in \text{tend}\, Q : (x - q(x), Aq(x) + Bu + \varepsilon p \leqq 0\}$ for $x \in E_n$.

Lemma 4. *$V(x)$ is non-empty and compact for $x \in E_n$. $V(x)$ is an upper semicontinuous in the sense of inclusion set-valued function on E_n* (cf. [1], [4], [7]).

Proof. The compactness of $V(x)$ is evident. From Lemma 2 it follows that $V(x)$ is non-empty. Let $x_n \to x$, $u_n \in V(x_n)$, $u_n \to u$. We have $u \in \text{tend}\, Q$, $(x - q(x), Aq(x) + Bu + \varepsilon p)) = \lim_{n \to \infty} (x_n - q(x_n), Aq(x_n) + Bu_n + \varepsilon p)$. Hence, $(x - q(x), Aq(x) + Bu + \varepsilon p) \leqq 0$, i.e. $u \in V(x)$. This proves the upper semicontinuity of $V(x)$ (cf. [1]).

Proof of the theorem. According to [10][1]), from Lemma 4 it follows the existence of a measurable function $u_0(x)$ such that $u_0(x) \in V(x)$ for $x \in E_n$. We shall prove that $u_0(x)$ is the sought ε-stabilizing control.

Denote $U_0(x) = \bigcap_{\delta > 0} \bigcap_{\text{mes} N = 0} \text{co}\, \overline{u(S(x, \delta) - N)}$. For every $x \in E_n$, $v \in U_0(x)$ and $p \in P$

(8) $$(x - q(x), Aq(x) + Bv + \varepsilon p) \leqq 0$$

is valid.

In order to prove this suppose the contrary. Then, sequences $\{x_n\} \to x$ and $\{p_n\}$ exist such that

(9) $$(x_n - q(x_n), Aq(x_n) + Bu_0(x_n) + \varepsilon p_n) > \eta > 0\,.$$

The sequences $\{x_n\}$, $\{u_0(x_n)\}$, $\{p_n\}$ are bounded, therefore we may choose a subsequence x_{n_k} such that $u_0(x_{n_k}) \to u^*$, $p_n \to p^* \in P$. From (9) it follows $(x - q(x), Aq(x) + Bu^* + \varepsilon p^*) \geqq \eta > 0$. This is impossible, as from the upper semicontinuity of $V(x)$ it follows $u^* \in V(x)$.

Now, suppose that a solution of (1) leaves G. Then, a boundary point x_0 of G_0 exists such that $x(t_0) = x_0$ and $x(t) \in E_n - G_0$ for $t \in (t_0, t_1\rangle$. Let $r(t) = q(x(t))$. For a.e. $t \in (t_0, t_1)$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - r(t)\|^2 = (x(t) - r(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) - r(t)\,) \in (x(t) - r(t)\,,$$

$$Ax(t) + BU_0(x(t)) + \varepsilon P - \dot{r}(t)) = (x(t) - r(t)\,,$$

$$Ar(t) + BU_0(x(t)) + \varepsilon P) + (x(t) - r(t)\,, \quad A(x(t) - r(t)) - (x(t) - r(t), \dot{r}(t))\,.$$

According to (8) we have $(x(t) - r(t), Ar(t) + Bu + \varepsilon p) \leqq 0$ for every $u \in U_0(x(t))$, $p \in P$. Due to this and lemma 3 we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t) - r(t)\|^2 \leqq (x(t) - r(t)\,, \quad A(x(t) - r(t)) \leqq \|A\|\,\|x(t) - r(t)\|^2\,.$$

Hence (cf. [11], Theorem 2.1 of chap. I),

$$\|x(t_1) - r(t_1)\| \leqq \|x(t_0) - r(t_0)\| \quad \exp \quad \{2\|A\|\,(t_1 - t_0)\}\,,$$

i.e. $x(t_1) - r(t_1) = 0$, which contradicts the assumption. This completes the proof.

Remark. The requirements, desired by the theorem are satisfied by every control, which is equal to $u_0(x)$ in a domain

$$H_\eta = \{x : x \in E_n - G_0, \varrho(G_0, x) < \eta\}\,,$$

$\eta > 0$ being arbitrarily small.

---

[1]) In fact, the existence of such a measurable function is proved in [10] for one-dimensional x. However, the proof may be transferred without complications to functions of x of an arbitrary finite dimension.

Corollary. *If Q is a polyhedron, then* tend *Q ist he set of the vertices of Q. From the theorem the bang-bang principle follows:*

For every ε-stabilizing control there exists a bang-bang ε-stabilizing control which is not worse. In particular, if a best ε-stabilizing control exists, then there is a best ε-stabilizing control, which is bang-bang.

References

[1] *Brunovský P.:* On the best stabilizing control under a given class of perturbations. Czech. math. journal, *15* (1965), 329—369.

[2] *Бруновски П.:* О стабилизации линейных систем при определенном классе постоянно действующих возмущений. Дифференциальные уравнения. (to appear)

[3] *Филиппов А. Ф.:* Дифференциальные уравнения с разрывной правой частью. Математический сборник *51* (1960), 99—128.

[4] *Барбашин Е. А., Алимов Ю. И.:* К теории релейных дифференциальных уравнений. Известия ВУЗ, Математика, 1962, 3—13.

[5] *Курцвейль Я.:* К аналитическому конструированию регуляторов. Автоматика и телемеханика *22* (1961), 836—844.

[6] *LaSalle J. P.:* The time optimal control problem. Contributions to the theory of nonlinear oscillations V, Princeton 1960, 1—25.

[7] *Ważewski T.:* On an optimal control problem. Differential equations and their applications, Prague 1963, 229—242.

[8] *Ważewski T.:* Sur la semicontinuité inferieur du „tendeur" d'un ensemble compact, variant d'une facon continue. Bull. Acad. Polon. Sci., Sér. Sci. Math. Phys. Astr. *9* (1961), 869—872.

[9] *Ważewski T.:* Systemes de commande et équations au contingent. Ibid. *9* (1961), 151—155.

[10] *Ważewski T.:* Sur une condition d'existence des fonctions implicites mesurables. Ibid. *9* (1961), 861—863.

[11] *Coddington E. A., Levinson N.:* Theory of ordinary differential equations. New York, 1955.

Authors address: Bratislava-Patrónka, Dúbravská cesta. (Ústav technickej kybernetiky SAV).

Výťah

PRINCÍP „BANG-BANG" V PROBLÉME ε-STABILIZÁCIE LINEÁRNYCH SYSTÉMOV RIADENIA

PAVOL BRUNOVSKÝ, Bratislava

V súhlase s [1] sa zavádza pojem ε-stabilizujúceho riadenia a (u, ε)-invariantnej oblasti pre sústavy riadenia ľubovoľnej konečnej dimenzie. Riadenie u_1 sa nazýva lepším ako riadenie u_2, ak minimálna v smysle inklúzie (u_1, ε)-invariantná oblasť je v istom smysle menšia ako minimálna v smysle inklúzie (u_2, ε)-invariantná oblasť. Dokazuje sa veta, ktorej dôsledkom je bang-bang princíp:

K ľubovoľnému ε-stabilizujúcemu riadeniu u existuje ε-stabilizujúce riadenie typu bang-bang, ktoré nie je horšie ako u. Špeciálne, ak existuje najlepšie ε-stabilizujúce riadenie, potom existuje najlepšie ε-stabilizujúce riadenie typu bang-bang.

Резюме

ПРИНЦИП РЕЛЕЙНОСТИ УПРАВЛЕНИЯ ДЛЯ ПРОБЛЕМЫ ε-СТАБИЛИЗАЦИИ ЛИНЕЙНЫХ СИСТЕМ УПРАВЛЕНИЯ

ПАВЕЛ БРУНОВСКИ (Pavol Brunovský), Братислава

В соответствии с [1] вводится понятие ε-стабилизирующего управления и (u, ε)-инвариантной области для систем управления произвольной конечной размерности. Управление u_1 называется лучшим по сравнению с управлением u_2, если минимальная по включению (u_1, ε)-инвариантная область в определенном смысле меньше минимальной (u_2, ε)-инвариантной области. Доказывается теорема, следствием которой является принцип релейности управления:

Для всякого ε-стабилизирующего управления u существует релейное ε-стабилизирующее управление, которое не хуже u. В частности, если существует наилучшее ε-стабилизируещее управление, то существует наилучшее ε-стабилизирующее управление, являющееся релейным.