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SOME PROPERTIES OF NON-CANONIC SYSTEMS OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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This article deals with a certain class of non-canonic systems of linear integro-differential equations with variable coefficients; theorems concerning the existence, uniqueness and stability of the solution are given.

0. In the paper the vector equation

$$(L(t)\,x)' + R(t)\,x + S(t)\int_0^t x\,\mathrm{d}\tau = f(t)\,, \tag{0.1}$$

where $L(t)$, $R(t)$, $S(t)$ are square-matrices defined for $t \geqq 0$, will be considered. Such an equation describes the behavior of every linear physical system with lumped time-varying elements, particularly of every electrical network. From the physical point of view the most important case occurs if the matrices appearing in (0.1) and certain matrices related to them are symmetric and positive semidefinite.

1. First let us consider a more general equation than (0.1). For the sake of brevity we shall introduce the following notation:

Let A, B be constant $r \times r$ matrices and let $\det A = 0$; the matrix B will be called subjoint to A, if there is a constant $r \times k$ matrix U whose columns u_i constitute a complete set of linearly independent solutions of $Au = 0$ and a constant $k \times r$ matrix V whose rows v_j constitute a complete set of linearly independent solutions of $vA = 0$ such that the $k \times k$ matrix VBU is non-singular.

Obviously, if B is subjoint to A, then for any matrices $\tilde{U}$, $\tilde{V}$ with rank k which fulfill the equations $A\tilde{U} = 0$, $\tilde{V}A = 0$ we have $\det \tilde{V}B\tilde{U} \neq 0$.

Let $A(t)$ be an $r \times r$ matrix which has a continuous derivative $A'(t)$ everywhere in $\langle 0, \infty)$, $f(t)$ an r-dimensional vector defined in $\langle 0, \infty)$.

a) $A(t)$ and $f(t)$ will be called compatible, if $f(t)$ is integrable and if for every $l \times r$ matrix $Y(t)$, $l \leqq r$, which has a continuous derivative on an interval $\langle t_1, t_2 \rangle \subset \langle 0, \infty)$ and fulfills the equality $Y(t)\,A(t) = 0$ there, the vector $Y(t)\,f(t)$ is absolutely continuous on $\langle t_1, t_2 \rangle$.

b) $A(t)$ and $f(t)$ will be called strongly compatible, if $f(t)$ is continuous on $\langle 0, \infty)$ and if for every matrix $Y(t)$ with properties stated in a) the vector $Y(t) f(t)$ has a contínous derivative on $\langle t_1, t_2 \rangle$.

Let $A(t)$ be an $r \times r$ matrix defined for $t \geqq 0$, $W(t, \tau)$ an $r \times r$ matrix defined for $0 \leqq \tau \leqq t < \infty$, and $f(t)$ an r-dimensional vector defined for $t \geqq 0$. The r-dimensional vector $x(t)$ will be called the solution of the equation

$$A(t)\, x(t) + \int_0^t W(t, \tau)\, x(\tau)\, d\tau = f(t)\,, \tag{1.1}$$

if (1.1) is fulfilled almost everywhere in $\langle 0, \infty)$.

Then we have

Theorem 1.1. *Let the following assumptions be satisfied:*

1) *The $r \times r$ matrix $A(t)$ has a continuous derivative $A'(t)$ everywhere in $\langle 0, \infty)$ and there is a fixed integer $h < r$ such that* rank $A(t) = h$ *for every $t \geqq 0$.*

2) *The $r \times r$ matrices $W(t, \tau)$, $\partial W(t, \tau)/\partial t$ are continuous everywhere in the region $0 \leqq \tau \leqq t < \infty$.*

3) *For every $t \geqq 0$ the matrix $W^*(t) = W(t, t)$ is subjoint to $A(t)$.*

4) *The matrix $A(t)$ and the vector $f(t)$ are compatible.*

5) *There is a constant r-dimensional vector ξ such that*

$$A(0)\, \xi = f(0)\,. \tag{1.2}$$

Then there is a unique integrable solution $x(t)$ of (1.1). If in addition $A(t)$ and $f(t)$ are strongly compatible, then $x(t)$ is continuous in $\langle 0, \infty)$.

Moreover, assumption 3) is fulfilled, if both $A(t)$ and $W^*(t)$ are symmetric and for every $t \geqq 0$ either $W^*(t)$ or $A(t) + W^*(t)$ is positive definite.

Note 1. If rank $A(t) = r$ for every $t \geqq 0$, (1.1) is obviously equivalent to a Volterra's equation; then, of course, for the existence and uniqueness of a solution it suffices to assume that $A(t)$, $W(t, \tau)$ are continuous and $f(t)$ is integrable.

Proof of Th. 1.1.: Referring to the Theorem in [1], construct $r \times r$ matrices $M(t)$, $N(t)$ which have a continuous derivative in $\langle 0, \infty \rangle$ and fulfill the conditions $\det M(t) \neq 0$, $\det N(t) \neq 0$ in $\langle 0, \infty)$ and $A(t)\, M(t) = [B(t) \mid 0]$, $N(t)\, A(t) = \begin{bmatrix} C(t) \\ \hline 0 \end{bmatrix}$, where $B(t)$, $C(t)$ is an $r \times h$ and $h \times r$ matrix, respectively. Using substitution

$$x(t) = M(t)\, y(t)\,, \tag{1.3}$$

it is obvious that (1.1) is equivalent to the equation

$$N(t)\, A(t)\, M(t)\, y(t) + \int_0^t N(t)\, W(t, \tau)\, M(\tau)\, y(\tau)\, d\tau = N(t)\, f(t)\,, \tag{1.4}$$

$t \geqq 0$. Let us subdivide matrices $M(t)$, $N(t)$ into blocks as follows

(1.5) $$M(t) = \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, \quad N(t) = \begin{bmatrix} N_{11}(t) & N_{12}(t) \\ N_{21}(t) & N_{22}(t) \end{bmatrix},$$

where $M_{11}(t)$, $N_{11}(t)$ are $h \times h$ matrices, and denote

(1.6) $$M_2(t) = \begin{bmatrix} M_{12}(t) \\ M_{22}(t) \end{bmatrix}, \quad N_2(t) = [N_{21}(t) \mid N_{22}(t)] ;$$

moreover, let

(1.7) $$B(t) = \begin{bmatrix} B_{11}(t) \\ B_{21}(t) \end{bmatrix}, \quad W(t, \tau) = \begin{bmatrix} w_{11}(t, \tau) & w_{12}(t, \tau) \\ w_{21}(t, \tau) & w_{22}(t, \tau) \end{bmatrix},$$

$$N(t)\, W(t, \tau)\, M(\tau) = \begin{bmatrix} \tilde{w}_{11}(t, \tau) & \tilde{w}_{12}(t, \tau) \\ \tilde{w}_{21}(t, \tau) & \tilde{w}_{22}(t, \tau) \end{bmatrix},$$

where $B_{11}(t)$, $w_{11}(t, \tau)$, $\tilde{w}_{11}(t, \tau)$ are $h \times h$ matrices.

Then from the equation $A(t)\, M(t) = [B(t) \mid 0]$ we have

(1.8) $$B_{11} = A_{11}M_{11} + A_{12}M_{21}, \quad A_{11}M_{12} + A_{12}M_{22} = 0,$$
$$B_{21} = A_{21}M_{11} + A_{22}M_{21}, \quad A_{21}M_{12} + A_{22}M_{22} = 0.$$

Analogously, from the equation for $N(t)\, A(t)$ it follows that

(1.9) $$N_{21}A_{11} + N_{22}A_{21} = 0, \quad N_{21}A_{12} + N_{22}A_{22} = 0.$$

Thus we have

(1.10) $$N(t)\, A(t)\, M(t) = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{12} & 0 \end{bmatrix},$$

where

(1.11) $$\tilde{A}_{12} = N_{21}B_{11} + N_{22}B_{21} =$$
$$= N_{21}A_{11}M_{11} + N_{21}A_{12}M_{21} + N_{22}A_{21}M_{11} + N_{22}A_{22}M_{21}.$$

From (1.9), however, we get immediately $\tilde{A}_{12}(t) = 0$, and consequently, $\det \tilde{A}_{11}(t) \neq 0$ in $\langle 0, \infty)$.

On the other hand, (1.5) and (1.7) yield

(1.12) $$\tilde{w}_{22}(t, \tau) = N_{21}(t)\, w_{11}(t, \tau)\, M_{12}(\tau) + N_{21}(t)\, w_{12}(t, \tau)\, M_{22}(\tau) +$$
$$+ N_{22}(t)\, w_{21}(t, \tau)\, M_{12}(\tau) + N_{22}(t)\, w_{22}(t, \tau)\, M_{22}(\tau).$$

Next observe that for any $t \geqq 0$ we may put $U = M_2(t)$, $V = N_2(t)$, since $A(t)\, M_2(t) = 0$ and $N_2(t)\, A(t) = 0$ identically and both $M_2(t)$ and $N_2(t)$ have rank $r - h$ for every $t \geqq 0$; thus, by assumption 3) we have

(1.13) $$\det N_2(t)\, W(t, t)\, M_2(t) \neq 0$$

for $t \geqq 0$. However, (1.6), (1.7) yield

$$N_2(t)\, W(t, t)\, M_2(t) = N_{21} w_{11}^* M_{12} + N_{21} w_{12}^* M_{22} + N_{22} w_{21}^* M_{12} + N_{22} w_{22}^* M_{22}$$

with $w_{jk}^* = w_{jk}(t, t)$, $j, k = 1, 2$; consequently, by (1.12),

$$\det \tilde{w}_{22}^*(t) \neq 0 \tag{1.14}$$

for every $t \geqq 0$.

Now, consider the vector $N(t) f(t) = g(t)$ appearing in (1.4). Putting $f(t) = \begin{bmatrix} f_1(t) \\ \hdashline f_2(t) \end{bmatrix}$, $g(t) = \begin{bmatrix} g_1(t) \\ \hdashline g_2(t) \end{bmatrix}$, where $f_1(t)$, $g_1(t)$ are h-dimensional vectors, we have by (1.5):

$$g_2(t) = N_{21}(t) f_1(t) + N_{22}(t) f_2(t)\,. \tag{1.15}$$

On the other hand, since the matrix $N_2(t)$ satisfies the requirements given in the definition of compatibility, we may put $Y(t) = N_2(t)$. But $N_2(t) f(t) = g_2(t)$; consequently, $g_2(t)$ is absolutely continuous in $\langle 0, \infty)$ by assumption 4) of the theorem. If in addition $A(t)$ and $f(t)$ are strongly compatible, $g_2(t)$ possesses a continuous derivative in $\langle 0, \infty)$.

Furthermore, it is readily seen that condition (1.2) is equivalent to the condition $g_2(0) = 0$. Indeed, (1.2) has a solution ξ, if and only if $\omega' f(0) = 0$ for any solution ω of $\omega' A(0) = 0$. Since $N_2(0) A(0) = 0$, the equality $N_2(0) f(0) = g_2(0) = 0$ is the necessary and sufficient condition for the validity of (1.2).

Summarizing the previous results and putting $y(t) = \begin{bmatrix} y_1(t) \\ \hdashline y_2(t) \end{bmatrix}$, where $y_1(t)$ is an h-dimensional vector, we can split (1.4) into the following equations

$$\tilde{A}_{11}(t)\, y_1(t) + \int_0^t \tilde{w}_{11}(t, \tau)\, y_1(\tau)\, d\tau + \int_0^t \tilde{w}_{12}(t, \tau)\, y_2(\tau)\, d\tau = g_1(t)\,, \tag{1.16}$$

$$\int_0^t \tilde{w}_{21}(t, \tau)\, y_1(\tau)\, d\tau + \int_0^t \tilde{w}_{22}(t, \tau)\, y_2(\tau)\, d\tau = g_2(t)\,. \tag{1.17}$$

In view of the above facts, however, (1.17) is equivalent to

(1.18)
$$\tilde{w}_{21}^*(t)\, y_1(t) + \tilde{w}_{22}^*(t)\, y_2(t) + \int_0^t \frac{\partial \tilde{w}_{21}(t, \tau)}{\partial t}\, y_1(\tau)\, d\tau + \int_0^t \frac{\partial \tilde{w}_{22}(t, \tau)}{\partial t}\, y_2(\tau)\, d\tau = g_2'(t)\,,$$

so that both (1.16) and (1.17) are equivalent to

$$\bar{A}(t)\, y(t) + \int_0^t \bar{W}(t, \tau)\, y(\tau)\, d\tau = h(t) \tag{1.19}$$

with

$$\bar{A}(t) = \left[\begin{array}{c|c} \tilde{A}_{11}(t) & 0 \\ \hline \tilde{w}^*_{21}(t) & \tilde{w}^*_{22}(t) \end{array}\right], \quad h(t) = \left[\begin{array}{c} g_1(t) \\ \hline g'_2(t) \end{array}\right].$$

Because $\det \bar{A}(t) = \det \tilde{A}_{11}(t) \,.\, \det \tilde{w}^*_{22}(t) \neq 0$ in $\langle 0, \infty)$, (1.19) is equivalent to a Volterra's equation; thus there is a unique integrable vector $y(t)$ which fulfills (1.19). Moreover, $y(t)$ is continuous, provided $h(t)$ is continuous, i.e. if $A(t), f(t)$ are strongly compatible. Hence, the same is true for the solution $x(t)$ of (1.1).

It remains to prove the last statement of the theorem. If for a $\bar{t} \geqq 0$ the matrix $W^*(\bar{t})$ is positive definite, then obviously $U' W^*(\bar{t})\, U$ is positive definite for any $r \times \times (r - h)$ matrix U with rank $U = r - h$; thus $\det U' W^*(\bar{t})\, U \neq 0$. If in addition the matrix U fulfills the equation $A(\bar{t})\, U = 0$, we have also $U' A(\bar{t}) = 0$. Consequently, $W^*(\bar{t})$ is subjoint to $A(\bar{t})$ by definition.

Next, let $A(\bar{t}) + W^*(\bar{t})$ be definite for a $\bar{t} \geqq 0$. Since $A(t)$ is symmetric we may put $N(t) = M'(t)$; moreover, using the above notation we have

$$M'(\bar{t})\,(A(\bar{t}) + W^*(\bar{t}))\, M(\bar{t}) = \left[\begin{array}{c|c} \tilde{A}_{11}(\bar{t}) + \tilde{w}^*_{11}(\bar{t}) & \tilde{w}^*_{12}(\bar{t}) \\ \hline \tilde{w}^*_{21}(\bar{t}) & \tilde{w}^*_{22}(\bar{t}) \end{array}\right]. \tag{1.20}$$

Because the left hand side of (1.20) is positive definite, $\tilde{w}^*_{22}(\bar{t})$ is positive definite, and consequently, $\det \tilde{w}^*_{22}(\bar{t}) \neq 0$.

On the other hand, it was shown earlier that $\tilde{w}^*_{22}(t) = N_2(t)\, W^*(t)\, M_2(t)$, i.e. $\tilde{w}^*_{22}(\bar{t}) = M'_2(\bar{t})\, W^*(\bar{t})\, M_2(\bar{t})$ in our case. However, we may put $U = M_2(\bar{t})$, $V = M'_2(\bar{t})$ so that $W^*(\bar{t})$ is subjoint to $A(\bar{t})$. Hence, Theorem 1.1 is proved.

Theorem 1.2. *Let the matrices $A(t)$, $W(t, \tau)$ and vectors $f_k(t)$, $k = 0, 1, 2, \ldots$ satisfy the assumptions of Th. 1.1; moreover, let $f_k(t)$, $k = 0, 1, \ldots$ be absolutely continuous in $\langle 0, \infty)$. If $x_k(t)$ denotes the solution of*

$$A(t)\, x_k(t) + \int_0^t W(t, \tau)\, x_k(\tau)\, \mathrm{d}\tau = f_k(t)\,, \tag{1.21}$$

$k = 0, 1, 2, \ldots,$ *and if for a* $T > 0$

$$\int_0^T \|f'_k(t) - f'_0(t)\|\, \mathrm{d}t \to 0 \tag{1.22}$$

and $f_k(0) \to f_0(0)$, *then*

$$\int_0^T \|x_k(t) - x_0(t)\|\, \mathrm{d}t \to 0\,. \tag{1.23}$$

Proof. First observe that if (1.22) and $f_k(0) \to f_0(0)$ are true then $f_k(t) \to f_0(t)$ uniformly on $\langle 0, T\rangle$. Moreover, since eq. (1.21) is linear it suffices to prove Th. 1.2 for a sequence $\bar{f}_k(t)$ such that each $\bar{f}_k(t)$ fulfills condition (1.2) and

$$\int_0^T \|\bar{f}'_k(t)\|\, \mathrm{d}t \to 0\,, \quad \bar{f}_k(0) \to 0\,. \tag{1.24}$$

Referring to the proof of Th. 1.1, the solution $\bar{x}_k(t)$ of (1.21) with $f_k(t) = \bar{f}_k(t)$ is given by $\bar{x}_k(t) = M(t)\,\bar{y}_k(t)$, $k = 0, 1, 2, \ldots$, where $\bar{y}_k(t)$ is the solution of

$$\bar{A}(t)\,\bar{y}_k(t) + \int_0^t \bar{W}(t, \tau)\,\bar{y}_k(\tau)\,\mathrm{d}\tau = \bar{h}_k(t) \tag{1.25}$$

with

$$\bar{h}_k(t) = \left[\frac{N_{11}(t)\,\bar{f}_{1k}(t) + N_{12}(t)\,\bar{f}_{2k}(t)}{(N_{21}(t)\,\bar{f}_{1k}(t) + N_{22}(t)\,\bar{f}_{2k}(t))'}\right]. \tag{1.26}$$

From (1.26), however, it is clear that $\int_0^T \|\bar{h}_k(t)\|\,\mathrm{d}t \to 0$. Moreover, since (1.25) is equivalent to Volterra's equation, we have also $\int_0^T \|\bar{y}_k(t)\|\,\mathrm{d}t \to 0$, and consequently, $\int_0^T \|\bar{x}_k(t)\|\,\mathrm{d}t \to 0$, Q.E.D..

Example 1. Let $A(t)$, $B(t)$ be $r \times r$ matrices defined on $\langle 0, \infty)$, $f(t)$ an r-dimensional vector defined on $\langle 0, \infty\rangle$, c an r-dimensional constant vector; as in the theory of differential equations, the vector $x(t)$ will be called a solution of the equation

$$A(t)\,x'(t) + B(t)\,x(t) = f(t) \tag{1.27}$$

with initial condition c, if 1) $x(t)$ is absolutely continuous in $\langle 0, \infty)$, 2) $x(0) = c$, 3) equation (1.27) is satisfied almost everywhere.

Then from Theorem 1.1 we have immediately the following assertion:

If 1) $A(t)$, $B(t)$ *have a continuous derivative everywhere in* $\langle 0, \infty)$ *and* $A(t)$ *has a fixed rank* $h < r$ *for every* $t \geqq 0$, 2) *for every* $t \geqq 0$ *the matrix* $B(t)$ *is subjoint to* $A(t)$, 3) $A(t)$ *and* $f(t)$ *are compatible,* 4) *there is a constant vector* ξ *such that*

$$A(0)\,\xi + B(0)\,c = f(0)\,, \tag{1.28}$$

then there is a unique solution $x(t)$ *of* (1.27) *with the initial condition* c.

Indeed, putting $x(t) = \int_0^t u(\tau)\,\mathrm{d}\tau + c$, eq. (1.27) is equivalent to

$$A(t)\,u(t) + \int_0^t B(t)\,u(\tau)\,\mathrm{d}\tau = f(t) - B(t)\,c\,. \tag{1.29}$$

However, applying Theorem 1.1 to (1.29) we get immediately the assertion just stated.

2. Theorem 1.1 can easily be extended to a case that $f(t)$ is a vector having distributions as its components.

Let $n \geqq 0$ be an integer and let the system $\mathbf{D}_n$ be defined as follows: for every $f \in \mathbf{D}_n$ there is an r-dimensional vector function $F(t)$ which is locally integrable in $(-\infty, \infty)$ and vanishes almost everywhere in $(-\infty, 0)$ such that $f = F^{(n)}$ (distributional derivative), i.e.

$$(f_i, \varphi) = (-1)^n \int_{-\infty}^{\infty} F_i(t)\,\varphi^{(n)}(t)\,\mathrm{d}t\,, \quad \varphi(t) \in \mathbf{K} \tag{2.1}$$

for each component f_i of f, where $F_i(t)$, $i = 1, 2, \ldots, r$ are components of $F(t)$ and $\boldsymbol{K}$ is the set of all infinitely differentiable functions with compact support.

It can be easily verified that for each $f \in \mathbf{D}_n$ the vector function $F(t)$ is determined uniquely up to a set of measure zero. (See [2].)

Let $\mathbf{W}_n$, $n \geqq 0$ be the set of all $r \times r$ matrices $W(t, \tau)$ which are defined for $0 \leqq \leqq \tau \leqq t < \infty$ and have the following properties: if $W(t, \tau) \in \mathbf{W}_n$, then a continuous $\partial^n W(t, \tau)/\partial \tau^n$ exists everywhere in $0 \leqq \tau \leqq t < \infty$ and $(\partial^i W/\partial \tau^i)^*$ possesses a continuous $n - i - 1$-th derivative in $\langle 0, \infty)$ for $i = 0, 1, \ldots, n - 1$.

If $W(t, \tau) \in \mathbf{W}_n$, $x \in \mathbf{D}_n$ with $x = X^{(n)}$, let

$$(2.2) \qquad [Wx] = \sum_{i=0}^{n-1} (-1)^i \left(\frac{\partial^i W}{\partial \tau^i}\right)^* X^{(n-i-1)} + (-1)^n \int_0^t \frac{\partial^n W(t, \tau)}{\partial \tau^n} X(\tau) \, d\tau \, .$$

Obviously, $[Wx] \in \mathbf{D}_{\max[n-1, 0]}$ and appears as a generalization of the integral $\int_0^t W(t, \tau) \, x(\tau) \, d\tau$.

Defining finally the product Ax, where $x \in \mathbf{D}_n$ and the matrix $A(t)$ possesses a continuous derivative of n-th order, in the manner commonly used in the theory of distributions, then the following assertion is true:

Theorem 2.1. *Let $f \in \mathbf{D}_m$, $m \geqq 1$, and let the following conditions be satisfied:*

a) *The matrix $A(t)$ possesses a continuous derivative of $m + 1$-th order everywhere in $\langle 0, \infty)$ and there is an integer $h < r$ such that rank $A(t) = h$ for every $t \geqq 0$.*

b) *$W(t, \tau) \in \mathbf{W}_{m+1}$ and for every $t \geqq 0$ the matrix $W^*(t)$ is subjoint to $A(t)$.*

Then there is a unique $x \in \mathbf{D}_{m+1}$ which fulfills the equation

$$(2.3) \qquad Ax + [Wx] = f \, .$$

For the proof the following assertion will be useful:

Lemma 2.1. *Let the $r \times r$ matrix $A(t)$ satisfy the conditions: $A(t)$ has a continuous derivative in $\langle 0, \infty)$ and there is an integer $h < r$ such that* rank *$A(t) = h$ for every $t \geqq 0$. If for every $t \geqq 0$ the matrix $B(t)$ is subjoint to $A(t)$, then for every $t \geqq 0$ the matrix $B(t) + \lambda A'(t)$, λ being any number, is subjoint to $A(t)$.*

Proof: Referring to the Theorem in [1], there are matrices $M_2(t)$, $N_2(t)$ possessing a continuous derivative in $\langle 0, \infty)$ such that rank $M_2(t)$ = rank $N_2(t) = r - h$ and

$$(2.4) \qquad A(t) \, M_2(t) = 0 \, , \quad N_2(t) \, A(t) = 0$$

for every $t \geqq 0$. Thus, by definition,

$$(2.5) \qquad \det N_2(t) \, B(t) \, M_2(t) \neq 0$$

for every $t \geqq 0$. On the other hand, from (2.4) we have $A'(t) \, M_2(t) + A(t) \, M_2'(t) = 0$, and consequently, $N_2(t)\, A'(t)\, M_2(t) = 0$. Therefore, for every $t \geqq 0$,

$$\det N_2(t)\,(B(t) + \lambda A'(t))\, M_2(t) \neq 0\,, \tag{2.6}$$

i.e. $B(t) + \lambda A'(t)$ is subjoint to $A(t)$, Q.E.D.

Proof of Th. 2.1. Let us choose any $\tilde{x} \in \mathbf{D}_{m+1}$ with $\tilde{x} = \tilde{X}^{(m+1)}$, and consider the expression $A\tilde{x} + [W\tilde{x}] = A\tilde{X}^{(m+1)} + [W\tilde{X}^{(m+1)}]$. Using formulas (5) and (8) in [2], which are obviously true also for the matrix conception, we easily obtain the following equality in the distributional sense:

$$A\tilde{X}^{(m+1)} + [W\tilde{X}^{(m+1)}] = \{A\tilde{X} + [Q\tilde{X}]\}^{(m+1)}\,, \tag{2.7}$$

where

$$Q(t, \tau) = -\, nA'(\tau) + W^*(\tau) + \int_\tau^t P(z, \tau)\, \mathrm{d}z\,, \tag{2.8}$$

$P(t, \tau)$ being continuous for $0 \leqq \tau \leqq t < \infty$.

Consider now the equation

$$A(t)\, X(t) + \int_0^t Q(t, \tau)\, X(\tau)\, \mathrm{d}\tau = \int_0^t F(\tau)\, \mathrm{d}\tau\,, \tag{2.9}$$

where $F(t)$ is given by $f = F^{(m)} \in \mathbf{D}_m$. By assumption, $W^*(t)$ is subjoint to $A(t)$ for every $t \geqq 0$; hence by Lemma 2.1, $Q^*(t) = -\, nA'(t) + W^*(t)$ is subjoint to $A(t)$ for every $t \geqq 0$. Since the remaining assumptions of Th. 1.1 are also satisfied, there is a unique solution $X(t)$ of (2.9). Because (2.9) is fulfilled almost everywhere in $\langle 0, \infty)$, it is also true in the distributional sense; thus, taking the $m + 1$-th distributional derivative of both sides of (2.9) and using the identity (2.7), we have

$$AX^{(m+1)} + [WX^{(m+1)}] = F^{(m)} = f\,. \tag{2.10}$$

Thus, $x = X^{(m+1)} \in \mathbf{D}_{m+1}$ is the unique solution of (2.3) and Th. 2.1 is proved.

3. Throughout this paragraph equation (0.1) will be treated.

Let $L(t)$, $R(t)$, $S(t)$ be $r \times r$ matrices defined on $\langle 0, \infty)$, $f(t)$ an r-dimensional vector defined on $\langle 0, \infty)$, c an r-dimensional constant vector; the locally integrable vector $x(t)$ will be called the solution of the equation

$$(L(t)\, x(t))' + R(t)\, x(t) + S(t) \int_0^t x(\tau)\, \mathrm{d}\tau = f(t) \tag{3.1}$$

with intial condition c, if $x(t)$ fulfills the equation

$$L(t)\, x(t) + \int_0^t R(\tau)\, x(\tau)\, \mathrm{d}\tau + \int_0^t S(\tau) \int_0^\tau x(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau = \int_0^t f(\tau)\, \mathrm{d}\tau + L(0)\, c \tag{3.2}$$

almost everywhere in $\langle 0, \infty)$.

Observing that (3.2) can be written as

$$L(t)\,x(t) + \int_0^t Q(t,\tau)\,x(\tau)\,\mathrm{d}\tau = \int_0^t f(\tau)\,\mathrm{d}\tau + L(0)\,c \tag{3.3}$$

with

$$Q(t,\tau) = R(\tau) + \int_\tau^t S(z)\,\mathrm{d}z\,, \tag{3.4}$$

we have by Th. 1.1:

Theorem 3.1. *Let $L'(t)$, $R(t)$, $S(t)$ be continuous in $\langle 0, \infty)$, and let an integer $h \leqq r$ exist such that* $\operatorname{rank} L(t) = h$ *for every $t \geqq 0$. Moreover, if $h < r$, let $R(t)$ be subjoint to $L(t)$ for every $t \geqq 0$. Then for every integrable $f(t)$ and c there is a unique integrable solution of* (3.1) *with initial condition c.*

Let us now investigate the properties of (3.1), if the matrices $L(t)$, $R(t)$, $S(t)$ are subjected to certain specific conditions.

Let $\mathscr{L}^2$ be the set of all r-dimensional vectors $x(t)$ defined on $\langle 0, \infty)$ such that $\int_0^T \|x(t)\|^2\,\mathrm{d}t < \infty$ for any finite T.

Lemma 3.1. *Let $A(t)$ be a symmetric $r \times r$ matrix which has a continuous derivative everywhere in $\langle 0, \infty)$, and let a fixed integer $h \leqq r$ exist such that* $\operatorname{rank} A(t) = h$ *for every $t \geqq 0$; moreover, let $x(t) \in \mathscr{L}^2$ be a vector such that there is an absolutely continuous vector $\{Ax\}(t)$ equal to $A(t)\,x(t)$ almost everywhere in $\langle 0, \infty)$. Then* 1) *$x'(t)\,\{Ax\}'(t)$ is integrable,* 2) *there is an absolutely continuous function $\{x'Ax\}(t)$ which is equal to $x'(t)\,A(t)\,x(t)$ almost everywhere in $\langle 0, \infty)$ and*

$$\int_0^t x'(\tau)\,\{Ax\}'(\tau)\,\mathrm{d}\tau = \tfrac{1}{2}[\{x'Ax\}(\tau)]_0^t + \tfrac{1}{2}\int_0^t x'(\tau)\,A'(\tau)\,x(\tau)\,\mathrm{d}\tau \tag{3.5}$$

for every $t \geqq 0$.

Moreover, if $\{Ax\}(0) = A(0)\,c$, c being a constant vector, then $\{x'Ax\}(0) = c'A(0)\,c$.

Proof. Consider first the case that $h < r$. For the sake of brevity denote $y(t) = \{Ax\}(t)$. Then there is a vector $u(t)$ which is absolutely continuous in $\langle 0, \infty)$ such that

$$A(t)\,u(t) = y(t) \tag{3.6}$$

everywhere in $\langle 0, \infty)$. Actually, by assumption we have $A(t)\,x(t) = y(t)$ almost everywhere in $\langle 0, \infty)$; moreover, by the Theorem in [1] there is a non-singular matrix $M(t) = [M_1(t) \mid M_2(t)]$, ($M_1(t)$ being an $r \times h$ matrix) which has a continuous derivative in $\langle 0, \infty)$ such that $A(t)\,M_2(t) = 0$ and $M_2'(t)\,A(t) = 0$ due to the symmetry of $A(t)$. Consequently, we have

$$M_2'(t)\,y(t) = 0 \tag{3.7}$$

almost everywhere in $\langle 0, \infty)$. However, since the left hand side of (3.7) is continuous, (3.7) is valid everywhere in $\langle 0, \infty)$.

On the other hand, (3.7) is a sufficient condition for the existence of a vector $u(t)$ which fulfills (3.6) everywhere in $\langle 0, \infty)$.

Thus, let $u(t)$ be a solution of (3.6) and define the vector $w(t)$ by

$$u(t) = M(t)\, w(t) . \tag{3.8}$$

Then (3.6) yields $M^{\text{`}}(t)\, A(t)\, M(t)\, w(t) = M^{\text{`}}(t)\, y(t)$, where, of course, $M^{\text{`}}(t)\, y(t)$ is absolutely continuous in $\langle 0, \infty)$. But

$$M^{\text{`}}(t)\, A(t)\, M(t) = \left[\begin{array}{c|c} B(t) & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $B(t)$ is a non-singular $h \times h$ matrix with a continuous derivative in $\langle 0, \infty)$; thus, putting $w = \left[\begin{array}{c} w_1(t) \\ \hline w_2(t) \end{array}\right]$, $M^{\text{`}}(t)\, y(t) = \left[\begin{array}{c} z_1(t) \\ \hline z_2(t) \end{array}\right]$, we get $B(t)\, w_1(t) = z_1(t)$, $z_2(t) = 0$. Consequently, putting $w_1(t) = B^{-1}(t)\, z_1(t)$, $w_2(t) = 0$ and defining $u(t)$ by (3.8), $u(t)$ will be absolutely continuous in $\langle 0, \infty)$ and our assertion is proved.

Next, for $u(t)$ already obtained define the vector $v(t)$ by $v(t) = x(t) - u(t)$; then we have

$$A(t)\, v(t) = 0 \tag{3.9}$$

almost everywhere in $\langle 0, \infty)$, and consequently, there is an $r - h$-dimensional vector $s(t)$ such that $v(t) = M_2(t)\, s(t)$ almost everywhere in $\langle 0, \infty)$. Thus we have $v^{\text{`}}(t)\, A'(t)\, v(t) = s^{\text{`}} M_2^{\text{`}} A' M_2 s$; but from $A(t)\, M_2(t) = 0$ we get $A' M_2 + A M_2' = 0$, and consequently, $M_2^{\text{`}} A' M_2 = 0$. Hence,

$$v^{\text{`}}(t)\, A'(t)\, v(t) = 0 \tag{3.10}$$

almost everywhere in $\langle 0, \infty)$.

Define the absolutely continuous function $\{x^{\text{`}} A x\}(t)$ by

$$\{x^{\text{`}} A x\}(t) = u^{\text{`}}(t)\, A(t)\, u(t) ; \tag{3.11}$$

then for almost every $t \geqq 0$ we have

$$x^{\text{`}}(t)\, A(t)\, x(t) = (u^{\text{`}} + v^{\text{`}})\, A(u + v) = u^{\text{`}} A u = \{x^{\text{`}} A x\}(t) . \tag{3.12}$$

Furthermore, by (3.9), (3.10) for almost every $t \geqq 0$,

$$\begin{aligned} &\{x^{\text{`}} A x\}'(t) - 2x^{\text{`}}(t)\, \{Ax\}'(t) + x^{\text{`}}(t)\, A'(t)\, x(t) = \\ &= (u^{\text{`}} A u)' - 2(u + v)^{\text{`}}\, (Au)' + (u + v)^{\text{`}}\, A'(u + v) = \\ &= 2u'^{\text{`}} A u + u^{\text{`}} A' u - 2(u + v)^{\text{`}}\, (A'u + Au') + u^{\text{`}} A' u + 2u^{\text{`}} A' v = 0 . \end{aligned} \tag{3.13}$$

Since both functions $\{x'Ax\}'(t)$ and $x'(t)\,A'(t)\,x(t)$ are locally integrable, the same is true for $x'(t)\,\{Ax\}'(t)$; thus, integrating (3.13) within limits 0, t, the equality (3.5) follows immediately.

It remains to prove the last statement of Lemma 3.1. Thus, let $\{Ax\}(0) = A(0)\,c$; from (3.6) we have $A(0)\,u(0) = y(0) = \{Ax\}(0)$. Consequently, $A(0)\,(u(0) - c) = 0$, i.e. $u(0) = c + \xi$, where ξ satisfies the equality $A(0)\,\xi = 0$. By (3.11), however,

$$\{x'Ax\}(0) = u'(0)\,A(0)\,u(0) = c'A(0)\,c\,, \tag{3.14}$$

Q.E.D.

Finally, if $\operatorname{rank} A(t) = r$, then it suffices to put $u(t) = A^{-1}(t)\,\{Ax\}(t)$, $v(t) = A^{-1}(t)\,(A(t)\,x(t) - \{Ax\}(t))$ and repeat the procedure presented above. Hence, Lemma 3.1 is proved.

Equation (3.1) will be called normal, if

1. matrices $L'(t)$, $R(t)$, $S'(t)$ are continuous in $\langle 0, \infty)$,
2. there is an integer $h \leqq r$ such that $\operatorname{rank} L(t) = h$ for every $t \geqq 0$,
3. for every $t \geqq 0$ each of matrices $L(t)$, $L'(t) + 2R(t)$, $S(t)$, $-S'(t)$ is symmetric and positive semidefinite.

Theorem 3.2. *Let equation* (3.1) *be normal and for every* $t \geqq 0$ *let either the matrix* $L + L' + 2R + S$ *or* $S - S'$ *be positive definite. If* (3.1) *has a solution* $x(t)$ *with initial condition* c *and* $x(t) \in \mathscr{L}^2$, *then* $x(t)$ *is the unique solution of* (3.1) *with initial condition* c *in* $\mathscr{L}^2$.

Proof. Since (3.1) is linear, then by the definition of a solution it suffices to prove that $x(t) \in \mathscr{L}^2$ and

$$L(t)\,x + \int_0^t R(\tau)\,x(\tau)\,\mathrm{d}\tau + \int_0^t S(\tau)\int_0^\tau x(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0 \tag{3.15}$$

implies $x(t) = 0$ almost everywhere in $\langle 0, \infty)$. Thus, putting $q(t) = \int_0^t x(\tau)\,\mathrm{d}\tau$, we have from (3.15):

$$L(t)\,q' + \int_0^t R(\tau)\,q'(\tau)\,\mathrm{d}\tau + \int_0^t S(\tau)\,q(\tau)\,\mathrm{d}\tau = 0\,. \tag{3.16}$$

Referring to Lemma 3.1 it is obvious that $L(t)$ and $q'(t)$ fulfill its assumptions so that we have

$$\{Lq'\}'(t) = -R(t)\,q'(t) - S(t)\,q(t) \tag{3.17}$$

and $\{Lq'\}(0) = 0$. Consequently, by formula (3.5),

$$\int_0^t q''(\tau)\,(-R(\tau)\,q'(\tau) - S(\tau)\,q(\tau))\,\mathrm{d}\tau = \tag{3.18}$$

$$= \tfrac{1}{2}[\{q''Lq'\}(\tau)]_0^t + \tfrac{1}{2}\int_0^t q''(\tau)\,L'(\tau)\,q'(\tau)\,\mathrm{d}\tau\,.$$

On the other hand, integrating by parts,

$$\int_0^t q^{\backprime\prime}(\tau)\, S(\tau)\, q(\tau)\, \mathrm{d}\tau = [q^{\backprime}(\tau)\, S(\tau)\, q(\tau)]_0^t - \int_0^t q^{\backprime}(\tau)\, (S(\tau)\, q(\tau))'\, \mathrm{d}\tau = \tag{3.19}$$

$$= q^{\backprime}(t)\, S(t)\, q(t) - \int_0^t q^{\backprime}(\tau)\, S'(\tau)\, q(\tau)\, \mathrm{d}\tau - \int_0^t q^{\backprime}(\tau)\, S(\tau)\, q'(\tau)\, \mathrm{d}\tau\,,$$

i.e., since $S(\tau)$ is symmetric,

$$\int_0^t q^{\backprime\prime}(\tau)\, S(\tau)\, q(\tau)\, \mathrm{d}\tau = \tag{3.20}$$

$$= \tfrac{1}{2} q^{\backprime}(t)\, S(t)\, q(t) - \tfrac{1}{2} \int_0^t q^{\backprime}(\tau)\, S'(\tau)\, q(\tau)\, \mathrm{d}\tau\,.$$

Introducing (3.20) into (3.18) and rearranging, we have

$$[\{q^{\backprime\prime} L q'\}(\tau)]_0^t + \int_0^t q^{\backprime\prime}(\tau)\, (L'(\tau) + 2R(\tau))\, q'(\tau)\, \mathrm{d}\tau + \tag{3.21}$$

$$+ q^{\backprime}(t)\, S(t)\, q(t) - \int_0^t q^{\backprime}(\tau)\, S'(\tau)\, q(\tau)\, \mathrm{d}\tau = 0$$

for every $t \geqq 0$. But, since by Lemma 3.1 $\{q^{\backprime\prime} L q'\}(t) = q^{\backprime\prime}(t)\, L(t)\, q'(t)$ almost everywhere in $\langle 0, \infty)$, and $\{q^{\backprime\prime} L q'\}(0) = 0$, we have for almost every $t \geqq 0$:

$$q^{\backprime\prime} L q' + \int_0^t q^{\backprime\prime}(L' + 2R)\, q'\, \mathrm{d}\tau + q^{\backprime} S q - \int_0^t q^{\backprime} S' q\, \mathrm{d}\tau = 0\,. \tag{3.22}$$

Due to the assumption of normality for (3.1), however, each term involved in (3.22) is non-negative so that for almost every $t \geqq 0$,

$$q^{\backprime\prime} L q' = 0\,, \quad q^{\backprime} S q = 0\,, \tag{3.23}$$

$$\int_0^t q^{\backprime\prime}(L' + 2R)\, q'\, \mathrm{d}\tau = 0\,, \quad \int_0^t q^{\backprime} S' q\, \mathrm{d}\tau = 0\,.$$

As the integrals in (3.23) are continuous, they are zero everywhere and we have

$$q^{\backprime\prime}(L' + 2R)\, q' = 0\,, \quad q^{\backprime} S' q = 0 \tag{3.24}$$

almost everywhere in $\langle 0, \infty)$. However, by a well-known theorem from algebra, (3.23) and (3.24) yield

$$Lq' = 0\,, \quad Sq = 0\,, \quad (L' + 2R)\, q' = 0\,, \quad S'q = 0 \tag{3.25}$$

almost everywhere in $\langle 0, \infty)$. At the same time, since both S and q are continuous, we have $Sq = 0$ everywhere, and consequently, $S'q + Sq' = 0$ almost everywhere.

Thus,

$$Sq' = 0 \tag{3.26}$$

almost everywhere.

If now $L + L' + 2R + S$ is definite for every $t \geqq 0$, then from (3.25) and (3.26) we have

$$(L + L' + 2R + S)\, q' = 0 \tag{3.27}$$

almost everywhere, and consequently, $x(t) = q'(t) = 0$ almost everywhere. Analogously, if $S - S'$ is definite for $t \geqq 0$, then again (3.25) yields $(S - S')\, q = 0$ everywhere, i.e. $q' = 0$ almost everywhere. Hence, Theorem 3.2 is proved.

Let us now consider the stability of a solution of (3.1).

Let $x(t)$ be the unique solution of (3.1) with initial condition c corresponding to the vector $f(t)$; the solution $x(t)$ will be called stable with respect to the initial condition if to every $\varepsilon > 0$ there is a $\delta > 0$ such that for every solution $\tilde{x}(t)$ of (3.1) with initial condition $\tilde{c}$ corresponding to $f(t)$, where $\tilde{c}$ fulfills the inequality $\|\tilde{c} - c\| < \delta$, we have $\|\tilde{x}(t) - x(t)\| < \varepsilon$ for every $t \geqq 0$.

Since eq. (3.1) is linear it is sufficient to investigate the dependence of the solution on c by the assumption that $f(t) = 0$. For this purpose, let us introduce the following notation:

Equation (3.1) will be said to satisfy one of the following conditions C_i, $i = 1, \ldots, 3$, if there is a positive number a_i, $i = 1, \ldots, 3$, such that for every $t \geqq 0$ and every constant vector ξ we have

$$\begin{aligned} \mathrm{C}_1 &: \quad \xi' L(t)\, \xi \geqq a_1 \|\xi\|^2\,, \\ \mathrm{C}_2 &: \quad \xi'(L'(t) + 2R(t))\, \xi \geqq a_2 \|\xi\|^2\,, \\ \mathrm{C}_3 &: \quad \xi' S(t)\, \xi \geqq a_3 \|\xi\|^2\,. \end{aligned}$$

Obviously, if a condition C_i is satisfied, the corresponding matrix is positive definite for every $t \geqq 0$.

Theorem 3.3. *Let* (3.1) *be a normal equation with* $f(t) = 0$; *if any one of conditions* C_i, $i = 1, \ldots, 3$, *is fulfilled and* $x(t) \in \mathscr{L}^2$ *is a solution of* (3.1) *with initial condition* c, *then* $x(t)$ *is determined uniquely in* $\mathscr{L}^2$ *and the following estimates are true*:

1. *If* C_1 *is true, then* $\|x(t)\| \leqq (a_1^{-1} c' \mathrm{L}(0)\, c)^{\frac{1}{2}}$ *for every* $t \geqq 0$.
2. *If* C_2 *is true, then* $\int_0^t \|x(\tau)\|^2 \, d\tau \leqq a_2^{-1} c' \mathrm{L}(0)\, c$ *for every* $t \geqq 0$.
3. *If both* C_1 *and* C_2 *are true, then*

$$\int_0^t \|x(\tau)\|^2 \, d\tau \leqq a_2^{-1}\, c' L(0)\, c \left(1 - \exp\left(-\frac{a_2}{a_1} t\right)\right) \tag{3.28}$$

for every $t \geqq 0$.

4. *If* C_3 *is true, then* $\|\int_0^t x(\tau)\,\mathrm{d}\tau\| \leqq (a_3^{-1}c'L(0)\,c)^{\frac{1}{2}}$ *for every* $t \geqq 0$.

Note 2. If C_1 is true, then, of course, a unique solution always exists, is continuous and $x(0) = c$.

Corollary. *If* (3.1) *is normal and condition* C_1 *is satisfied then every solution of* (3.1) *is stable with respect to the initial condition.*

For the proof of Theorem 3.3 the following assertion will be necessary:

Lemma 3.2. *Let* $a(t) \geqq 0$ *be non-decreasing in* $\langle 0, \infty)$, $\varphi(t)$ *non-negative in* $\langle 0, \infty)$ *and* $\kappa > 0$; *if for every* $t \geqq 0$

$$\varphi(t) + \kappa \int_0^t \varphi(\tau)\,\mathrm{d}\tau \leqq a(t)\,, \tag{3.29}$$

then

$$\int_0^t \varphi(\tau)\,\mathrm{d}\tau \leqq \kappa^{-1} a(t)\,(1 - \exp(-\kappa t)) \tag{3.30}$$

for every $t \geqq 0$.

Proof. Choose a $t \geqq 0$; then for every $\xi \in \langle 0, t\rangle$ and $\varepsilon > 0$ we have

$$\varphi(\xi) + \kappa \int_0^\xi \varphi(\tau)\,\mathrm{d}\tau \leqq a(\xi) < a(t) + \varepsilon\,. \tag{3.31}$$

From this it follows that

$$\frac{\kappa\varphi(\xi)}{a(t) + \varepsilon - \kappa \int_0^\xi \varphi(\tau)\,\mathrm{d}\tau} < \kappa\,; \tag{3.32}$$

thus, integrating (3.32) between 0 and ξ, we get

$$\ln \frac{a(t) + \varepsilon}{a(t) + \varepsilon - \kappa \int_0^\xi \varphi(\tau)\,\mathrm{d}\tau} < \kappa\xi\,, \tag{3.33}$$

i.e.

$$\int_0^\xi \varphi(\tau)\,\mathrm{d}\tau < \kappa^{-1}(a(t) + \varepsilon)\,(1 - e^{-\kappa\xi})\,.$$

Putting $\xi = t$ and letting $\varepsilon \to 0$, (3.30) follows.

Proof of Theorem 3.3: The first statement of the theorem is a direct consequence of Theorem 3.2. In order to derive the estimates let $x(t)$ be the solution of (3.1) with initial condition c, i.e., by definition,

$$L(t)\,x(t) + \int_0^t R(\tau)\,x(\tau)\,\mathrm{d}\tau + \int_0^t S(\tau) \int_0^\tau x(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau = L(0)\,c \tag{3.34}$$

almost everywhere in $\langle 0, \infty)$. Recalling Lemma 3.1, we have with the substitution $q(t) = \int_0^t x(\tau)\,\mathrm{d}\tau$:

$$\{Lq'\}(t) = L(0)\,c - \int_0^t R(\tau)\,q'(\tau)\,\mathrm{d}\tau - \int_0^t S(\tau)\,q(\tau)\,\mathrm{d}\tau\,, \tag{3.35}$$

so that $\{Lq'\}(0) = L(0)\,c$, and consequently, $\{q'^{\backprime}Lq'\}(0) = c^{\backprime}L(0)\,c$. Applying formula (3.5) to (3.35), we obtain similarly as in the proof of Theorem 3.2 that

$$q'^{\backprime}(t)\,L(t)\,q'(t) + \int_0^t q'^{\backprime}(L' + 2R)\,q'\,\mathrm{d}\tau + q^{\backprime}(t)\,S(t)\,q(t) - \\ - \int_0^t q^{\backprime}S'q\,\mathrm{d}\tau = c^{\backprime}L(0)\,c \tag{3.36}$$

almost everywhere. Since all the terms in (3.36) are non-negative, we have

$$q'^{\backprime}(t)\,L(t)\,q'(t) + \int_0^t q'^{\backprime}(L' + 2R)\,q'\,\mathrm{d}\tau \leqq c^{\backprime}L(0)\,c\,, \tag{3.37}$$

$$q^{\backprime}(t)\,S(\tau)\,q(t) - \int_0^t q^{\backprime}S'q\,\mathrm{d}\tau \leqq c^{\backprime}L(0)\,c \tag{3.38}$$

almost everywhere.

Now, if C_1 is fulfilled, then $q'(t)$ is continuous and (3.37) is satisfied everywhere in $\langle 0, \infty)$; hence

$$a_1\|q'(t)\|^2 = a_1\|x(t)\|^2 \leqq c^{\backprime}L(0)\,c\,,$$

Q.E.D.

If C_2 is satisfied, then from (3.37) it follows that

$$a_2\int_0^t \|q'(\tau)\|^2\,\mathrm{d}\tau \leqq c^{\backprime}L(0)\,c$$

for every $t \geqq 0$.

If both C_1 and C_2 are true, (3.37) is fulfilled everywhere and we have

$$a_1\|x(t)\|^2 + a_2\int_0^t \|x(\tau)\|^2\,\mathrm{d}\tau \leqq c^{\backprime}L(0)\,c\,. \tag{3.39}$$

But (3.28) follows from (3.39) by Lemma 3.2.

The remaining estimate can be deduced in the same manner from (3.38).

The method of proof of Theorem 3.3 permits us to establish an estimate for the solution of (3.1) also if $f(t) \neq 0$.

Theorem 3.4. *Let* (3.1) *be a normal equation fulfilling conditions* C_1 *and* C_2, *and let* $f(t) \in \mathscr{L}^2$. *Then the solution* $x(t)$ *with zero initial condition belongs to* $\mathscr{L}^2$

and

$$(3.40) \qquad \int_0^t \|x(\tau)\|^2 \, d\tau \leqq 4a_2^{-2} \left(1 - \exp\left(-\frac{a_2}{a_1} t\right)\right)^2 \int_0^t \|f(\tau)\|^2 \, d\tau$$

for every $t \geqq 0$.

Proof. Since C_1 is satisfied, a unique solution exists and obviously $x(t) \in \mathcal{L}^2$. By definition,

$$(3.41) \qquad L(t)\, x(t) + \int_0^t R(\tau)\, x(\tau)\, d\tau + \int_0^t S(\tau) \int_0^\tau x(\sigma)\, d\sigma\, d\tau = \int_0^t f(\tau)\, d\tau$$

almost everywhere in $\langle 0, \infty)$. In terms of Lemma 3.1 we have

$$(3.42) \qquad \{Lq'\}'(t) = f(t) - R(t)\, q'(t) - S(t)\, q(t)$$

with $q(t) = \int_0^t x(\tau)\, d\tau$, and $\{Lq'\}(0) = 0$. Using again formula (3.5) and arranging the equation as before, we get

$$(3.43) \qquad q'^{\backprime}(t)\, L(t)\, q'(t) + \int_0^t q'^{\backprime}(L' + 2R)\, q'\, d\tau + q'(t)\, S(t)\, q(t) - \\ - \int_0^t q^{\backprime} S' q\, d\tau = 2 \int_0^t q'^{\backprime} f\, d\tau$$

everywhere in $\langle 0, \infty)$. ($q'(t)$ is continuous.) Since all the terms of the left-hand side of (3.43) are non-negative, we have

$$(3.44) \qquad \int_0^t q'^{\backprime} f\, d\tau \geqq 0,$$

and, at the same time, $\int_0^t q'^{\backprime} f\, d\tau \leqq \int_0^t \|q'\|\, \|f\|\, d\tau \leqq \{\int_0^t \|q'\|^2\, d\tau\}^{\frac{1}{2}} \{\int_0^t \|f\|^2\, d\tau\}^{\frac{1}{2}} = h(t)$. Thus, (3.43) yields by C_1, C_2:

$$(3.45) \qquad a_1 \|q'\|^2 + a_2 \int_0^t \|q'(\tau)\|^2\, d\tau \leqq 2h(t).$$

By Lemma 3.2 we have from (3.45),

$$\int_0^t \|q'\|^2\, d\tau \leqq 2a_2^{-1}\left(1 - e^{-(a_2/a_1)t}\right) \left\{\int_0^t \|f\|^2\, d\tau\right\}^{\frac{1}{2}} \left\{\int_0^t \|q'\|^2\, d\tau\right\}^{\frac{1}{2}}.$$

But from this (3.40) follows immediately.

Note 3. The integral $\int_0^t q'^{\backprime} f\, d\tau$ has the physical meaning of energy supplied by outer forces to the system in the time-span $0 \div t$. Since it is non-negative for any $t \geqq 0$ by (3.44), the physical system described by (3.1) is incapable of producing energy. Thus, the normality defined above characterizes the passivity of the system.

4. The solution $x(t)$ of equation (1.1), which satisfies the conditions of Th. 1.1, depends, roughly speaking, on $f(t)$ and $f'(t)$. However, the conditions given in Th. 1.1 may easily be generalized to the case that $x(t)$ depends on $f(t), f'(t), \ldots, f^{(n)}(t)$. For this purpose let us introduce the following notation:

Let $A(t)$ be an $r \times r$ matrix having a continuous derivative on $I = \langle t_1, t_2 \rangle$, and let $W(t, \tau)$ be an $r \times r$ matrix continuous on $I^2 = \underset{t,\tau}{E}[t_1 \leqq \tau \leqq t \leqq t_2]$, which has a continuous derivative $\partial W / \partial t$ on I^2.

Let regular constant $r \times r$ matrices C_1, C_2 exist such that

$$\tilde{A}(t) = C_1\, A(t)\, C_2 = \left[\begin{array}{c|c} A_{11}(t) & A_{12}(t) \\ \hline A_{21}(t) & A_{22}(t) \end{array}\right], \tag{4.1}$$

where rank $A_{11}(t) =$ rank $A(t)$ on I and $|\det A_{11}(t)| \geqq c > 0$.

Let $P(t)$ be a matrix fulfilling the equalities

$$P(t)\, A_{11}(t) + A_{21}(t) = 0\,, \quad P(t)\, A_{12}(t) + A_{22}(t) = 0 \tag{4.2}$$

on I, and let

$$Y(t) = \left[\begin{array}{c|c} 0 & 0 \\ \hline P(t) & I \end{array}\right]. \tag{4.3}$$

(Obviously, $P(t)$, $Y(t)$ possess a continuous derivative on I). Moreover, let

$$\begin{aligned} \hat{A}(t) &= \tilde{A}(t) + Y(t)\, \tilde{W}^*(t)\,, \\ \hat{W}(t, \tau) &= \tilde{W}(t, \tau) + \frac{\partial}{\partial t}(Y(t)\, \tilde{W}(t, \tau))\,, \end{aligned} \tag{4.4}$$

where $\tilde{W}(t, \tau) = C_1\, W(t, \tau)\, C_2$. Then $\hat{P} = (\hat{A}, \hat{W})$ will be called the derived pair to $P = (A, W)$ and this fact will be symbolized by $P \to \hat{P}$ on I.

If particurlarly for a pair $P = (A, W)$ we have $|\det A(t)| \geqq d > 0$ on I, P will be called regular.

Furthermore, let $f(t)$ be an r-dimensional integrable vector function defined on I. Denoting $\tilde{f}(t) = C_1 f(t)$, let $Y(t) \tilde{f}(t)$ be absolutely continuous on I and let

$$\hat{f}(t) = \tilde{f}(t) + (Y(t)\tilde{f}(t))'\,. \tag{4.5}$$

Then the triple $\hat{T} = (\hat{A}, \hat{W}, \hat{f})$ will be called the derived triple to $T = (A, W, f)$ and we shall write $T \to \hat{T}$ on I. If in addition $(Y\tilde{f})(t_1) = 0$, then $\hat{T}$ will be called equivalent to T, and we shall use the notation $T \Leftrightarrow \hat{T}$ on I.

The triple T will be called regular, if $|\det A(t)| \geqq d > 0$ on I.

Lemma 4.1. *Let*

$$A(t)\, x(t) + \int_{t_1}^{t} W(t, \tau)\, x(\tau)\, d\tau = f(t)\,, \tag{4.6}$$

$$\hat{A}(t)\, \hat{x}(t) + \int_{t_1}^{t} \hat{W}(t, \tau)\, \hat{x}(\tau)\, d\tau = \hat{f}(t)\,, \quad t \in I\,. \tag{4.7}$$

Then

a) *if* $x(t)$ *is a solution of* (4.6) *and* $T \to \hat{T}$, $\hat{x}(t) = C_2^{-1}x(t)$ *is a solution of* (4.7) *and* $(Y\hat{f})(t_1) = 0$.

b) *if* $\hat{x}(t)$ *is a solution of* (4.7) *and* $T \Leftrightarrow \hat{T}$, $x(t) = C_2\hat{x}(t)$ *is a solution of* (4.6).

Proof: a) Putting $x = C_2\hat{x}$ into (4.6) and multiplying by C_1, we have

$$\tilde{A}\hat{x} + \int_{t_1}^{t} \tilde{W}\hat{x}\, d\tau = \hat{f}. \tag{4.8}$$

Due to the equality

$$Y(t)\, \tilde{A}(t) = 0 \tag{4.9}$$

we have

$$\int_{t_1}^{t} Y(t)\, \tilde{W}(t, \tau)\, \hat{x}(\tau)\, d\tau = (Y\hat{f})(t), \tag{4.10}$$

and consequently

$$Y(t)\, \tilde{W}^*(t)\, \hat{x} + \int_{t_1}^{t} \frac{\partial}{\partial t}(Y(t)\, \tilde{W}(t, \tau))\, \hat{x}(\tau)\, d\tau = (Y\hat{f})'(t). \tag{4.11}$$

Adding this to (4.8), we get (4.7). Moreover, from (4.10), $(Y\hat{f})(t_1) = 0$.

In order to prove b), define the matrix $N(t)$ on I by

$$N(t) = \left[\begin{array}{c|c} I & 0 \\ \hline P(t) & I \end{array}\right]; \tag{4.12}$$

obviously, $\det N(t) = 1$ and $N(t)\, Y(t) = Y(t)$. Now, let $\hat{x}(t)$ be a solution of (4.7) and let $T \Leftrightarrow \hat{T}$. Multiplying (4.7) by $N(t)$, we get

$$(N\tilde{A} + Y\tilde{W}^*)\, \hat{x} + \int_{t_1}^{t} \left(N\tilde{W} + N\frac{\partial}{\partial t}(Y\tilde{W})\right) \hat{x}\, d\tau = N\hat{f} + N(Y\hat{f})'. \tag{4.13}$$

But it can be easily verified that

$$N\tilde{A} + Y\tilde{W}^* = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline U_1^* & U_2^* \end{array}\right], \tag{4.14}$$

$$N\tilde{W} + N\frac{\partial}{\partial t}(Y\tilde{W}) = \left[\begin{array}{c|c} \tilde{W}_{11} & \tilde{W}_{12} \\ \hline U_1 + \dfrac{\partial U_1}{\partial t} & U_2 + \dfrac{\partial U_2}{\partial t} \end{array}\right], \tag{4.15}$$

$$N\hat{f} + N(Y\hat{f})' = \left[\begin{array}{c} \hat{f}_1 \\ \hline g + g' \end{array}\right], \quad \hat{f} = \left[\begin{array}{c} \hat{f}_1 \\ \hline \hat{f}_2 \end{array}\right], \tag{4.16}$$

where $U_1 = P\tilde{W}_{11} + \tilde{W}_{21}$, $U_2 = P\tilde{W}_{12} + \tilde{W}_{22}$, $g = P\tilde{f}_1 + \tilde{f}_2$ and

$$\tilde{W} = \left[\begin{array}{c|c} \tilde{W}_{11} & \tilde{W}_{12} \\ \hline \tilde{W}_{21} & \tilde{W}_{22} \end{array}\right].$$

Putting $\hat{x}' = [\hat{x}_1' \mid \hat{x}_2']$ and using (4.14), (4.15), (4.16), we have from (4.13)

$$A_{11}\hat{x}_1 + A_{12}\hat{x}_2 + \int_{t_1}^{t} \tilde{W}_{11}\hat{x}_1 \, d\tau + \int_{t_1}^{t} \tilde{W}_{12}\hat{x}_2 \, d\tau = \tilde{f}_1 , \tag{4.17}$$

$$U_1^*\hat{x}_1 + U_2^*\hat{x}_2 + \int_{t_1}^{t} \left(U_1 + \frac{\partial U_1}{\partial t}\right) \hat{x}_1 \, d\tau + \int_{t_1}^{t} \left(U_2 + \frac{\partial U_2}{\partial t}\right) \hat{x}_2 \, d\tau = g + g' . \tag{4.18}$$

However, with $u(t) = \int_{t_1}^{t} U_1\hat{x}_1 \, d\tau + \int_{t_1}^{t} U_2\hat{x}_2 \, d\tau$ we get from (4.18),

$$u' + u = g + g' . \tag{4.19}$$

Consequently, $u = g + c \exp(-t)$ on I, c being a constant vector. Since by assumption $(Y\tilde{f})(t_1) = 0$, i.e. $g(t_1) = 0$, and $u(t_1) = 0$, it follows that $c = 0$.

Hence, equations (4.17), (4.18) are equivalent to

$$A_{11}\hat{x}_1 + A_{12}\hat{x}_2 + \int_{t_1}^{t} \tilde{W}_{11}\hat{x}_1 \, d\tau + \int_{t_1}^{t} \tilde{W}_{12}\hat{x}_2 \, d\tau = \tilde{f}_1 , \tag{4.20}$$

$$\int_{t_1}^{t} (P\tilde{W}_{11} + \tilde{W}_{21}) \hat{x}_1 \, d\tau + \int_{t_1}^{t} (P\tilde{W}_{12} + \tilde{W}_{22}) \hat{x}_2 \, d\tau = P\tilde{f}_1 + \tilde{f}_2 , \tag{4.21}$$

i.e., by (4.12), to

$$N\tilde{A}\hat{x} + \int_{t_1}^{t} N\tilde{W}\hat{x} \, d\tau = N\tilde{f} . \tag{4.22}$$

Finally, multiplying (4.22) by $N^{-1}(t)$ and putting $\tilde{A} = C_1AC_2$, $\tilde{W} = C_1WC_2$, $\tilde{f} = C_1f$ and $C_2\hat{x} = x$, we get (4.6), Q.E.D.

Theorem 4.1. *Let $A(t)$ be an $r \times r$ matrix defined on $\langle 0, \infty)$, $W(t, \tau)$ an $r \times r$ matrix defined on $0 \leqq \tau \leqq t < \infty$; let an integer $n \geqq 1$ exist such that*

a) *$A(t)$ has a continuous n-th derivative in $\langle 0, \infty)$,*

b) *$\partial^n W(t, \tau)/\partial t^n$ is continuous on $0 \leqq \tau \leqq t < \infty$ and $(\partial^i W/\partial t^i)^*$ has a continuous $n - i - 1$-th derivative in $\langle 0, \infty)$ for $i = 0, 1, \ldots, n - 1$. Let $f(t)$ be an r-dimensional integrable vector on $\langle 0, \infty)$.*

Moreover, to each $t > 0$ let a closed interval I_t containing t inside exist such that

$$T_0 = (A, W, f) \to T_1^t \to T_2^t \to \ldots \to T_{n_t}^t \quad on \quad I_t , \tag{4.23}$$

where $0 \leqq n_t \leqq n$ *and* $T^t_{n_t}$ *is regular; furthermore, let an interval* $I_0 = \langle 0, \bar{t} \rangle$ *exist such that*

$$T_0 \Leftrightarrow T^0_1 \Leftrightarrow T^0_2 \Leftrightarrow \ldots \Leftrightarrow T^0_{n_0} \quad on \quad I_0 \tag{4.24}$$

with $0 \leqq n_0 \leqq n$ *and* $T^0_{n_0}$ *regular.*

Then the equation

$$A(t)\, x(t) + \int_0^t W(t, \tau)\, x(\tau)\, \mathrm{d}\tau = f(t) \tag{4.25}$$

possesses a unique integrable solution.

If in addition $f(t)$ *possesses an integrable n-th derivative in* $\langle 0, \infty \rangle$, *then condition* (4.23) *may be replaced by*

$$P_0 = (A, W) \to P^t_1 \to P^t_2 \to \ldots \to P^t_{n_t} \quad on \quad I_t\,. \tag{4.26}$$

Proof. From Borel's theorem it follows that there is a sequence of closed intervals $I_i = \langle t_i, t^*_i \rangle$, $i = 1, 2, \ldots$ such that

a) $t_1 = 0,\ t_i < t_{i+1} < t^*_i < t^*_{i+1},\ i = 1, 2, \ldots,$

b) $T_0 \to T^i_1 \to T^i_2 \to \ldots \to T^i_{n_i}$ on I_i, $i = 2, 3, \ldots,$

c) $T_0 \Leftrightarrow T^1_1 \Leftrightarrow T^1_2 \Leftrightarrow \ldots \Leftrightarrow T^1_{n_1}$ on I_0,

where $T^i_{n_i}$ are regular and $n_i \leqq n$.

Assume that the solution $x(t)$ of (4.25) has already been established on $\langle 0, t^*_{k-1} \rangle = \bigcup_{i=1}^{k-1} I_i$, that it is integrable and uniquely determined. Thus, almost everywhere on $\langle 0, t^*_{k-1} \rangle$ we have

$$A(t)\, x(t) + \int_{t_k}^t W(t, \tau)\, x(\tau)\, \mathrm{d}\tau = g_{k-1}(t) \tag{4.27}$$

with

$$g_{k-1}(t) = f(t) - \int_0^{t_k} W(t, \tau)\, x(\tau)\, \mathrm{d}\tau\,. \tag{4.28}$$

Observe that the vector $\int_0^{t_k} W(t, \tau)\, x(\tau)\, \mathrm{d}\tau$ in (4.28) is defined for every $t \geqq 0$ and possesses a continuous derivative of n-th order.

On the other hand, (4.27) should be satisfied on I_k, i.e. at least on $\langle t^*_{k-1}, t^*_k \rangle$, for it is satisfied on $\langle t_k, t^*_{k-1} \rangle \subset I_k$ by assumption. Multiplying (4.27) by $Y^k_1(t)$, we have

$$Y^k_1(t) \int_{t_k}^t W(t, \tau)\, x(\tau)\, \mathrm{d}\tau = Y^k_1(t)\, g_{k-1}(t)\,. \tag{4.29}$$

Thus, from (4.29),

$$(Y^k_1 g_{k-1})\,(t_k) = 0 \tag{4.30}$$

so that by definition, $T_0 = (A, W, g_{k-1}) \Leftrightarrow (A_1, W_1, g^1_{k-1}) = T^k_1$. Consequently, by Lemma 4.1, (4.27) is equivalent to

$$A_1(t)\, x_1(t) + \int_{t_k}^{t} W_1(t, \tau)\, x_1(\tau)\, d\tau = g^1_{k-1}(t) \tag{4.31}$$

on I_k.

Since (4.31) is satisfied on $\langle t_k, t^*_{k-1} \rangle$, we obtain further

$$(Y^k_2 g^1_{k-1})(t_k) = 0 \tag{4.32}$$

so that $T^k_1 \Leftrightarrow T^k_2$. Continuing this process we get finally $T_0 \Leftrightarrow T^k_1 \Leftrightarrow T^k_2 \Leftrightarrow \ldots \Leftrightarrow T^k_{n_k}$; hence, by Lemma 4.1, (4.27) is equivalent to

$$A_{n_k}(t)\, x_{n_k}(t) + \int_{t_k}^{t} W_{n_k}(t, \tau)\, x_{n_k}(\tau)\, d\tau = g^{n_k}_{k-1}(t)\,. \tag{4.33}$$

As $|\det A_{n_k}(t)| \geqq p_k > 0$ on I_k by assumption, (4.33) has a unique integrable solution on I_k by Volterra's theorem. Hence, there is a unique integrable vector $x(t)$ on I_k which fulfills (4.27) and consequently, (4.25).

Repeating this procedure for I_0 making use of property c) instead of b), we easily conclude that there is a unique integrable vector $x(t)$ which fulfills (4.25) on I_0.

Finally, if $f(t)$ has an integrable n-th derivative in $\langle 0, \infty)$, then obviously (4.26) implies (4.23), since $Y(t) f(t)$ has an n-th integrable derivative, etc. Hence, Th. 4.1 is proved.

Note 4. Observe that the following statement is true:

Let the assumptions of Th. 4.1 *be satisfied; then there are fixed integers* $1 \leqq \leqq r_0 \leqq r_1 \leqq r_2 \leqq \ldots \leqq r_n = r$ *such that for any choice of* $\bar{\imath} \geqq 0$ *we have*

$$\operatorname{rank} A^i_{\bar{\imath}}(t) = r_i \tag{4.34}$$

on $I_{\bar{\imath}}$ *with* $T^{\bar{\imath}}_i = (A^i_{\bar{\imath}}, W^i_{\bar{\imath}}, f^i_{\bar{\imath}})$, $T^{\bar{\imath}}_0 = (A, W, f)$.

Proof. From Borel's theorem it follows that it suffices to prove the equality of ranks only for intervals I_1, I_2 such that $I_1 = \langle a, b \rangle$, $I_2 = \langle c, d \rangle$, $a < c < b < d$. Suppose that $\alpha = \operatorname{rank}_{I_1} A(t) \neq \operatorname{rank}_{I_2} A(t) = \beta$ and that, for example, $\alpha < \beta$. Then, of course, equalities (4.2) cannot be true on $\langle c, b \rangle$, i.e. they cannot be true on the entire interval I_1, which is a contradiction. Thus, we have rank $A(t) = r_0$ everywhere on $\langle 0, \infty)$. By the same argument we get $\operatorname{rank} A^i_{I_1}(t) = \operatorname{rank} A^i_{I_2}(t)$, $i = 1, 2, \ldots, n$.

The inequality $r_k \leqq r_{k+1}$ is an obvious consequence of (4.4).

It is apparent that the statement just proved may be reversed, i.e. that equalities (4.34) imply (4.26).

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Résumé

NĚKTERÉ VLASTNOSTI NEKANONICKÝCH SYSTÉMŮ LINEÁRNÍCH INTEGRO-DIFERENCIÁLNÍCH ROVNIC

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V práci se vyšetřuje vektorová integrální rovnice (1.1) a její speciální případ (0.1).

V prvé části jsou stanoveny podmínky pro existenci a jednoznačnost řešení rovnice (1.1) v případě, kdy její pravá strana je integrovatelný vektor, a je odvozena věta o závislosti řešení na pravé straně.

Druhá část je věnována podmínkám existence a jednoznačnosti řešení rovnice (1.1), kdy její pravá strana je vektorem distribucí.

Ve třetí části se uvažuje rovnice (0.1), zejména její „normální" typ, tj. kdy matice $L(t)$, $L'(t) + 2R(t)$, $S(t)$, $-S'(t)$ jsou symetrické a positivně semidefinitní pro všechna $t \geqq 0$. Je dokázána věta o jednoznačnosti řešení, jehož norma je lokálně integrovatelná s kvadrátem, věta o stabilitě a konečně některé odhady pro čtverec normy řešení.

V poslední části práce je pak poukázáno na jedno zobecnění výsledků prvé části.

Резюме

НЕКОТОРЫЕ СВОЙСТВА НЕКАНОНИЧЕСКИХ СИСТЕМ ЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal), Прага

В работе исследуется векторное интегральное уравнение (1.1) и его частный случай (0.1).

В первой части установлены условия существования и единственности решения уравнения (1.1) в случае, когда его правая часть — интегрируемый вектор, и выводится теорема о зависимости решения от правой части.

Вторая часть посвящается условиям существования и единственности решения уравнения (1.1) в случае, когда правая часть — вектор обобщенных функций.

В третьей части исследуется уравнение (0.1), в особенности его „нормальный" тип, т.е. случай, когда матрицы $L(t)$, $L'(t) + 2R(t)$, $S(t)$, $-S'(t)$ симметрические и положительно полуопределенные для всех $t \geqq 0$. Доказывается теорема о единственности решения, норма которого локально интегрируема с квадратом, теорема об устойчивости и, наконец, даются некоторые оценки для квадрата нормы решения.

В последней части работы проводится одно обобщение результатов первой части.