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ON A CERTAIN ORDERING OF THE VERTICES OF A TREE

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This paper proves the necessary and sufficient condition under which it is possible to order the vertices of given finite tree T into a simple sequence, every two neighbour vertices of this sequence having the distance at most 2 in the metric of the tree T.

Our considerations will refer to finite, connected, non-oriented graphs without circles, which have at least two vertices. We call these graphs *trees* (see for example C. BERGE [1], p. 165). Consider a tree T. Let $\{T\}$ denote the set of the vertices of this tree. By the *order of vertex* z in the tree T we mean the number of different edges of the tree T, which the vertex z coincides with. When vertices $z_1, z_2 \in \{T\}$ are joined with an edge, denote this edge by (z_1, z_2). We say that $(z_1, \ldots, z_{n+1})$, $n \geqq 1$, is a *path* of length n of the tree T when z_i are mutually different vertices of the tree T for $i = 1, \ldots, n + 1$ and (z_i, z_{i+1}) are edges of the tree T for $i = 1, \ldots, n$. The path $(z_1, \ldots, z_{n+1})$ connects vertices z_1 and z_{n+1}.

Let $(z_1, \ldots, z_{n+1})$ be a path of the tree T. By the *distance* between vertices z_1 and z_{n+1} we mean the number $\mu(z_1, z_{n+1}) = n$.[1]) $z_2, \ldots, z_n$ are the *inner vertices* of the path $(z_1, \ldots, z_{n+1})$.

Let $a, b \in \{T\}$, $a \neq b$. We say that the tree T is $2 - (a, b)$*-orderable* when there exists an ordering of the set $\{T\}$ into the simple sequence $a = t_1, t_2, \ldots, t_s = b$, where $s = \operatorname{card}\{T\}$, $t_i \in \{T\}$ for $i = 1, \ldots, s$, $\mu(t_i, t_{i+1}) \leqq 2$ for $i = 1, \ldots, s - 1$.

We say that the tree T is 2-*orderable* when there exist vertices $a, b \in \{T\}$ such that T is $2 - (a, b)$-orderable. In this paper the necessary and sufficient conditions under which the given tree T is $2 - (a, b)$-orderable or 2-orderable are proved. The more general ordering of vertices of a general graph were worked out by M. Sekanina but does not include $2 - (a, b)$-ordering [2], [3].

In what follows the letter T, with appropriate indices will denote trees, and small letters, with appropriate indices will denote their vertices. Insofar as we shall mention sequences or subsequences, we shall have in mind only finite ones. Introducing in what

[1]) Note that for $a, b \in \{T\}$, $a \neq b$, there exists in the graph T (which is a tree) just one path connecting them (see C. BERGE [1], p. 165, theorem 1 (6)).

follows a sequence $\ldots, x, a_1, a_2, \ldots, a_n, y, \ldots$, if $n = 0$, we shall mean a sequence $\ldots, x, y, \ldots$

Let (x, y) be an edge of the tree T and let the order of a vertex x be 1. Construct the tree T_1 from the tree T in such a way that $\{T_1\} = \{T\} - x$ and let edges of the tree T_1 be just those edges of the tree T, which do not coincide with the vertex x. Then we say that the tree T_1 was constructed from the tree T by omitting the vertex x and the other way around that the tree T was constructed from the tree T_1 by adding the edge (x, y) to the vertex y.

Lemma 1. *Let T_0 be 2-(a, b)-orderable. Let T be the tree which we get from T_0 by adding further edges to vertices $u_i \in \{T_0\}$, $i = 1, \ldots, q$ when there exists a subset M of vertices u_i $(i = 1, \ldots, q)$ and a $1-1$ mapping φ of the set M into the set of edges of a tree T_0 such that either α) $u_i \in M$ and $\varphi(u_i) = (u_i, v_i)$, where v_i is a neighour vertex to a vertex u_i in a given 2-(a, b)-ordering of the tree T_0, or β) in a given 2-(a, b)-ordering of T_0 there are successive vertices t_j, t_{j+1}, for which in T_0 $\mu(t_j, u_i) = \mu(t_{j+1}, u_i) = 1$.*

Then the tree T is 2-(a, b)-orderable.

Proof of lemma 1. Let a tree T_1 be constructed from the tree T_0 by adding k edges (a_p, u_1) to a vertex u_1, $p = 1, \ldots, k$, $k > 0$.

Case α). A 2-(a, b)-ordering of T_0 has the form $\ldots, x, u_1, z, \ldots$ and let $\varphi(u_1) = (x, u_1)$ without the loss of generality, so $\mu(x, u_1) = 1$. Then $\ldots, x, a_1, \ldots, a_k, u_1, z, \ldots$ is a 2-(a, b)-ordering of the tree T_1, because $\mu(x, u_1) = 1$, $\mu(a_1, u_1) = 1$, or $\mu(x, a_1) = 2$; $\mu(a_p, u_1) = 1$, $\mu(a_{p+1}, u_1) = 1$, or $\mu(a_p, a_{p+1}) = 2$ for $p = 1, \ldots, k - 1$; $\mu(a_k, u_1) = 1$.

Case β). A 2-(a, b)-ordering of the tree T_0 has the form $\ldots, t_j, t_{j+1}, \ldots$, when $\mu(t_j, u_1) = \mu(t_{j+1}, u_1) = 1$. Then $\ldots, t_j, a_1, \ldots, a_k, t_{j+1}, \ldots$ is a 2-(a, b)-ordering of the three T_1 because

$$\mu(t_j, u_1) = 1,\ \mu(u_1, a_1) = 1, \text{ or } \mu(t_j, a_1) = 2;$$
$$\mu(a_p, u_1) = 1,\ \mu(a_{p+1}, u_1) = 1, \text{ or } \mu(a_p, a_{p+1}) = 2 \text{ for } p = 1, \ldots, k - 1;$$
$$\mu(a_k, u_1) = 1,\ \mu(u_1, t_{j+1}) = 1, \text{ or } \mu(a_k, t_{j+1}) = 2.$$

Let a tree T_i be constructed from a tree T_{i-1} by adding edges to the vertex u_i, $i = 1, \ldots, q$. Evidently T_q is the tree T. Analogous to the manner in which we got from a 2-(a, b)-ordering of the tree T_0 to a 2-(a, b)-ordering of the tree T_1, we can also find a 2-(a, b)-ordering of trees $T_2, \ldots, T_q$, because the described construction of a 2-(a, b)-ordering has the following property: When for a vertex u_i conditions α) or β) hold with respect to a 2-(a, b)-ordering of the tree T_0, then they hold for u_i in all trees $T_1, T_2, \ldots, T_{i-1}$ too, in which a 2-(a, b)-ordering is introduced by means of the above mentioned construction. This last statement will be evident if we realize that in both cases α) and β) a 2-(a, b)-ordering of the tree T_i differs from a 2-(a, b)-ordering of the tree T_{i-1} only between two members of the sequence, which are simply assigned to the vertex u_i.

Lemma 2. *When T is 2-(a, b)-orderable, then the tree T_0, which we can get by omitting (some or all) vertices of the order 1 of T with the exception of vertices a, b, is also 2-(a, b)-orderable.*

Proof of lemma 2. Let $a, \ldots, x, a_1, \ldots, a_k, y, \ldots, b$ be a 2-(a, b)-ordering of T and $a_i \in \{T\}$, $a_i \bar{\in} \{T_0\}$ for $i = 1, \ldots, k$; $x, y \in \{T_0\}$. The order of vertices a_i is 1 and therefore to every a_i there exists just one vertex $c_i \in \{T_0\}$ such that $\mu(a_i, c_i) = 1$ for $i = 1, \ldots, k$. From the 2-(a, b)-ordering of T it follows that $\mu(a_i, a_{i+1}) \leqq 2$ for $i = 1, \ldots, k - 1$. For this reason $c_i = c_{i+1}$ for $i = 1, \ldots, k - 1$, because otherwise $\mu(a_i, a_{i+1}) \geqq 3$. So there exists $c = c_i \in \{T_0\}$, $i = 1, \ldots, k$ such that $\mu(a_i, c) = 1$ for $i = 1, \ldots, k$. It follows from the 2-(a, b)-ordering of T that $\mu(x, a_1) \leqq 2$, $\mu(a_k, y) \leqq 2$. Further we have shown that $\mu(a_1, c) = 1$, $\mu(a_k, c) = 1$, so $\mu(x, c) \leqq 1$, $\mu(y, c) \leqq 1$, or $\mu(x, y) \leqq 2$. Consequently $a, \ldots, x, y, \ldots, b$ is a 2-(a, b)-ordering of the tree T_0.

Note. From lemma 2 it is easy to obtain: *When T is 2-(a, b)-orderable then each subtree T_0 of the tree T for which $a, b \in \{T_0\}$ is also 2-(a, b)-orderable. When the tree T_0 is not 2-(a, b)-orderable then any tree T, which has T_0 as its subtree is not 2-(a, b)-orderable.*

Lemma 3. *Let (c_0, c_1) be an edge of a tree T, the orders of c_0 and of c_1 being greater than 1. Let $a, b \in \{T\}$, $a \neq b$, be given such that the path connecting a and b does not contain c_1. Denote by $\{S\}$ the set of vertices of the tree T, which can be connected with the vertex c_1 by paths not containing the vertex c_0. Let $a \bar{\in} \{S\}$, $b \bar{\in} \{S\}$.*

If there exists a 2-(a, b)-ordering of the tree T then it must be either of the form α: $a, \ldots, c_0, s_1, \ldots, s_k, c_1, \ldots, b$ or of the form β: $a, \ldots, c_1, s_1, \ldots, s_k, c_0, \ldots, b$, where $s_1, \ldots, s_k$ is a suitable ordering of the set $\{S\}$, $k = \text{card}\,\{S\}$.

Proof of lemma 3. Notice at first that $k \geqq 1$, because the order of c_1 is at least 2. Assume that the assertion of lemma 3 does not hold. Then either there exists at least one vertex not belonging to the set $\{S\}$ between vertices c_0 and c_1 in a 2-(a, b)-ordering of the tree T, or there exists at least one vertex of the set $\{S\}$, which does not occur in a 2-(a, b)-ordering of the tree T between vertices c_0 and c_1. Then in each of these cases neighbour vertices x, y in a 2-(a, b)-ordering of the tree T can be found such that $x \in \{S\}$, $y \bar{\in} \{S\}$, $x \neq c_0 \neq y$, $x \neq c_1 \neq y$. According to the construction of the set $\{S\}$ the path connecting vertices x and y must contain vertices c_0 and c_1, so $\mu(x, y) \geqq 3$. But this is a contradiction because vertices x and y are neighbouring in a given 2-(a, b)-ordering of the tree T.

Choose a vertex $c_0 \in \{T\}$ and different edges (c_0, c_1), (c_0, c_2), (c_0, c_3) of the tree T, the orders of vertices c_1, c_2, c_3 being greater than 1. For $i = 1, 2, 3$ denote by $\{S_i\}$ the set of all those vertices of the tree T, which can be connected with a vertex c_i by a path not containing the vertex c_0. Let vertices $a \neq b$ of the tree T be given such that they do not belong to the set $\{S_1\} \cup \{S_2\} \cup \{S_3\} \cup \{c_1\} \cup \{c_2\} \cup \{c_3\}$. Notice that $\{S_i\} \neq 0$ for $i = 1, 2, 3$, because the order of c_i is greater than 1. Call such a vertex c_0 of the tree T a *vertex of type I with respect to vertices a, b*. In what follows we shall omit, "with respect to vertices a, b", when mentioning both vertices of type I and vertices of further types.

Lemma 4. *If a tree T contains a vertex c_0 of type I then it is not 2-(a, b)-orderable.*

P r o o f o f l e m m a 4. Consider a tree T_1 constructed from the tree T in such a way that instead of $\{S_i\}$ we retain only one vertex s_i lying in $\{S_i\}$, which is connected with c_i by an edge, for $i = 1, 2, 3$. For illustration the tree T is in fig. 1 and from it is con-

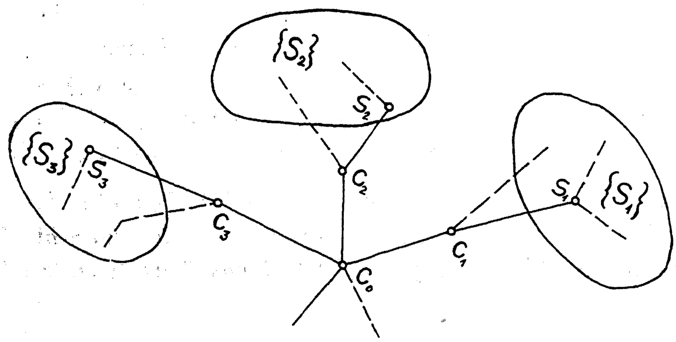

Fig. 1.

structed the tree T_1 in fig. 2. The other vertices and edges are without any change. According to the note following lemma 2 it suffices to show that the tree T_1 is not 2-(a, b)-orderable. For this purpose we make full use of lemma 3. According to this lemma every 2-(a, b)-ordering of the tree T_1 must contain three subsequences:

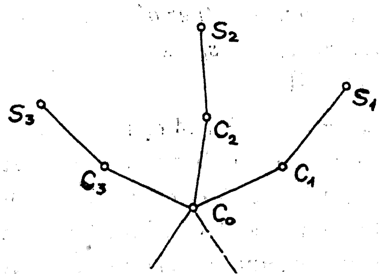

Fig. 2.

$$c_0, s_1, c_1 \quad \text{resp.} \quad c_1, s_1, c_0 ,$$
$$c_0, s_2, c_2 \quad \text{resp.} \quad c_2, s_2, c_0 ,$$
$$c_0, s_3, c_3 \quad \text{resp.} \quad c_3, s_3, c_0 .$$

As a 2-(a, b)-ordering of the tree T_1 is a simple ordering of the set $\{T_1\}$, we conclude that it is not possible to place these tree sequences in such a way that the vertex c_0 may occur only once; therefore it is not possible to 2-(a, b)-order the tree T_1 nor consequently the tree T.

Let $a \neq b$, $a, b \in \{T\}$. The following consideration is quite analogous for either a vertex a or for a vertex b. Therefore let us consider a vertex $a \in \{T\}$. Let (a, c_1), (a, c_2) be two different edges of the tree T, the order of c_1 and c_2 being greater than 1. For $i = 1, 2$ denote by $\{S_i\}$ the set of vertices of the tree T, which can be connected with the vertex c_i by a path not containing the vertex a. Do not let the vertex b belong to the set $\{S_1\} \cup \{S_2\} \cup \{c_1\} \cup \{c_2\}$. Then call the vertex a a *vertex of type II*. Evidently $\{S_i\} \neq \emptyset$, because the order of a vertex c_i is greater than 1 for $i = 1, 2$. (For illustration see fig. 3.) Notice that according to the definition a vertex of type II can be solely the vertex a or b.

Lemma 5. *If the vertex a or b of the tree T is of type II, then the tree T is not 2-(a, b)-orderable.*

Proof of lemma 6. Without any loss of generality let a be the vertex of type II. Consider the tree T_1, constructed from the tree T in such a way that instead of $\{S_i\}$ we keep only one vertex s_i lying in $\{S_i\}$, which is connected with c_i by an edge, for $i = 1, 2$. For illustration we give in fig. 3 the tree T, and in fig. 4 the tree T_1, constructed from T. The remaining vertices and edges we leave without any change. In the sense of the note following lemma 2 it suffices to show that the tree T_1 is not 2-(a, b)-orderable. In the sense of lemma 3 every 2-(a, b)-ordering of the tree T_1 is bound to contain two sequences:

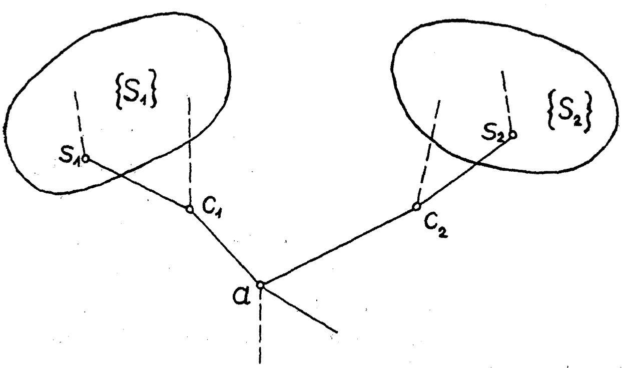

Fig. 3.

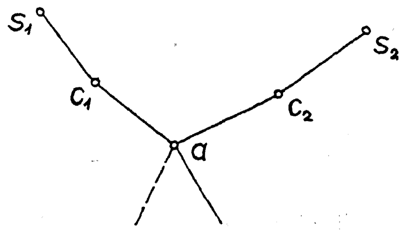

Fig. 4.

$$a, s_1, c_1\,, \quad \text{resp.} \quad a, c_1, s_1\,,$$

$$a, s_2, c_2\,, \quad \text{resp.} \quad a, c_2, s_2\,.$$

As a 2-(a, b)-ordering starts with the vertex a, this is not possible and consequently no 2-(a, b)-ordering of the tree T_1 nor of the tree T exists.

Let $a \neq b$, $a, b \in \{T\}$. Do not let $c_0 \in \{T\}$ lie on the path connecting vertices a and b. Let (c_0, c_1), (c_0, c_2), (c_0, c_3) be tree different edges of the tree T, the orders of vertices c_i being greater than 1 for $i = 1, 2, 3$. For $i = 1, 2, 3$ denote by $\{S_i\}$ the set of all vertices of

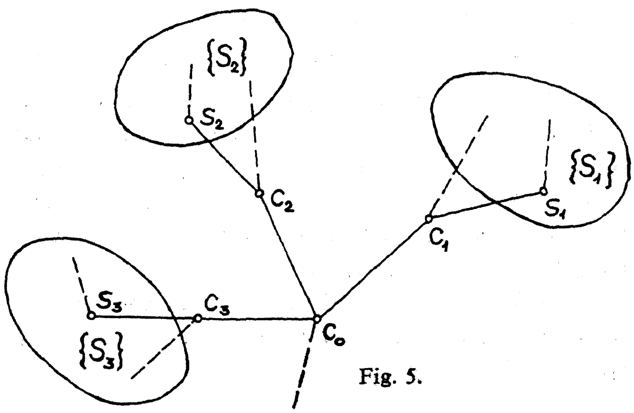

Fig. 5.

the tree T, which can be connected with a vertex c_i by a path not containing a vertex c_0. Let $a, b \in \{S_1\} \cup \{c_1\}$. Call the vertex c_0 a *vertex of type III*. For illustration see fig. 5. Evidently $\{S_i\} \neq \emptyset$ for $i = 1, 2, 3$.

Lemma 6. *If a tree T contains a vertex c_0 of type III, then it is not 2-(a, b)-orderable.*

Proof of lemma 6. Consider the tree T_1 constructed from the tree T so that instead of $\{S_i\}$ we keep only one vertex $s_i \in \{S_i\}$, which is connected with c_i by an edge for $i = 2, 3$. For illustration see fig. 6. The other vertices and edges we keep without any change.

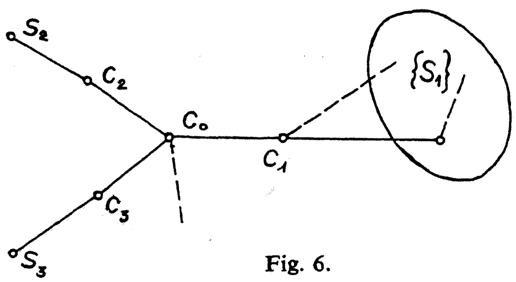

Fig. 6.

According to the note following lemma 2 it suffices to show that the tree T_1 is not 2-(a, b)-orderable. According to lemma 3 every 2-(a, b)-ordering of T_1 must be of the form either $\ldots, c_0, \ldots, c_1, \ldots$ or $\ldots, c_1, \ldots, c_0, \ldots$, where between c_0 and c_1 must lie all vertices of the set $\{s_2\} \cup \{c_2\} \cup \{s_3\} \cup \{c_3\}$. Again in accordance with lemma 3 this 2-(a, b)-ordering must contain two subsequences:

$$c_0, s_2, c_2 \quad \text{resp.} \quad c_2, s_2, c_0 ,$$

$$c_0, s_3, c_3 \quad \text{resp.} \quad c_3, s_3, c_0 .$$

These two requirements cannot hold simultaneously because a 2-(a, b)-ordering of the tree T_1 is simple. So T_1 and also T are not 2-(a, b)-orderable.

Lemma 7. *Let two vertices $c_0, d_0 \in \{T\}$, $c_0 \neq d_0$ be given, lying inside the path connecting two different vertices a, b of the tree T. Let $(c_0, c_1), (c_0, c_2), (d_0, d_1), (d_0, d_2)$ be mutually different edges of the tree T, the orders of vertices c_1, c_2, d_1, d_2 being greater than 1. Let none of edges $(c_0, c_i), (d_0, d_i)$ for $i = 1, 2$ lie on a path $(a, \ldots, b)$. Let the order of the inner vertices of a path connecting vertices c_0 and d_0 not be 2. Then T is not 2-(a, b)-orderable.* (For illustration see fig. 7.)

Proof of lemma 7. Let $(a, \ldots, e_0, c_0, e_1, e_2, \ldots, e_n, d_0, \ldots, b)$, $n \geqq 0$, denote the path connecting vertices a and b. For $i = 1, \ldots, n$ denote by f_i some of the vertices connected with e_i by an edge and not lying on the path $(c_0, \ldots, d_0)$. This is possible because the order of vertices e_i is at least 3. Further, denote subsequently by s_1, s_2, r_1, r_2 vertices of the tree T connected with vertices c_1, c_2, d_1, d_2 by an edge but not lying on the path $(a, \ldots, b)$.

Form the tree T_1 from the tree T so that we omit from the tree T all vertices with the exception of vertices on a path $(a, \ldots, b)$ and vertices $s_1, c_1, s_2, c_2, r_1, d_1, r_2, d_2, f_1, f_2, \ldots, f_n$ (see fig. 8).

Consider that there exists a 2-(a, b)-ordering of the tree T_1 and without any loss of generality let c_0 precede d_0 in this ordering. In accordance with lemma 3 this ordering must include four sequences:

$$c_1\ s_1,\ c_0 \quad \text{resp.} \quad c_0,\ s_1,\ c_1\ ,$$
$$c_2,\ s_2,\ c_0 \quad \text{resp.} \quad c_0,\ s_2,\ c_2\ ,$$
$$d_1,\ r_1,\ d_0 \quad \text{resp.} \quad d_0,\ r_1,\ d_1\ ,$$
$$d_2,\ r_2,\ d_0 \quad \text{resp.} \quad d_0,\ r_2,\ d_2\ .$$

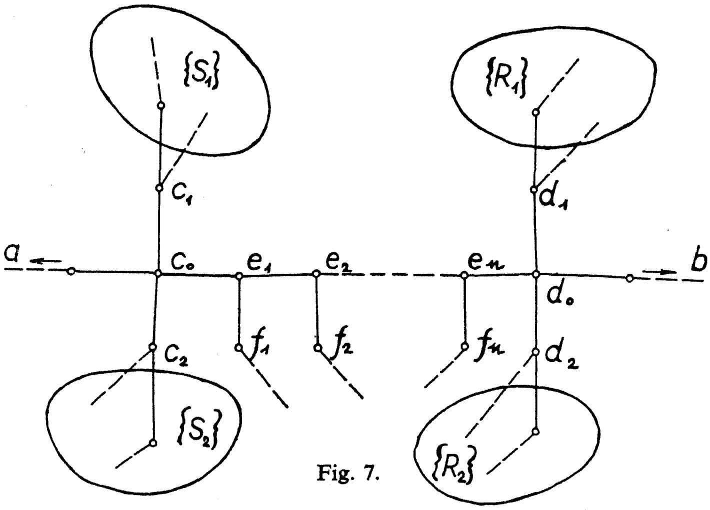

Fig. 7.

As both, the vertex c_0 and d_0 can occur in a 2-(a, b)-ordering only once, this ordering must contain two subsequences

$$c_1,\ s_1,\ c_0,\ s_2,\ c_2 \quad \text{resp.} \quad c_2,\ s_2,\ c_0,\ s_1,\ c_1\ ,$$
$$d_1,\ r_1,\ d_0,\ r_2,\ d_2 \quad \text{resp.} \quad d_2,\ r_2,\ d_0,\ r_1,\ d_1\ .$$

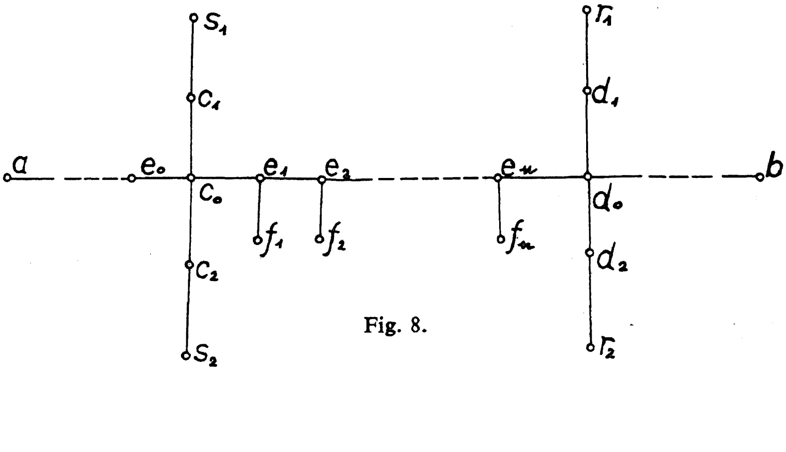

Fig. 8.

Immediately before a vertex c_1 resp. c_2 and immediately behind c_2 resp. c_1 only vertices e_0 and e_1 can occur in a 2-(a, b)-ordering because except for vertices s_1, c_0, s_2, which are bound to occur between vertices c_1 and c_2, all the others have a distance of at least 3 from vertices c_1 and c_2. A vertex e_0 must precede a vertex e_1 in this 2-(a, b)-ordering, because otherwise it would not be possible to finish the 2-(a, b)-ordering in such a way that it could end at the vertex b. Consequently a 2-(a, b)-ordering contains the subsequence

$$e_0, c_1, s_1, c_0, s_2, c_2, e_1, \quad \text{resp.} \quad e_0, c_2, s_2, c_0, s_1, c_1, e_1 .$$

It is easy to see immediately, that $n = 0$ cannot hold otherwise after c_2 resp. c_1 would have to follow immediately the vertex d_0, which nevertheless is bound to be between vertices r_1 and r_2. In a 2-(a, b)-ordering the vertex f_1 must follow immediately after the vertex e_1. If it could occur later in the ordering, the vertex e_2 should precede it immediately (or d_0, as far as $n = 1$) because it would be the only vertex not yet mentioned which has a distance from f_1 smaller than or equal to 2. The vertex f_1 could not be succeeded by any vertex in a 2-(a, b)-ordering, because none of the vertices already mentioned in the ordering would have a distance from f_1 smaller than 2 or equal to 2. So $f_1 = b$, which is a contradiction to our assumption. Considering analogously e_2, f_2 etc., we get that a 2-(a, b)-ordering contains a subsequence

$$c_1, s_1, c_0, s_2, c_2, e_1, f_1, e_2, f_2, \ldots, e_n, f_n, d_0$$

resp.

$$c_2, s_2, c_0, s_1, c_1, e_1, f_1, e_2, f_2, \ldots, e_n, f_n, d_0 .$$

But this is a contradiction because d_0 must lie between vertices r_1 and r_2. Consequently it is not possible to 2-(a, b)-order the tree T_1 and accordingly in the sense of the note following lemma 2 the tree T is not 2-(a, b)-orderable.

Lemma 8. *Let $a, b \in \{T\}$, $a \neq b$ and let (a, c_1) denote an edge of the tree T, which is not an edge of the path $(a, \ldots, b)$. Let d be an inner vertex of the path $(a, \ldots, b)$ and (d, d_1), (d, d_2) two different edges of the tree T which are not edges of the path $(a, \ldots, b)$, the orders of vertices d_1 and d_2 being at least* 2. *Let the order of the inner vertices of the path $(a, \ldots, d)$ be not* 2. *Then T is not* 2-(a, b)-*orderable.* (For illustration see fig. 9.) *An analogous lemma holds for the vertex b.*

Proof of lemma 8. Let $(a, e_1, e_2, \ldots, e_n, d)$, $n \geqq 0$ denote the path connecting vertices a and d. For $i = 1, \ldots, n$ denote by f_i some of the vertices connected with e_i by an edge but not lying on the path $(a, \ldots, d)$. Further denote subsequently by r_1, r_2 vertices of the tree T connected with vertices d_1, d_2 by an edge but not lying on the path $(a, \ldots, b)$.

Form the tree T_1 from the tree T in such a way that we omit from the tree T all vertices except those on the path $(a, \ldots, b)$ and vertices $c_1, r_1, d_1, r_2, d_2, f_1, f_2, \ldots, f_n$ (see fig. 10).

Suppose that there exists a 2-(a, b)-ordering of the tree T_1. According to lemma 3 this ordering must contain two subsequences

$$d, r_1, d_1 \quad \text{resp.} \quad d_1, r_1, d,$$
$$d, r_2, d_2 \quad \text{resp.} \quad d_2, r_2, d.$$

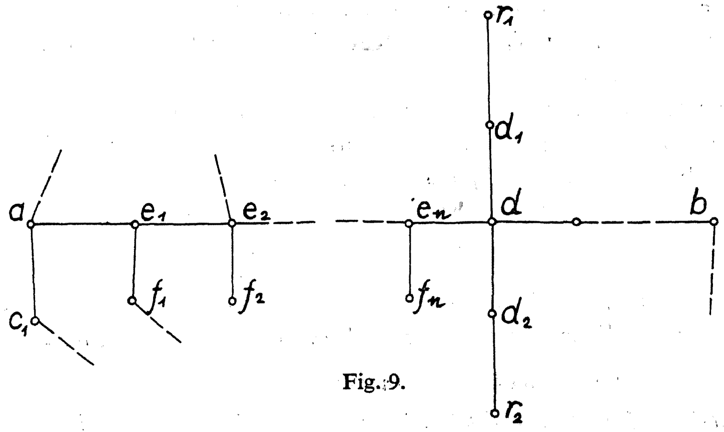

Fig. 9.

As the vertex d can occur in this sequence only once, this 2-(a, b)-ordering has to contain a subsequence

$$d_1, r_1, d, r_2, d_2 \quad \text{resp.} \quad d_2, r_2, d, r_1, d_1.$$

By analogous considerations as in the end of the proof of lemma 7, when we investigate the possibility of ordering vertices $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_n$, we find that this 2-(a, b)-ordering is bound to start with a subsequence

$$a, c_1, e_1, f_1, e_2, f_2, \ldots, e_n, f_n, d.$$

As d has to be directly between r_1 and r_2 according to what precedes, we get a contradiction. Consequently T_1 is not 2-(a, b)-orderable and neither is T.

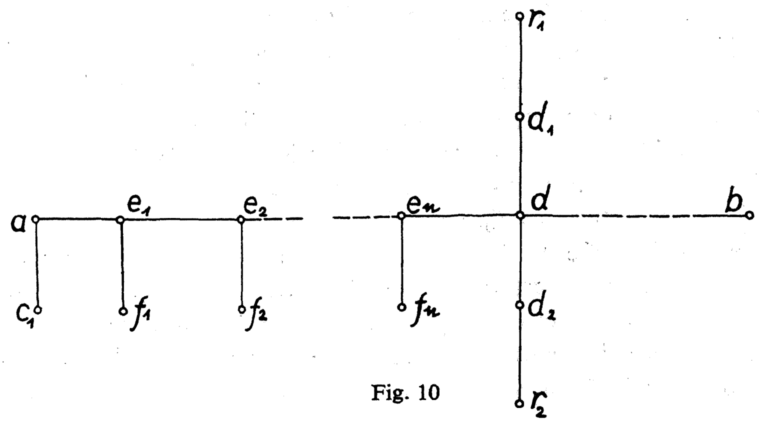

Fig. 10

Lemma 9. *Let the orders of all inner vertices of a path $(a, e_1, e_2, \ldots, e_n, b)$, $n \geqq 0$, of the tree T be greater than 2 and let orders of end-vertices a, b be greater than 1. Then the tree T is not 2-(a, b)-orderable.*

Proof of lemma 9. Denote by $(a, c), (e_1, f_1), (e_2, f_2), \ldots, (e_n, f_n), (b, g)$ mutually different edges of the tree T not lying on the path $(a, \ldots, b)$. Construct the subtree T_1 from the tree T in such a way that we omit all vertices and edges of the tree T except vertices $a, b, c, g, e_1, \ldots, e_n, f_1, \ldots, f_n$, edges of the path $(a, \ldots, b)$ and edges (a, c), $(e_1, f_1), \ldots, (e_n, f_n), (b, g)$.

Proceeding in a manner analogous to the proof of lemma 8, we ascertain that a 2-(a, b)-ordering of the tree T_1 has to start with a subsequence $a, c, e_1, f_1, \ldots, e_n, f_n, b$. A 2-$(a, b)$-ordering of the tree T_1 has to end at the vertex b and in spite of this, the vertex g does not occur between the vertices a and b. So T_1 is not 2-(a, b)-orderable and, in the sense of the note following lemma 2, neither is T.

Theorem. *The tree T is 2-(a, b)-orderable iff the tree T_0 which we get from the tree T by omitting all vertices of the order 1 except for vertices a, b satisfies:*

1° *the order of each vertices is $\leqq 4$ (in T_0),*

2° *the vertices of order 3 and 4 (in T_0) occur only inside the path connecting vertices a and b,*

3° *between every two vertices of order 4 (in T_0), there exists at least one vertex of order 2 (in T). If the order of vertex a is greater than 1 (in T) then between it and the nearest vertex of order 4 (in T_0) there exists at least one vertex of order 2 (in T), and similarly for vertex b. If the orders of both vertices a and b are greater than 1 (in T) then there exists at least one vertex of order 2 (in T) between them.*

Proof. *Necessity.* From the construction of the tree T_0 from the tree T it is easy to see that if condition 1° is not fulfilled, there exists a vertex $c_0 \in \{T\}$ of type I, so according to lemma 4, T is not 2-(a, b)-orderable. Consequently condition 1° is necessary. When condition 2° is not fulfilled, that means that either the order of vertex a (or of b) is at least 3 (in T_0) consequently that it is of type II, or there exists a vertex $c_0 \in \{T_0\}$, not lying on a path connecting vertices a and b and the order of c_0 is at east 3l (in T_0), consequently c_0 is of type III. According to lemma 5 or 6 T is not 2-(a, b)-orderable. Therefore condition 2° is necessary. Since condition 3° is not valid T satisfies the assumption of lemma 7 or lemma 8 or 9. This means that T is not 2-(a, b)-orderable and consequently condition 3° is necessary.

The *sufficency* can be shown by a construction of a 2-(a, b)-ordering of the tree T. First we 2-(a, b)-order the tree T_0. Consider the path (a, ..., b). According to the assumption no vertices of order $\geqq 5$ exist in T_0 and if vertices of order 4 occur in T_0, then they must be the inner vertices of the path $(a, \ldots, b)$. Denote these vertices (the order of which is 4) subsequently by $p_1, p_2, \ldots, p_k$, $k \geqq 0$ following their occuring on the path $(a, \ldots, b)$ starting with the vertex nearest to vertex a and ending with the one closest to vertex b. Each vertex p_i $(i = 1, \ldots, k)$ is an end vertex of two paths having no common edges with the path $(a, \ldots, b)$ and with themselves. Denote as p_i^j ($\bar{p}_i^j$) that vertex on one of these paths, which has the distance j ($j \geqq 1$) from a vertex p_i. Now construct for every i a sequence

$$(P_i) \qquad p_i^1, p_i^3, \ldots, p_i^4, p_i^2, p_i, \bar{p}_i^2, \bar{p}_i^4, \ldots, \bar{p}_i^3, \bar{p}_i^1,$$

where first the upper indices increase in odd numbers and these being exhausted, they fall in even numbers, again increase in odd numbers and fall in even numbers, all vertices p_i^j and $\bar{p}_i^j$ for all j. It is obvious that the distance of two neighbour vertices in a sequence (P_i) is at most 2.

Further notice the vertex a or b, respectively, in the tree T_0. If the order of it is 2 (in T_0) there issues from it a path $(a, a^1, a^2, \ldots, a^l)$ resp. $(b, b^1, \ldots, b^m)$, $l \geqq 1$ resp. $m \geqq 1$, the edges of which do not lie on the path $(a, \ldots, b)$. The order of the vertex a^l resp. b^m is 1. Now form the sequence

$$(\mathrm{Z}) \quad a^2, a^4, \ldots, a^3, a^1, \qquad \text{resp.} \qquad (\mathrm{K}) \quad b^1, b^3, b^5, \ldots, b^2.$$

When the order of a, b, respectively, is 1 in T_0, then we consider (Z), (K), respectively, to be empty sequences. It is obvious that the distance (in T_0) of two subsequent vertices in a sequence (Z), (K), respectively, is at most equal to 2.

Let for $i = 1, \ldots, k - 1$ $(p_i, u_{i1}, \ldots, u_{in_i}, p_{i+1})$ denote the path connecting vertices p_i and p_{i+1}. According to supposition 3° at least one vertex of the order 2 in T is bound to occur among u_{ij}, $j = 1, \ldots, n_i$; let it be the vertex u_{iv_i}. Let $(a, u_{01}, u_{02}, \ldots, u_{0n_0}, p_1)$ resp. $(p_k, u_{k1}, \ldots, u_{kn_k}, b)$ denote the path of the tree T_0. Then according to 3° the vertex a (b) is of order 1 in T or among $u_{01}, u_{02}, \ldots, u_{0n_0}$ (among $u_{k1}, \ldots, u_{kn_k}$), there is bound to occur a vertex of the order 2 in T. Denote it by u_{0v_0} (u_{kv_k}).

When x denotes some of the inner vertices of the path $(a, \ldots, b)$ of the tree T_0 of the order 3 in T_0 then denote as $(x, x^1, \ldots, x^q)$ the path issuing from the vertex x, the edges of which do not lie on the path $(a, \ldots, b)$ and where the order of x^q is 1 ($q \geqq 1$). Then form the sequences:

$$(X^-) \quad x^2, x^4, \ldots, x^3, x^1, \qquad (X^+) \quad x^1, x^3, \ldots, x^4, x^2.$$

When a vertex x is an inner vertex of the path $(a, \ldots, b)$ of the order 2 in T_0 then the symbols (X^-), (X^+) will denote empty sequences. From the assumption it follows that except for inner vertices of the path $(a, \ldots, b)$ the order of all other vertices must be at most 2 (in T_0). Consequently if the order of vertices a and b is not 1 (in T) then

$$\begin{aligned}(1) \quad & a, (\mathrm{Z}), u_{01}, (U_{01}^-), u_{02}, (U_{02}^-), u_{03}, \ldots, (U_{0,v_0-1}^-), u_{0v_0}, (U_{0,v_0+1}^+), u_{0v_0+1}, \ldots, \\ & (U_{0,n_0}^+), u_{0n_0}, (P_1), u_{11}, (U_{11}^-), u_{12}, (U_{12}^-), \ldots, \\ & (U_{1,v_1-1}^-), u_{1,v_1}, (U_{1,v_1+1}^+), u_{1,v_1+1}, \ldots, (U_{1,n_1}^+), u_{1,n_1}, (P_2), u_{21}, (U_{21}^-), \ldots \\ & \ldots, u_{k-1,n_{k-1}}, (P_k), u_{k,1}, (U_{k,1}^-), u_{k,2}, \ldots, (U_{k,v_k-1}^-), u_{k,v_k}, \\ & (U_{k,v_k+1}^+), u_{k,v_k+1}, \ldots, (U_{k,n_k}^+), u_{k,n_k}, (K), b\end{aligned}$$

is a 2-(a, b)-ordering of the tree T_0. If the order of vertex a, (b) is equal to 1 in T let us put $a = u_{0,v_0}$ $(b = u_{k,v_k})$ and we can get a 2-(a, b)-ordering of the tree T_0 from the above mentioned ordering by omitting vertices appearing before u_{0,v_0} (after u_{k,v_k}).

The tree T can be constructed from the tree T_0 by adding edges in stated vertices of the tree T_0. But definitely we do not add edges in vertices u_{i,v_i}, $i = 0, \ldots, k$ because these form the inner vertices of the path $(a, \ldots, b)$ and are of order 2 in T. We shall define a subset M of vertices of the tree T_0, to which we shall add further edges, and the $1-1$ mapping φ of the set M in the edges of the tree T_0. We find that each of the vertices of the tree T_0 to which we add further edges fulfils case α) or β) of lemma 1 with respect to the ordering (1) of the tree T_0, the set M and the map φ. We can see further that φ is a $1-1$ map, which means that the assumption of lemma 1 will be fulfilled, so T will be 2-(a, b)-orderable.

Now let $y \in \{T_0\}$ be an arbitrary but fixed vertex, to which we add further edges. We differentiate possible locations of the vertex y:

a) Do not let y lie on the path $(a, \ldots, b)$, and let the order of y be 1 in T_0. So y occurs in subsequences of sequence (1) of the type (P_i), or (Z), or (K), or (X^-), or (X^+). If y is an inner member of some of the mentioned subsequences then the upper index belonging to it changes parity with respect to some of the adjoint members of the sequence. When y is the first or last member, then in the ordering (1) y immediately precedes or closely follows a vertex on the path $(a. \ldots, b)$ from which y has the distance 1. Put $y \in M$ and define $\varphi(y)$ as the only edge of the tree T_0, which coincides with y. Evidently y satisfies case α) of lemma 1.

b) Do not let y lie on the path $(a, \ldots, b)$, and let the order of y be 2 in T_0. So y occurs in one of the subsequences mentioned in a). Hence y satisfies case β) of lemma 1, because suitable t_j, t_{j+1} must exist in the subsequence $u_{i-1,n_{i-1}}(P_i)$, $u_{i,1}$, or a, (Z), or (K), b, or x, (X^-), or (X^+), x.

c) $y = a$ or b and a or b do not coincide with u_{0,v_0} or u_{k,v_k}. Consider the case $y = a$. If (Z) is empty then put $y \in M$, $\varphi(y) = (a, u_{0,1})$ and y satisfies case α) of lemma 1, If (Z) is not empty, the vertex y fulfils case β) of lemma 1, because it suffices to put $t_j = a^1$, $t_{j+1} = u_{0,1}$. Analogous for $y = b$.

d) $y = p_i$, where $1 \leqq i \leqq k$. Then y fulfils case β) of lemma 1, because it suffices to put $t_j = u_{i-1,n_{i-1}}$, $t_{j+1} = p_i^1$.

e) $y = u_{i,m_i}$, where $m_i \neq v_i$, $0 \leqq i \leqq k$, $1 \leqq m_i \leqq n_i$. As $m_i \neq v_i$, a sequence (U^-_{i,m_i}) or (U^+_{i,m_i}) occurs in the ordering (1). If this sequence is empty then there occur on the ordering (1) closely succeeding vertices u_{i,m_i}, u_{i,m_i+1} or u_{i,m_i-1}, u_{i,m_i}. Put $y \in M$ and $\varphi(y) = (u_{i,m_i}, u_{i,m_i+1})$ or $\varphi(y) = (u_{i,m_i-1}, u_{i,m_i})$. Evidently y satisfies case α) of lemma 1. The sequence (U^-_{i,m_i}) or (U^+_{i,m_i}) being nonempty, it then suffices to put $t_j = u^1_{i,m_i}$, $t_{j+1} = u_{i,m_i+1}$ or $t_j = u_{i,m_i-1}$, $t_{j+1} = u^1_{i,m_i}$ and the vertex y satisfies case β) of lemma 1.

It can be shown easily that φ is a $1-1$ mapping of the set M in the set of edges of the tree T_0. So the assumptions of lemma 1 are fulfilled and T is 2-(a, b)-orderable. How to obtain from the ordering (1) (which is a 2-(a, b)-ordering of the tree T_0) a 2-(a, b)-ordering of the tree T is also described in lemma 1. The proof of the theorem is thus finished.

Note. The sufficiency of the theorem can also be proved by induction with respect to the number of vertices of the tree T. Note at first that every tree containing just two vertices (and just vertices a, b) is 2-(a, b)-orderable. Further make an assumption that every tree with less than n vertices, $n \geqq 3$, satisfying assumptions of the theorem, is 2-(a, b)-orderable. Let T denote an arbitrary but fixed tree with n vertices, which satisfies assumptions of the theorem. Let $(a, e_1, e_2, \dots, e_k, b)$ be the path of the tree T, $k \geqq 0$. By omitting the edge (a, e_1) for $k > 0$ or edge (a, b) for $k = 0$ we obtain two connected components of the tree T, of which the one containing the vertex a is denoted as G^a, the other as G^b. Let p denote the order of the vertex a in the tree T. Since T satisfies the assumptions of the theorem, the graph G^a has the following form: G^a contains vertices $a, a_1^1, a_1^2, \dots, a_1^q, a_2, a_3, \dots, a_{p-1}$, $q \geqq 0$, and contains edges $(a, a_1^1), (a_1^1, a_1^2), \dots, (a_1^{q-1}, a_1^q), (a, a_2), (a, a_3), \dots, (a, a_{p-1})$. Assume the sequence (2): $a, a_1^2, a_1^4, \dots, a_1^q, \dots, a_1^3, a_1^1, a_2, \dots, a_{p-1}$ with respect to G^a. For $p = 1$ a G^a contains only the vertex a and the corresponding sequence (2) is only a. Differentiate in what follows two cases.

Consider first that $k = 0$. Then let r be the order of the vertex b in the tree T. With respect to the fact that T fulfils the assumptions of the theorem, G^b has the following form: it contains vertices $b, b_1^1, b_1^2, \dots, b_1^t, b_2, b_3, \dots, b_{r-1}$, $t \geqq 0$, and contains edges $(b, b_1^1), (b_1^1, b_1^2), \dots, (b_1^{t-1}, b_1^t), (b, b_2), (b, b_3), \dots, (b, b_{r-1})$. Assume the sequence (3): $b_{r-1}, b_{r-2}, \dots, b_2, b_1^1, b_1^3, \dots, b_1^t, \dots, b_1^4, b_1^2, b$ with respect to G^b. As T fulfils the assumptions of the theorem, especially 3°, either $p = 1$ or $r = 1$, consequently (2), (3) is a 2-(a, b)-ordering of the tree T.

Now consider $k \geqq 1$. G^b is a tree with smaller number of vertices than n.

Case a). When the order of e_1 is 2 in T then it is of order 1 in G^b and it can be easily shown according to the induction-assumption that G^b can be 2-(e_1, b)-ordered. As the distance of the last vertex of the sequence (2) from the vertex e_1 in the tree T is at most 2, we can get a 2-(a, b)-ordering of the tree T when connecting the sequence (2) and a 2-(e_1, b)-ordering of the tree G^b in the order mentioned.

Let T_0 have the same significance as in the theorem.

Case b). The order of the vertex e_1 is greater than 2 in T but at most 3 in T_0 and the order of the vertex a equals 1 in T. Let $f_1, f_2, \dots, f_n$ be vertices of the tree G^b, which do not lie on a path $(e_1, \dots, b)$ and are connected with e_1 by an edge. Further let $f_1 \in \{T_0\}$, when the order of e_1 is 3 in T_0. Then the tree which we get from the tree G^b by omitting vertices $f_2, f_3, \dots, f_n$ can be 2-(f_1, b) ordered with respect to the induction-assumption. Note especially that the order of the vertex f_1 in T_0 is at most equal to 2 and further that the order of the vertex e_1 equals 2, in a tree constructed in this way. As the order of the vertex a is 1 in T, G^a contains only the vertex a and consequently a

2-(a, b)-ordering of the tree T can be obtained only by connecting the sequence $a, f_2, f_3, \ldots, f_n$ and the 2-(f_1, b)-ordering in the order mentioned.

Case c). Let the order of e_1 be greater than 2 in T but at most 3 in T_0 and the order of the vertex a be greater than 1 in T. The tree G^b is 2-(e_1, b)-orderable according to the induction-assumption. Notice especially that condition 3° is fulfilled, because the tree T fulfils it and the order of the vertex a in the tree T and also the order of the vertex e_1 in the tree G^b are greater than 1. As the distance of the last vertex of sequence (2) from the vertex e_1 is at most 2 in the tree T, we get a 2-(a, b)-ordering of the tree T when we connect the sequence (2) and the 2-(e_1, b)-ordering of the tree G^b in the order mentioned.

Case d). Let the order of the vertex e_1 be 4 in T_0. As T satisfies the assumptions of the theorem, the order of the vertex a is 1 in T, so G^a contains only the vertex a. Let $f_1 \in \{T_0\} \cap \{G^b\}$ and let (f_1, e_1) be an edge of G^b not lying on the path $(e_1, \ldots, b)$. Then G^b can be 2-(f_1, b)-ordered because the order of f_1 is at most 2 in T_0 (and in the tree G_0^b as well, which can be obtained from G^b by omitting the vertices of order 1 except vertices f_1 and b) and further the existence of a vertex of order 2 in G^b between the vertex f_1 and the nearest vertex of order 4 in G_0^b or between the vertex f_1 and the vertex b (in case the order of b is greater than 1 in G^b) is guaranteed. A 2-(a, b)-ordering of T can be obtained if we put the vertex a before the 2-(f_1, b)-ordering of the tree G^b.

Corollary. *The tree T is 2-ordered iff the graph obtained from the tree T by omitting all vertices of order* 1 *satisfies the following condition: either this graph contains at most one vertex, or this graph is the tree T_0' in which*

1'° *vertices of order greater than* 4 *do not exist* (*in* T_0'),

2'° *a path exists in T_0' such that all vertices of order* 3 *and* 4 (*in* T_0') *lie on it,*

3'° *between every two vertices of order* 4 (*in* T_0') *there lies at least one vertex of order* 2 (*in* T).

Proof of corollary. Note at first that the graph constructed from the tree T by omitting all vertices of order 1 may not be a tree, because it may contain just one vertex, or it may not contain any vertex, so according to the definition it is not a tree which has to contain at least two vertices. Therefore the necessity of the corollary follows immediately either from this note, or from the definition of a 2-ordering and from the above mentioned theorem.

We shall show the sufficiency. Assume at first a graph containing at most one vertex. Then it was obtained by means of the construction mentioned in the corollary solely from the tree containing the vertices $x_0, x_1, \ldots, x_n$, ($n \geqq 1$) and the edges (x_0, x_1), $(x_0, x_2), \ldots, (x_0, x_n)$. Such tree can be 2-ordered for example in this way: $x_0, x_1, x_2, \ldots, x_n$.

Let a tree T_0' be given which satisfies 1'°, 2'° and 3'° mentioned in the corollary. Denote as $(u_1, u_2, \ldots, u_n)$ a path, the existence of which is given in 2'°, ($n \geqq 2$).

Choose a path $(v_1, \ldots, v_m)$ of the tree T_0' such that it contains a path $(u_1, \ldots, u_n)$ and the order of vertices v_1 and v_m in T_0' is 1. Choose a path $(w_1, \ldots, w_p)$ in T such that it may contain a path $(v_1, \ldots, v_m)$ and the order of vertices w_1 and w_p may equal 1 in T. Note that in certain cases $v_1 = u_1$ or $v_m = u_n$. Construct the tree T_0 from the tree T_0' in such a way that we add to the tree T_0' those vertices and edges of the path $(w_1, \ldots, w_p)$ which are not yet contained in T_0'. It is obvious that T_0 can also be obtained if we omit all vertices of order 1 of the tree T with the exception of vertices w_1 and w_p.

We shall show that T_0 fulfils conditions 1°, 2° and 3° of the theorem when we put w_1 and w_p instead of a and b and when the tree T_0' satisfies conditions 1'°, 2'° and 3'° mentioned in the corollary.

The tree T_0' does not contain a vertex of order greater than 4 (in T_0') and the tree T_0 can be constructed from T_0' so that we add edges (w_1, v_1) and (w_p, v_m) to T_0'. Consequently except for vertices w_1, w_p, the order of which equals 1 (in T_0) and except for vertices v_1, v_m, the order of which is 2 (in T_0), the order of the other vertices is the same in T_0' and even in T_0, so that T_0 fulfils condition 1° of the theorem.

Consider the path $(w_1, \ldots, w_p)$ of the tree T_0. According to its construction all vertices of order 3 and 4 (in T_0) are lying on it. Then these vertices of order 3 and 4 (in T_0) must be inner vertices of the path $(w_1, \ldots, w_p)$, because the order w_1 and w_p in T_0 equals 1.

As condition 3'° of the corollary is fulfilled for the tree T_0', between every two vertices of order 4 (in T_0) there lies at least one vertex of order 2 (in T). This is an immediate consequence of the fact that insofar as the order of vertices in T_0' differ from the order of vertices in T_0, then the difference must be in vertices w_1, v_1, v_m, w_p the order of none of which can be greater than 2 either in T_0' or in T_0. The order of the vertices w_1 and w_p is 1 in T according to their construction. So the tree T_0 fulfils condition 3° as well.

If we put in the theorem $a = w_1$, and $b = w_p$, it can be easily seen that the tree T_0 satisfies conditions 1°, 2° and 3° mentioned in this theorem, so there exists a 2-(w_1, w_p)-ordering of the tree T, and consequently T is 2-orderable.

The question arises under what conditions a general graph is 2-(a, b)-orderable or only 2-orderable. I shall deal with this question in my following paper.

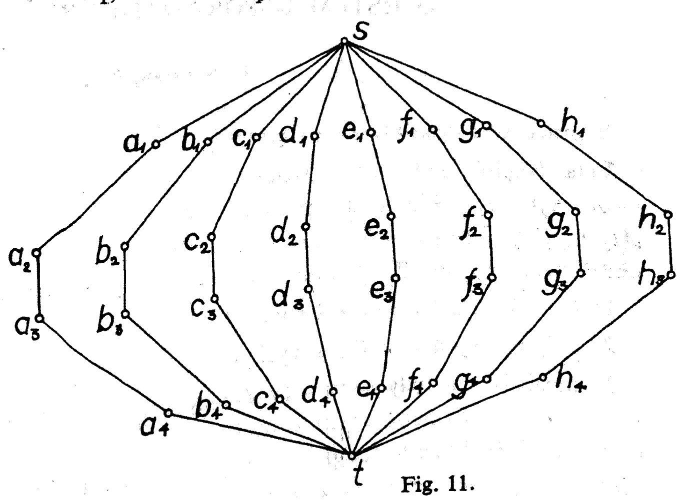

Fig. 11.

The proof of M. Sekanina [2] makes full use of the fact that instead of 3-ordering of general finite connected graph, it is possible to look only for 3-ordering of a certain skeleton of this graph. An analogous process referring to a 2-ordering is not possible, because a graph on fig. 11 can be 2-ordered (i. g. $s, a_1, a_2, a_3, a_4, t, b_4, b_3, b_2, b_1, c_1, \ldots, c_4, d_4, \ldots, d_1, e_1, \ldots, e_4, f_4, \ldots, f_1, g_1, \ldots, g_4, h_4, \ldots, h_1$) whereas no skeleton of it is 2-orderable. Indeed, in order to obtain a skeleton of the graph on fig. 11, it is necessary to omit one edge on every path connecting vertices s and t except for just one path (without any loss of generality this path is $(s, h_1, h_2, h_3, h_4, t)$). In the opposite case this skeleton would either contain a circle or it would not be connected, which is impossible. Whatever way we omit one edge from each of the paths $(s, a_1, a_2, a_3, a_4, t), (s, b_1, \ldots, b_4, t), \ldots, (s, g_1, \ldots g_4, t)$, we always obtain a skeleton in which either the vertex s or the vertex t will have the property that from it will issue at least five mutually disjunct paths having a length of at least 2. If in this skeleton we omit all vertices of order 1, we can see that in the tree constructed in this way either the order of the vertex s or of the vertex t is at least 5, so according to the corollary the skeleton is not 2-orderable.

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[1] *Berge Claude*: Théorie des graphes et ses applications. Russian translation, Moscow 1962.

[2] *Sekanina Milan*: On an Ordering of the Set of Vertices of a Connected Graph. Publ. Fac. Sci. Univ. Brno, Tchécoslovaquie, No *412* (1960), 137–142.

[3] *Sekanina Milan*: On an Ordering of the Vertices of Graphs. Čas. pro pěst. matematiky, *88* (1963), 265–282.

Výtah

O JISTÉM USPOŘÁDÁNÍ UZLŮ STROMU

F. Neuman, Brno

V práci jsou dokázána tato tvrzení.

Věta. Nechť T je konečný strom, a, b dva jeho různé uzly. Množina uzlů stromu T může být uspořádána do prosté posloupnosti $a = t_1, t_2, \ldots, t_s = b$ takové, že $\mu(t_i, t_{i+1}) \leqq 2$ pro $i = 1, \ldots, s - 1$, tehdy a jen tehdy, když pro podstrom T_1, který obdržíme ze stromu T odebráním koncových uzlů s výjimkou uzlů a a b, platí:

1° řád všech uzlů je nejvýš roven 4 (v T_1),

2° uzly řádu 3 a 4 (v T_1) se vyskytují pouze uvnitř cesty spojující a a b,

3° mezi dvěma uzly řádu 4 (v T_1) existuje alespoň jeden uzel řádu 2 (v T). Řád uzlu a je 1 (v T) nebo existuje uzel řádu 2 (v T) mezi a a nejbližším uzlem řádu 4 (v T_1). Podobně pro b. Když současně řád uzlu a i uzlu b je větší než 1(v T), pak mezi nimi existuje alespoň jeden uzel řádu 2 (v T).

Důsledek. Množinu vrcholů konečného stromu T lze uspořádat do prosté posloupnosti $t_1, t_2, \ldots, t_s$ tak, že $\mu(t_i, t_{i+1}) \leqq 2$ pro $i = 1, \ldots, s - 1$, tehdy a jen tehdy, když pro podstrom T_2 (prázdný strom a strom obsahující jeden uzel je nyní dovolen), který dostaneme ze stromu T vynecháním koncových uzlů, platí:

1'° řád všech uzlů T_2 je nejvýše 4 (v T_2),

2'° v T_2 existuje cesta, na které leží všechny uzly řádu 3 a 4 (v T_2),

3'° mezi každými dvěma uzly řádu 4 (v T_2) existuje uzel řádu 2 (v T).

Резюме

ОБ ОДНОМ УПОРЯДОЧЕНИИ ВЕРШИН ДЕРЕВА

Ф. НЕЙМАН, Брно

В работе доказано:

Теорема. Пусть T — конечное дерево, a и b — его различные вершины. Множество вершин дерева T можно упорядочить в простую последовательность $a = t_1, t_2, \ldots, t_s = b$ такую, что $\mu(t_i, t_{i+1}) \leqq 2$ для $i = 1, 2, \ldots, s - 1$, тогда и только тогда, если поддерево T_1, которое мы получим удалением висячих вершин дерева T за исключением вершин a, b, удовлетворяет следующим условиям:

1. степень всех вершин дерева T_1 не более 4 (в T_1),

2. вершины степени 3 и 4 (в T_1) появляются только внутри простой цепи, связывающей вершины a и b,

3. между каждыми двумя вершинами степени 4 (в T_1) существует по крайней мере одна вершина степени 2 (в T). Степень вершины a равна 1 (в T), или существует вершина степени 2 (в T) между a и ближайшей вершиной степени 4 (в T_1). Аналогично для b. Если одновременно степени a и b больше 1 (в T), то между a и b существует одна вершина степени 2 (в T).

Следствие. Множество вершин конечного дерева T можно упорядочить в простую последовательность $t_1, t_2, \ldots, t_s$ так, что $\mu(t_i, t_{i+1}) \leqq 2$ для $i = 1, 2, \ldots, s - 1$, тогда и только тогда, если для поддерева T_2 (пустое и одну вершину содержащее дерево здесь допускается), которое мы получим из дерева T удалением висячих вершин, выполнено:

1'. степень всех вершин дерева T_2 не более 4 (в T_2),

2'. в T_2 существует цепь, на которой находятся все вершины степени 3 и 4 (в T_2),

3'. между каждыми двумя вершинами степени 4 (в T_2) существует вершина степени 2 (в T).