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PARTITIONS IN CARTESIAN SYSTEMS

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In the opening part of [2], O. BORŮVKA described his theory of set partitions which he enriched in the sequel of [2] by a study of one binary operation in a given set.

Analogously, it is possible to apply this theory of set pertitions to the case of a set with one ν-ary operation (ν any ordinal) or, more generally, to the case of a map of a cardinal product of a family of sets onto a given set. This last topic forms the object of study in the present paper.

1. Chainings and bindings. Let S be a fixed non-void set and $\mathfrak{S}(S)$ the semilattice of all partitions in S with the usual ordering. If $\mathscr{P} = (\mathscr{P}^\iota)_{\iota \in I}$ is a family of partitions in S then we define a *chaining* in $\mathscr{P}$ between two $\mathscr{P}$-blocks A, B as any finite sequence of $\mathscr{P}$-blocks $A = A_0 \between A_1 \between \ldots \between A_{n-1} \between A_n = B$. If n is even and each member with even index is a $\mathscr{P}^\tau$-block for a fixed $\tau \in I$, then the sequence $A = A_0, A_2, \ldots, A_n = B$ will be called a *binding* of $\mathscr{P}^\tau$-blocks between A, B with cementing $\mathscr{P}$-blocks $A_1, A_3, \ldots, A_{n-1}$. We shall also say that A, B are *chained* or *bound*, respectively.

We begin with two elementary lemmas.

Lemma 1. *Let $\mathscr{P} = (\mathscr{P}^1, \mathscr{P}^2)$ be a pair of partitions in S. Then every chaining in $\mathscr{P}$ between two $\mathscr{P}^1$-blocks A, B becomes a binding of $\mathscr{P}^1$-blocks between A, B if the $\mathscr{P}^2$-blocks are omitted.*

Lemma 2. *Let $\mathscr{P} = (\mathscr{P}^\iota)_{\iota \in I}$ be a family of partitions on S. Then to each chaining between $A, B \in \mathscr{P}^\tau$ (for a fixed $\tau \in I$) there exists a binding of $\mathscr{P}^\tau$-blocks between A, B with cementing blocks belonging to the initial chaining.*

The proof of lemma 1 is clear. For the proof of lemma 2 it suffices to insert a $\mathscr{P}^\tau$-block $B_l \between A_l \cap A_{l+1}$ between all consecutive $A_l \between A_{l+1}$ of a given chaining. Such a $B_l \in \mathscr{P}^\tau$ must exist because now the partitions are *on* S. In such an enlarged chaining between A, B omit all $\mathscr{P}$-blocks not in $\mathscr{P}^\tau$ to obtain the required binding between A, B.

[1]) Cf. [9] for the notions used. The set of $\mathscr{P}$-blocks is $\{P | P \in \mathscr{P}^\iota, \iota \in I\}$. By $\between$ we denote the non-empty intersecting of two sets.

Let $\mathscr{P} = (\mathscr{P}^\iota)_{\iota \in I}$ be a family of partitions in S. Then the partition $\sup \mathscr{P} \in \mathfrak{S}(S)$ has the following characteristic property [3, pp. 16–17]: Each $\sup \mathscr{P}$-block is a union of a maximal set of $\mathscr{P}$-blocks chained in $\mathscr{P}$. The partition $\inf \mathscr{P} \in \mathfrak{S}(S)$ exists iff for each $\iota \in I$ there exist $A_\iota \in \mathscr{P}^\iota$ such that $\bigcap_{\iota \in J} A_\iota \neq \emptyset$. If $\inf \mathscr{P}$ exists, then every $\inf \mathscr{P}$-block has the form $\bigcap_{\iota \in J} B_\iota \neq \emptyset$ with $B_\iota \in \mathscr{P}^\iota$, $\iota \in I$.

2. Cartesian systems. Let Γ be a fixed index set. Put $\Gamma_0 = \Gamma \cup \{o\}$ where $o \notin \Gamma$. [2]) Let $(S_\alpha)_{\alpha_0}$ be a family of non-void sets and $f : \prod_\alpha S_\alpha \to S_0$ a surjection.[3]) Then $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$ will be called a *Cartesian system* or briefly a *system* (cf. [12], pp. 38–39).

If $\emptyset \neq S'_{\alpha_0} \subseteq S_{\alpha_0}$ for all α_0 and if f' is a restriction of f with domain $\prod S'_\alpha$, where $S'_0 = f'(\prod_\alpha S'_\alpha)$, then $\mathbf{C}' = ((S'_{\alpha_0})_{\alpha_0}, f')$ will be called a *subsystem* of $\mathbf{C}$.

A *map* σ between two systems $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f_0)$, $\mathbf{C}^* = ((S^*_{\alpha_0})_{\alpha_0}, f^*)$ is a family $(\sigma_{\alpha_0})_{\alpha_0}$ of maps $\sigma_{\alpha_0} : S_{\alpha_0} \to S^*_{\alpha_0}$ for all α_0; σ will be called *regular* if $\sigma_{\alpha_0} a = \sigma_{\beta_0} a$ for all $a \in S_{\alpha_0} \cap S_{\beta_0}$; σ will be called a *homomorphism* if $\sigma_0 f((a_\alpha)_\alpha) = f^*((\sigma_\alpha a_\alpha)_\alpha)$ for every choice $a_\alpha \in S_\alpha$ for all α.

A *partition* $\mathscr{P}$ in a system $\mathbf{C}$ is defined as a family $(\mathscr{P}_{\alpha_0})_{\alpha_0}$ where $\mathscr{P}_{\alpha_0}$ is a partition in S_{α_0} for all α_0. If, moreover, $\mathscr{P}_{\alpha_0}$ is a partition on S_{α_0} for all α_0, then we speak about a partition *on* $\mathbf{C}$.

Let $\sigma = (\sigma_{\alpha_0})_{\alpha_0}$ be an epimorphism between the systems $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$, $\mathbf{C}^* = (S^*_{\alpha_0})_{\alpha_0}, f^*)$. We say that the partition $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ on $\mathbf{C}$ is *induced* by σ if for each α_0 the $\mathscr{P}_{\alpha_0}$-blocks are $\sigma_{\alpha_0}^{-1} a$ for all $a \in S^*_{\alpha_0}$.

A partition $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ in a system $\mathbf{C}$ is said to be *generating* if, for each choice $A_\alpha \in \mathscr{P}_\alpha$ for all α, there exists a $\mathscr{P}_0$-block A_0 containing $f(\prod_\alpha A_\alpha)$.

If $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ is a generating partition in a system $\mathbf{C}$, then we define a subsystem $\mathbf{C}' = ((S'_{\alpha_0})_{\alpha_0}, f')$ in $\mathbf{C}$ *corresponding* to $\mathscr{P}$ as a system such that, for every α_0, S'_{α_0} is is the union of all $\mathscr{P}_{\alpha_0}$-blocks, and that f' is the portion of f with domain $\prod_\alpha S'_\alpha$.

The results for regular partitions in a Cartesian system may be specialized to the most customary case of any $\mathbf{C}$ with all S_α equal to a fixed set S and $S_0 \subseteq S$.

3. Generating partitions in Cartesian systems. We shall denote by $\mathscr{P} = (\mathscr{P}^\iota)_{\iota \in I}$ an arbitrary family of partitions in a given system $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$, and put $\mathscr{P}^\iota = (\mathscr{P}^\iota_{\alpha_0})_{\alpha_0}$ for all ι and $\mathscr{P}_{\alpha_0} = (\mathscr{P}^\iota_{\alpha_0})_\iota$ for all α_0. [4])

The set $\mathfrak{S}(\mathbf{C})$ of all partitions in $\mathbf{C}$ will be ordered $\mathbf{C}$ as follows: For $\mathscr{P}^1 = (\mathscr{P}^1_{\alpha_0})_{\alpha_0}$, $\mathscr{P}^2 = (\mathscr{P}^2_{\alpha_0})_{\alpha_0}$ in $\mathfrak{S}(\mathbf{C})$ set $\mathscr{P}^1 \leqq \mathscr{P}^2$ iff $\mathscr{P}^1_{\alpha_0} \leqq \mathscr{P}^2_{\alpha_0}$ in $\mathfrak{S}(S_{\alpha_0})$ for all α_0. Then $\mathfrak{S}(\mathbf{C})$ becomes a complete semilattice: For each family $\mathscr{P}$ of partitions in $\mathbf{C}$ there is a parti-

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[2]) In the following text, $\alpha, \beta, \gamma, \ldots$ vary over Γ, while $\alpha_0, \beta_0, \gamma_0, \ldots$ vary over Γ_0.

[3]) $\prod$ denotes the cardinal product in the sense of [6, p. 15].

[4]) ι varies over the same index set I.

tion $\sup \mathscr{P} = (\sup \mathscr{P}_{\alpha_0})_{\alpha_0} \in \mathfrak{S}(\mathbf{C})$; on the other hand, the partition $\inf \mathscr{P}$ need not exist. The existence of the partition $\inf \mathscr{P}$ is equivalent to the existence of $\inf \mathscr{P}_{\alpha_0}$ for all α_0; then $\inf \mathscr{P} = (\inf \mathscr{P}_{\alpha_0})_{\alpha_0} \in \mathfrak{S}(\mathbf{C})$.

Theorem 1. *Let σ be an epimorphism between systems $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$, $\mathbf{C}^* = ((S^*_{\alpha_0})_{\alpha_0}, f^*)$. Then the partition $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ in $\mathbf{C}$, induced by σ, is necessarily generating.*

Proof. Let $A_\alpha \in \mathscr{P}_\alpha$ for all α. Then for each α there is an element $a^*_\alpha \in S^*_\alpha$ such that $A_\alpha = \sigma_\alpha^{-1} a^*_\alpha$. Each element $b \in f(\prod_\alpha A_\alpha)$ has the form $f((a_\alpha)_\alpha)$ for some $a_\alpha \in A_\alpha$. Thus $\sigma_0 b = \sigma_0 f((a_\alpha)_\alpha) = f^*((\sigma_\alpha a_\alpha)_\alpha) = f^*((a^*_\alpha)_\alpha)$, and $b \in S_0$ is contained in $\sigma_0^{-1} f^*((a^*_\alpha)_\alpha) = B$. This yields $f(\prod_\alpha A_\alpha) \subseteq B$.

Theorem 2. *Let $\mathbf{C}^* = ((S^*_{\alpha_0})_{\alpha_0}, f^*)$ be a subsystem in a given system $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$, and $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ a generating partition in $\mathbf{C}$ with corresponding subsystem $\mathbf{C}' = ((S'_{\alpha_0})_{\alpha_0}, f')$ such that $S^*_{\alpha_0} \between S'_{\alpha_0}$ for all α_0. If one puts $\overline{\mathscr{P}}_{\alpha_0} = \mathscr{P}_{\alpha_0}] S^*_{\alpha_0}$ for all α_0 then $\overline{\mathscr{P}} = (\overline{\mathscr{P}}_{\alpha_0})_{\alpha_0}$ is a generating partition in $\mathbf{C}$.* [5])

Proof. Let $A_\alpha \in \mathscr{P}_\alpha$, $A_\alpha \between S^*_\alpha$ for all α. The partition is generating, so that a $\mathscr{P}_0$-block $A_0 \supseteq f(\prod_\alpha A_\alpha)$ exists. If $a_\alpha \in S^*_\alpha \cap A_\alpha$ for all α, then $f((a_\alpha)_\alpha) \in f(\prod_\alpha S^*_\alpha) \cap f(\prod_\alpha A_\alpha \subseteq$ $\subseteq S^*_0 \cap A_0$ because $\mathbf{C}^*$ is a subsystem of $\mathbf{C}$. Thus $S^*_0 \between A_0$, and consequently $\overline{\mathscr{P}}$ must be generating.

Theorem 3. *Let $\mathscr{P} = (\mathscr{P}^\iota)_\iota$ be a family of generating partitions in $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$ and $\mathbf{C}^\iota = ((S^\iota_{\alpha_0})_{\alpha_0})_{\alpha_0}, f^\iota)$ the corresponding subsystem with regard to $\mathscr{P}^\iota$ (for all ι). Then $\bigcap_\iota S^\iota_{\alpha_0} \neq \emptyset$ for all α_0 implies the existence of the partition* in $\mathscr{P} \in \mathfrak{S}(\mathbf{C})$, *and this partition is generating.*

Proof. The assumption $\bigcap_\iota S^\iota_{\alpha_0} \neq \emptyset$ for all α_0 implies the existence of $\inf \mathscr{P} \in \mathfrak{S}(\mathbf{C})$. Let $A_{\alpha_0} \in \inf \mathscr{P}_{\alpha_0}$ for all α_0. Then for all α_0, ι there exist $A^\iota_{\alpha_0} \in \mathscr{P}^\iota_{\alpha_0}$ such that $A_{\alpha_0} = \bigcap_\iota A^\iota_{\alpha_0}$. As $\mathscr{P}^\iota$ is generating, there is a $\mathscr{P}^\iota_0$-block $A^\iota_0 \supseteq f(\prod_\alpha A^\iota_\alpha)$ for each ι. Therefore $f(\prod_\alpha A_\alpha) \subseteq \bigcap_\iota f(\prod_\alpha A^\iota_\alpha) \subseteq \bigcap_\iota A^\iota_0 \in \inf \mathscr{P}_0$, so that the partition $\mathscr{P}$ is generating.

Theorem 4. *Let $\mathscr{P} = (\mathscr{P}^\iota)_\iota$ be a family of generating partitions in a given system $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$ with $\Gamma = \{1, \ldots, n\}$. Then* $\sup \mathscr{P}$ *is generating.*

Proof.[6]) Choose $x_\alpha, y_\alpha \in S_\alpha$ in the same $\sup \mathscr{P}_\alpha$-block for all α. The existence of

[5]) The symbol $\mathscr{A} [B$ is used to denote a *packing* in the sense of Borůvka, i.e. for a partition consisting of those blocks of a given partition $\mathscr{A}$ which intersect a given set B. Cf. [2, p. 23].

[6]) Cf. [11, pp. 190–191].

a chaining in $\mathscr{P}_\alpha$ between two $\mathscr{P}_\alpha$-blocks, of which the first contains x_α and the second y_α, may be expressed as the existence of a sequence

$$(*) \qquad x_\alpha = z_{\alpha,0}, z_{\alpha,1}, \ldots, z_{\alpha,r_\alpha} = y_\alpha$$

of elements in S_α. The elements $z_{\alpha,k-1}, z_{\alpha,k}$ must be contained in the same $P^{\alpha,k}$-block for some $\mathscr{P}^{\alpha,k} \in \mathscr{P}_\alpha$ (for all $k = 1, \ldots, r_\alpha$ and all α). From this one deduces, in turn that there exist $\mathscr{P}_0$-blocks such that

$f(z_{10}, z_{20}, \ldots, z_{n0}), f(z_{11}, z_{20}, \ldots, z_{n0})$ belong to the same $\mathscr{P}_0^{1,1}$-block, $\mathscr{P}_0^{1,1}$ from $\mathscr{P}_0$,
$f(z_{11}, z_{20}, \ldots, z_{n0}), f(z_{12}, z_{20}, \ldots, z_{n0})$ belong to the same $\mathscr{P}_0^{1,2}$-block, $\mathscr{P}_0^{1,2}$ from $\mathscr{P}_0$,
$\vdots$
$f(z_{1,r_1-1}, z_{20}, \ldots, z_{n0}), f(z_{1r_1}, z_{20}, \ldots, z_{n0})$ belong to the same $\mathscr{P}_0^{1,r_1}$-block, $\mathscr{P}_0^{1,r_1}$ from $\mathscr{P}_0$.

These and analogous relations for further sequences (*) ($\alpha = 1, \ldots, n$) yield that

$f(z_{10}, z_{20}, \ldots, z_{n0}), f(z_{1r_1}, z_{20}, \ldots, z_{n0})$ belong to the same $\sup\limits_{k=1,\ldots,r_1} \mathscr{P}_0^{1,k}$-block
$f(z_{1r_1}, z_{20}, \ldots, z_{n0}), f(z_{1r_1}, z_{2r_2}, \ldots, z_{n0})$ belong to the same $\sup\limits_{k=1,\ldots,r_2} \mathscr{P}_0^{2,k}$-block
$\vdots$
$f(z_{1r_1}, z_{2r_2}, \ldots, z_{n-1,r_{n-1}}, z_{n0}), f(z_{1r_1}, z_{2r_2}, \ldots, z_{n-1,r_{n-1}}, z_{nr_n})$ belong to the same $\sup\limits_{k=1,\ldots,r_n} \mathscr{P}_0^{n,k}$-block

Thus, finally, $f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)$ both belong to the same block of the partition $\sup\limits_{\alpha=1,\ldots,n} (\sup\limits_{k=1,\ldots,r_\alpha} \mathscr{P}_0^{n,k}) \leqq \sup \mathscr{P}_0$, as it was required to prove.

Remark. I do not know under what further conditions theorem 4 holds also for infinite index set Γ.

Now we shall investigate the possibly less familiar notion of the *Goldie composition* $\diamond$ of two partitions. Let $\mathscr{A}, \mathscr{B} \in \mathfrak{S}(S)$. Then $\mathscr{A} \diamond \mathscr{B}$ is a partition from $\mathfrak{S}(S)$ defined as follows: The elements $a, a' \in S$ belong to the same $\mathscr{A} \diamond \mathscr{B}$-block iff there exists a finite sequence $a = a_0, a_1, \ldots, a_r, a_{r+1} = a'$ of elements in S such that $a_0, a_1; a_2, a_3; \ldots; a_r, a_{r+1}$ belong to common $\mathscr{B}$-blocks, and $a_1, a_2; a_3, a_4; \ldots; \ldots; a_{r-1}, a_r$ belong to common $\mathscr{A}$-blocks. Another formulation is that the $\mathscr{A} \diamond \mathscr{B}$-blocks are the maximal unions of mutually bound $\mathscr{B}$-blocks with cementing $\mathscr{A}$-blocks (cf. § 1).

Now return to a system $\mathbf{C} = ((S_{\alpha o})_{\alpha o}, f)$, and for partitions $\mathscr{P}^i = (\mathscr{P}^i_{\alpha o})_{\alpha o}$; $i = 1, 2$ in $\mathbf{C}$ define the composition $\diamond$ by $\mathscr{P}^2 \diamond \mathscr{P}^1 = (\mathscr{P}^2_{\alpha o} \diamond \mathscr{P}^1_{\alpha o})_{\alpha o}$.

Theorem 5. *Let $\mathscr{P}^1, \mathscr{P}^2$ be generating partitions in a system $\mathbf{C} = ((S_{\alpha o})_{\alpha o}, f)$ with $\Gamma = \{1, \ldots, n\}$. Then $\mathscr{P} = \mathscr{P}^2 \diamond \mathscr{P}^1$ is also generating.*

Proof.[7]) For each α choose two elements a_α, a'_α in the same $\mathscr{P}_\alpha$-block. Then for each α there is a finite sequence $\alpha_\alpha = a_{\alpha,0}, a_{\alpha,1}, \ldots, a_{\alpha,r}, a_{\alpha,r+1} = a'_\alpha$ of elements in S_α such that consecutive members belong to common $\mathscr{P}^1$-blocks or $\mathscr{P}^2$-blocks. As Γ is finite, it may be supposed without the loss of generality that all considered sequences have the same length not depending on α. Therefore $f(a_{10}, \ldots, a_{n0})$, $f(a_{11}, \ldots, a_{n1})$ are in the same $\mathscr{P}^1_0$-block, $f(a_{11}, \ldots, a_{n1})$, $f(a_{12}, \ldots, a_{n2})$ are in the same $\mathscr{P}^2_0$-block, ..., $f(a_{1r}, \ldots, a_{nr})$, $f(a_{1,r+1}, \ldots, a_{n,r+1})$ are in the same $\mathscr{P}^1_0$-block. By definition of $\diamondsuit$, $f(a_1, \ldots, a_n)$, $f(a'_1, \ldots, a'_n)$ must lie in the same $\mathscr{P}_0$-block, as required.

Remark. I do not know the modifications of Theorem 5 necessary to make it apply to the case of an infinite index set Γ.

4. Factor systems. Let $\mathscr{P} = (\mathscr{P}_{\alpha 0})_{\alpha 0}$ be a generating partition on a given system $\mathbf{C} = ((S_{\alpha 0})_{\alpha 0}, f)$. A factor system $\mathbf{C}/\mathscr{P}$ is defined as a system $((\mathscr{P}_{\alpha 0})_{\alpha 0}, f/\mathscr{P})$ where $f/\mathscr{P}$ is a surjection of $\prod_\alpha \mathscr{P}_\alpha$ onto $\mathscr{P}_0$, determined by $f/\mathscr{P}((A_\alpha)_\alpha) = A_0$ where $A_\alpha \in \mathscr{P}_\alpha$ for all α and A_0 is a $\mathscr{P}_0$-block which contains $f(\prod_\alpha A_\alpha)$.

The concepts of a cover, refinement, cut, pairing, etc. (in the sense of Borůvka, [2], § 15.2-4) may be extended to Cartesian systems if they are simultaneously imposed on all $S_{\alpha 0}$.

Theorem 6. *Let $\mathscr{P} = (\mathscr{P}_{\alpha 0})_{\alpha 0}$ be a generating partition on a system $\mathbf{C} = ((S_{\alpha 0})_{\alpha 0}, f)$ with $\mathbf{C}/\mathscr{P} = \mathbf{C}' = ((S'_{\alpha 0})_{\alpha 0}, f')$. Let $\mathscr{P}' = (\mathscr{P}'_{\alpha 0})_{\alpha 0}$ be a partition on $\mathbf{C}'$ and $\mathscr{P}^* = (\mathscr{P}^*_{\alpha 0})_{\alpha 0}$ the cover of $\mathscr{P}$ enforced by $\mathscr{P}'$. Then $\mathscr{P}'$ is generating iff $\mathscr{P}^*$ is generating.*[8])

Proof. Let $\mathscr{P}'$ be generating. Choose $A^*_\alpha \in \mathscr{P}^*_\alpha$ for each α, and show that there exists a $\mathscr{P}^*$-block $A^*_0 \supseteq f(\prod_\alpha A^*_\alpha)$. Each $A^*_{\alpha 0}$ consists of all $\mathscr{P}_{\alpha 0}$-blocks contained in some $\mathscr{P}'_{\alpha 0}$-block $A''_{\alpha 0}$ (for each α_0). As $\mathscr{P}'$ is generating, for $A''_\alpha \in \mathscr{P}'_\alpha$ there must exist a $\mathscr{P}'_0$-block A''_0 which contains $f'(\prod_\alpha A''_\alpha)$. If A^*_0 consists of all $\mathscr{P}'_0$-blocks contained in A''_0, then $f'\prod_\alpha A''_\alpha) \subseteq A''_0$ implies $f\prod_\alpha A^*_\alpha) \subseteq A^*_0$. Conversely, let $\mathscr{P}^*$ be generating. If $A''_\alpha \in \mathscr{P}'_\alpha$ for all α, it is necessary to find a $\mathscr{P}'_0$-block $A''_0 \supseteq f'(\prod_\alpha A''_\alpha)$. Because $\mathscr{P}^*$ is generating, there exists a $\mathscr{P}^*_0$-block $A^*_0 \supseteq f(\prod_\alpha A^*_\alpha)$ where again A^*_α is the union of all $\mathscr{P}_\alpha$-blocks contained in A''_α (for each α). From $f(\prod_\alpha A''_\alpha) \subseteq A^*_0$ it follows again $f'(\prod_\alpha A''_\alpha) \subseteq A''_0$.

Theorem 7. *Between the systems $\mathbf{C} = ((S_{\alpha 0})_{\alpha 0}, f)$, $\mathbf{C}^* = ((S^*_{\alpha 0})_{\alpha 0}, f^*)$ there exists an epimorphism $\sigma = (\sigma_{\alpha 0})_{\alpha 0}$ iff there is an isomorphism $\varrho = (\varrho_{\alpha 0})_{\alpha 0}$ between a certain*

[7]) Cf. [7, § 1].

[8]) Let $\mathscr{A} \in \mathfrak{S}(S)$, $\mathscr{B} \in \mathfrak{S}(\mathscr{A})$. If $\mathscr{C} \in \mathfrak{S}(S)$ has the blocks which are unions of all $\mathscr{A}$-blocks contained in the same $\mathscr{B}$-block, then $\mathscr{C}$ will be termed a cover of $\mathscr{A}$ enforced by $\mathscr{B}$. – If $\mathscr{P} \in \mathfrak{S}(C)$, $\mathscr{P}' \in \mathfrak{S}(C/\mathscr{P})$ (cf. § 4) then $\mathscr{P}''$ will be termed a cover of $\mathscr{P}$ enforced by $\mathscr{P}'$ if each $\mathscr{P}''_{\alpha 0}$ is the cover of $\mathscr{P}_{\alpha 0}$ enforced by $\mathscr{P}'_{\alpha 0}$.

factor system $\mathbf{C}' = \mathbf{C}/\mathscr{P}$ and $\mathbf{C}^*$. This ϱ is such that ϱ_{α_0} maps each $\mathscr{P}_{\alpha_0}$-block A'_{α_0} onto $\sigma_{\alpha_0}A'_{\alpha_0} \in S^*_{\alpha_0}$ (for all α_0).

Proof. Let σ be an epimorphism between $\mathbf{C}$, $\mathbf{C}^*$. The partition $\mathscr{P}$ on $\mathbf{C}$ induced by σ is necessarily generating (theorem 1). Now determine a surjection ϱ: $\mathbf{C}/\mathscr{P} \to \mathbf{C}^*$. For each α_0, ϱ_{α_0} sends $A_{\alpha_0} \in \mathscr{P}_{\alpha_0}$ onto $a^*_{\alpha_0} \in S^*_{\alpha_0}$ with $\sigma^{-1}_{\alpha_0}A^*_{\alpha_0} = A_{\alpha_0}$. Thus $\varrho_{\alpha_0}A_{\alpha_0} = \sigma_{\alpha_0}a_{\alpha_0}$ for all $a_{\alpha_0} \in A_{\alpha_0}$. Choose $a_\alpha \in A_\alpha \in \mathscr{P}_\alpha$ for all α. Then $f((a_\alpha)_\alpha) \in f(\prod_\alpha A_\alpha) \subseteq f/\mathscr{P}(A_\alpha)_\alpha) \in \mathscr{P}$, so that $\varrho_0 f/\mathscr{P}((A_\alpha)_\alpha) = \sigma_0\, f((a_\alpha)_\alpha) = f/\mathscr{P}(\varrho_\alpha A_\alpha)_\alpha)$ and ϱ is even an isomorphism between $\mathbf{C}/\mathscr{P}$, $\mathbf{C}^*$. Conversely, let $\mathbf{C}/\mathscr{P}$ be an arbitrary factor system of $\mathbf{C}$ modulo the generating partition $\mathscr{P}$ on $\mathbf{C}$. Let σ be the surjection between $\mathbf{C}$ and $\mathbf{C}/\mathscr{P}$ such that $a_{\alpha_0} \in \sigma_{\alpha_0}a_{\alpha_0} = A_{\alpha_0} \in \mathscr{P}_{\alpha_0}$ for all α_0. According to $f((a_\alpha)_\alpha) \in f(\prod_\alpha A_\alpha) \subseteq f/\mathscr{P}((A_\alpha)_\alpha) \in \mathscr{P}_0$, there is also $\sigma_0\, f((a_\alpha)_\alpha) = f/\mathscr{P}((A_\alpha)_\alpha) = f/\mathscr{P}((\sigma_\alpha a_\alpha)_\alpha)$, so that σ is an epimorphism between $\mathbf{C}$ and $\mathbf{C}/\mathscr{P}$. If there is an isomorphism between $\mathbf{C}/\mathscr{P}$ and $\mathbf{C}^*$, then there is also an epimorphism between $\mathbf{C}$ and $\mathbf{C}^*$.

Theorem 8. *Let $\mathscr{P}^i = (\mathscr{P}^i_{\alpha_0})_{\alpha_0}$; $i = 1, 2$ be generating partitions in a given system $\mathbf{C} = ((S_{\alpha_0})_{\alpha_0}, f)$. If $\mathscr{P}^1, \mathscr{P}^2$ are paired,*[10]*) then there exists an isomorphism $\varrho = (\varrho_{\alpha_0})_{\alpha_0}$ between $\mathbf{C}/\mathscr{P}_1$and $\mathbf{C}/\mathscr{P}_2$ such that, for all α_0, to each $\mathscr{P}^1_{\alpha_0}$-block $A^1_{\alpha_0}$ there corresponds by ϱ_{α_0} the $\mathscr{P}^2_{\alpha_0}$-block $A^2_{\alpha_0} \between A^1_{\alpha_0}$.*

Proof. Let $\mathbf{C}/\mathscr{P}^1$, $\mathbf{C}/\mathscr{P}^2$ be paired factor systems. This means that, for all α_0, each $A^1_{\alpha_0} \in \mathscr{P}^1_{\alpha_0}$ intersects exactly one $A^2_{\alpha_0} \in \mathscr{P}^2_{\alpha_0}$. Thus for each α_0 one has a surjection ϱ_{α_0} under which $A^1_{\alpha_0} \to A^2_{\alpha_0}$ as before. Set $B_\alpha = A^1_\alpha \cap A^2_\alpha$ for each α. Thus $f(\prod_\alpha B_\alpha) \subseteq f(\prod_\alpha A^i_\alpha) \subseteq f/\mathscr{P}^i((A^i_\alpha)_\alpha) \in \mathscr{P}^i_0$; $i = 1, 2$. It follows that $f(\prod_\alpha B_\alpha) \subseteq f/\mathscr{P}^1((A^1_\alpha)_\alpha \cap f/\mathscr{P}^2((A^2_\alpha)_\alpha)$, so that $f/\mathscr{P}^1((A^1_\alpha)_\alpha) \between f/\mathscr{P}^2((A^2_\alpha)_\alpha) = \varrho_0 f/\mathscr{P}^1((A^1_\alpha)_\alpha) = f/\mathscr{P}^2((\varrho_\alpha A^1_\alpha)_\alpha)$ as required.

Theorem 9. *Let $\mathscr{P} = (\mathscr{P}_{\alpha_0})_{\alpha_0}$ be a generating partition on a given system $\mathbf{C} = (S_{\alpha_0})_{\alpha_0}, f)$, and $\mathscr{P}' = (\mathscr{P}'_{\alpha_0})_{\alpha_0}$ a generating partition on $\mathbf{C}' = \mathbf{C}/\mathscr{P} = ((S'_{\alpha_0})_{\alpha_0}, f')$. Then there is an isomorphism $\varrho = (\varrho_{\alpha_0})_{\alpha_0}$ between $\mathbf{C}'/\mathscr{P}'$ and the cover $\mathbf{C}^* = ((S^*_{\alpha_0})_{\alpha_0}, f^*)$ of $\mathbf{C}'$ enforced by $\mathscr{P}'$:*[11]*) For all α_0, each $A''_{\alpha_0} \in \mathscr{P}'_{\alpha_0}$ is mapped onto the union of all $\mathscr{P}_{\alpha_0}$-blocks contained in A''_{α_0}.*

Proof. Let $\mathbf{C}$, $\mathscr{P}$, $\mathscr{P}'$ be given and $\mathbf{C}^*$ be the cover of $\mathbf{C}'$ enforced by $\mathscr{P}'$. Each $A^*_{\alpha_0} \in S^*_{\alpha_0}$ consists of all $\mathscr{P}_{\alpha_0}$-blocks contained in the same $A''_{\alpha_0} \in \mathscr{P}'_{\alpha_0}$. Map each $A''_{\alpha_0} \in \mathscr{P}'_{\alpha_0}$ into the preceding $A^*_{\alpha_0} \in S^*_{\alpha_0}$ by a surjection $\varrho_{\alpha_0} : \mathscr{P}'_{\alpha_0} \to S^*_{\alpha_0}$ (for each α_0). The map $\varrho = (\varrho_{\alpha_0})_{\alpha_0}$ is necessarily an isomorphism between $\mathbf{C}'/\mathscr{P}'$ and $\mathbf{C}^*$. Indeed,

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[9]) We speak about a factor system *induced* by σ.

[10]) Two partitions $\mathscr{A}$, $\mathscr{B} \in \mathfrak{S}(S)$ are said to be *paired* if to each $\mathscr{A}$-block A ($\mathscr{B}$-block B) there exists exactly one $\mathscr{B}$-block $A' \between A$ ($\mathscr{A}$-block $B' \between B$). Two partitions $\mathscr{P}^1$, $\mathscr{P}^2$ in $\mathbf{C}$ are said to be *paired* if $\mathscr{P}^1_{\alpha_0}$, $\mathscr{P}^2_{\alpha_0}$ are paired for all α_0.

[11]) This is to mean that $\mathbf{C}^* = \mathbf{C}/\mathscr{P}^*$, where $\mathscr{P}^*$ is the cover of $\mathscr{P}$ enforced by $\mathscr{P}'$.

choose $A''_\alpha \in \mathscr{P}'_\alpha$ for all α so that $f'/\mathscr{P}'((A''_\alpha)_\alpha) = A''_0 \in \mathscr{P}'_0$. For each $A'_\alpha \in \mathscr{P}_\alpha$ with $A'_\alpha \subseteq A''_\alpha$ there is $f'((A'_\alpha)_\alpha = A'_0 \subseteq A''_0$, and consequently $f^*((A^*_\alpha)_\alpha) = A^*_0, f^*((\varrho_\alpha A''_\alpha)_\alpha) =$ $= \varrho_0 A''_0$, as required.

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Výtah

ROZKLADY V KARTÉZSKÝCH STRUKTURÁCH

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Zobecněním algebraické operace na dané množině je surjekce tvaru $\prod_{\alpha\in\Gamma} S_\alpha \to S_0$, kde S_α, S_0 jsou neprázdné množiny. Je provedena aplikace Borůvkovy teorie rozkladů množin na takovéto zobecněné operace (do nové situace jsou přeneseny pojmy vytvořujícího rozkladu a homomorfismu a jsou nalezeny příslušné teorémy). Speciálně pro $S_\alpha = S\,(\alpha \in \Gamma)$, $S_0 \subseteq S$, dávají nalezené výsledky obecnější teorii než je obvyklá teorie rozkladů množin s algebraickou operací.

Резюме

РАЗЛОЖЕНИЯ В ДЕКАРТОВЫХ СТРУКТУРАХ

ВАЦЛАВ ГАВЕЛ (Václav Havel), Брно

Обобщением алгебраической операции на данном множестве является сырьекция вида $\prod_{\alpha \in \Gamma} S_\alpha \to S_0$, где S_α, S_0 – непустые множества. К таким обобщенным операциям приложены основания теории разложения множеств О. Борувки (на новых началах определены понятия образующего разложения и гомоморфизма и выведены соответствующие теоремы). В частности, для $S_\alpha = S\ (\alpha \in \Gamma)$, $S_0 \leqq S$, дают найденные результаты более общую теорию, чем обычная теория разложений множеств с алгебраической операцией.