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THEORY OF KIRCHHOFF'S NETWORKS

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Equations describing the behaviour of Kirchhoff's networks (*e. g.* electrical networks with lumped elements) are examined from the viewpoint of existence, uniqueness and stability of their solutions and of compatibility of initial conditions. The common mathematical background of problems in the frequency and time domains is pointed out. The problem is approached from two sides: the theory of linear operators and that of ordinary linear differential equations with constant coefficients whose right hand sides are distributions.

Introduction. Several papers have appeared recently dealing with the mathematical problem of Kirchhoff's networks. As an example of Kirchhoff's network one can take an electrical network with lumped parameters, containing resistances, inductances and capacities as passive elements and voltage generators as the active ones. The fundamental problem is to decide whether the network has a unique solution for currents, *i. e.* whether there exists a unique solution of the equations describing the behaviour of currents and voltages in the network.

This problem has been solved by D. KÖNIG [1] for direct curent networks and by V. KNICHAL [2] for alternating current networks. For instance, in [2] the network problem can be reduced essentially to the following one: To find assumptions concerning the real constant square matrices R, L, S, under which there exists a unique solution $J(t) = J_0 \exp\{i\omega t\}$, with J_0 a complex vector and $\omega > 0$, of the system

$$(0,1) \qquad X' \left[R\,J(t) + L \frac{\mathrm{d}J(t)}{\mathrm{d}t} + S \int_0^t J(\tau)\,\mathrm{d}\tau \right] = X' E_0 \exp\{i\omega t\}\,, \quad a'\,J(t) = 0\,,$$

where X', a' are constant matrices, E_0 is a complex vector.

In the presented paper the approach is more general. For example, instead of (0,1) we can examine the system

$$(0,2) \qquad X' \left[RJ + L\left(\frac{\mathrm{d}J}{\mathrm{d}t} - J_0\delta_0\right) + S(J^{(-1)} + q_0 H_0) \right] = X' E\,, \quad a' J = 0\,,$$

where $E = E(t)$ is a vector distribution, $E(t) = 0$ for $t < 0$, J_0, q_0 constant vectors, H_0 is Heaviside's function and δ_0 its derivative in the distributional sense, $J(t) = 0$ for $t < 0$.

The main purpose of paragraph 1 is to give some conditions for the existence and uniqueness of solution of an abstract network. In paragraph 2 the theory of Heaviside's operators is presented in the extent needed in paragraphs 3 and 5. Here theorems on existence and uniqueness of solutions of Kirchhoff's networks in the time domain and in the frequency domain, on stability of solutions and on compatibility of initial conditions can be found. In paragraph 6 there are some extensions of the results found in [2].

There are two main physical approaches to the problem of Kirchhoff's networks: examination in the time domain on one hand and in the frequency domain on the other. The former mathematically reduces to the study of systems of type (0,2), the latter to the study of systems of linear algebraical equations with rational functions (in complex domain) as coefficients. As shown in paragraph 3, these two approaches are equivalent. If we study a system with a fixed frequency, we deal with systems of linear algebraical equations with numerical coefficients, as in paragraph 6.

In paragraph 7, the problem of Kirchhoff's networks is approached from the basis of the theory of ordinary differential equations. As a result one obtains a normal[1]) system of ordinary differential equations with constant coefficients and with Schwartz's distributions on the right sides. Compatibility of initial conditions is examined for the case when the right sides are integrable functions.

1. ABSTRACT NETWORKS

Let us start with some useful considerations. Let $\boldsymbol{P}$ be a linear space, i. e. $\boldsymbol{P}$ is a nonempty set and

1. there is some rule under which to each pair of elements $x, y \in \boldsymbol{P}$ there corresponds a uniquely determined element $x + y \in \boldsymbol{P}$,

2. there is some field $\boldsymbol{T}$ such that for every $A \in \boldsymbol{T}$ and every $x \in \boldsymbol{P}$ the product $Ax \in \boldsymbol{P}$ is defined,

3. the following conditions are satisfied:

a) $x + y = y + x$ for every $x, y \in \boldsymbol{P}$,

b) $(x + y) + z = x + (y + z)$ for every $x, y, z \in \boldsymbol{P}$,

c) there is an element $0 \in \boldsymbol{P}$ such that $x + 0 = x$ for each $x \in \boldsymbol{P}$,

d) for every $x \in \boldsymbol{P}$ an element $(-x) \in \boldsymbol{P}$ exists such that $x + (-x) = 0$,

e) if $1 \in \boldsymbol{T}$ is the unit then $1 . x = x$ for each $x \in \boldsymbol{P}$,

f) $A(Bx) = (AB)x$ for each $x \in \boldsymbol{P}$ and every $A, B \in \boldsymbol{T}$,

g) $(A + B)x = Ax + Bx$, $A(x + y) = Ax + Ay$ for each $x, y \in \boldsymbol{P}$, $A, B \in \boldsymbol{T}$.

[1]) A system of differential equations is called normal if it is solved with respect to the derivatives of the highest order.

It can be readily seen that the following statements are true:

Lemma 1,1. a) *In* $\mathbf{P}$ *there exists a unique element* 0,

b) *for each* $x \in \mathbf{P}$ *the element* $(-x)$ *is determined uniquely,*

c) $0x = 0$ *for each* $x \in \mathbf{P}$,

d) $A0 = 0$ *for each* $A \in \mathbf{T}$,

e) *if* $Ax = 0$, $A \in \mathbf{T}$, $x \in \mathbf{P}$, *then either* $A = 0$ *or* $x = 0$.

The expression "matrix A (or vector a) over Q" will mean that the elements of A (or a) belong to the set Q.

The following theorem can be easily proved:

Theorem 1,1. *Let A be a matrix over $\mathbf{T}$ with σ rows and ϱ columns, and with rank $h < \varrho$. Then there exists a matrix X over $\mathbf{T}$ with ϱ rows and v columns, $v = \varrho - h$, which has the maximal rank, such that the following conditions are fulfilled:*

1. *If x is a ϱ-dimensional vector over $\mathbf{P}$ fulfilling the equation*

$$Ax = 0 \tag{1,1}$$

then there is a v-dimensional vector y over $\mathbf{P}$ such that

$$x = Xy \tag{1,2}$$

holds.

2. *If y is any v-dimensional vector over $\mathbf{P}$, then the vector x defined by* (1,2) *fulfils equation* (1,1).

Moreover,

a) $AX = 0$ *and the columns of X form a complete set of linearly independent solutions of the equation* (1,1) *in* $\mathbf{T}$;

b) *for each pair of vectors x, y over $\mathbf{P}$ satisfying* (1,2) *the equivalence* $x \neq 0 \Leftrightarrow \Leftrightarrow y \neq 0$ *is true;*

c) *for X there can be taken any matrix whose columns form a complete set of linearly independent solutions of equation* (1,1) *in the field* $\mathbf{T}$.

If A is a matrix over $\mathbf{T}$ with σ rows and ϱ columns and rank ϱ, then the vector $x = 0$ over $\mathbf{P}$ is the unique solution of equation (1,1).

Let now $H = \{h_1, \ldots, h_r\}$ be a finite non-empty set, the elements of which will be called branches; let further $U = \{u_1, \ldots, u_s\}$ be a finite non-empty set, the elements of which will be called nodes. Let Γ be a function on H into $U \times U$ such that if $h_i \in H$, then

$$\Gamma(h_i) = (u_{i_1}, u_{i_2}), \quad u_{i_1}, u_{i_2} \in U, \quad u_{i_1} \neq u_{i_2}.$$

In this case the nodes u_{i_1}, u_{i_2} will be called the initial, terminal node of the branch h_i, respectively, and h_i will be said to be oriented from u_{i_1} to u_{i_2}. The branch $h_i \in H$ and the node $u_j \in U$ are said to be incident if u_j is either the initial or the terminal node of h_i. Two distinct branches h_i, $h_j \in H$ will be called adjacent if there exists an $u_k \in U$ incident with both h_i and h_j.

The triple $(H, U, \Gamma) = G$ will be called an oriented graph, if for each $u_j \in U$ there exists an $h_i \in H$ such that u_j and h_i are incident. In this case h_i and u_j will be called a branch and a node of the graph G, respectively.

A graph $G_1 = (H_1, U_1, \Gamma_1)$ is called a subgraph of G, if $H_1 \subset H, U_1 \subset U, \Gamma_1(h_i) = \Gamma(h_i)$ for $h_i \in H_1$. In this case we write $G_1 \subset G$.

The expression $K = \sum_{i=1}^{r} c_i h_i$ where $c_i \in \boldsymbol{T}$, $h_i \in H$, will be called a 1-complex. If $K' = \sum_{i=1}^{r} c'_i h_i$ is also a 1-complex, $\alpha, \alpha' \in \boldsymbol{T}$, let us define $\alpha K + \alpha' K' = \sum_{i=1}^{r} (c_i + c'_i) h_i$. We put $K = 0$, if and only if $c_i = 0$, $i = 1, \ldots, r$. Complexes $K_1, \ldots, K_q$ will be called linearly independent, if from $\sum_{i=1}^{q} \alpha_i K_i = 0$, $\alpha_i \in \boldsymbol{T}$, if follows that $\alpha_i = 0$, $i = 1, \ldots, q$.

In a similar manner, the expression $L = \sum_{i=1}^{s} c_i u_i$, where $c_i \in \boldsymbol{T}$, $u_i \in U$, will be called a 0-complex. The notions of $\alpha L + \alpha' L'$, $L = 0$ and linear independence are defined analogously.

For the sake of brevity the notation $(-1) h_i = - h_i$, $(-1) u_k = - u_k$, $-1 \in \boldsymbol{T}$, will be used.

On the system of all 1-complexes let us define the operation ∂ by the relation $\partial K = \sum_{i=1}^{r} c_i \, \partial h_i$, where $\partial h_i = u_{i_2} - u_{i_1}$ if $\Gamma(h_i) = (u_{i_1}, u_{i_2})$. If $\partial K = 0$ for some 1-complex K, then K will be called a cycle.

Let us now define the matrix a over $\boldsymbol{T}$ with r rows and s columns as follows ($a = [a_{ik}]$ will also be called the incidence matrix):

$$
\begin{aligned}
a_{ik} &= 1 \in \boldsymbol{T} \text{ if } u_k = u_{i_2}\,, \\
a_{ik} &= -1 \in \boldsymbol{T} \text{ if } u_k = u_{i_1}\,, \\
a_{ik} &= 0 \in \boldsymbol{T} \text{ if } u_{i_1} \neq u_k \neq u_{i_2}, \text{ where } \Gamma(h_i) = (u_{i_1}, u_{i_2})\,.
\end{aligned}
$$

According to the definition of an oriented graph, it is obvious that $r \geqq 1$, $s \geqq 2$ and, therefore, the rank of a is $\geqq 1$.

Let us further denote $h' = [h_1, \ldots, h_r]$, $h_i \in H$, $u' = [u_1, \ldots, u_s]$, $u_k \in U$. If $K = [K_{ik}]$ is any matrix the elements of which are 1-complexes, let $\partial K = [\partial K_{ik}]$. Then obviously

$$\partial h = a u \,. \tag{1,3}$$

If K is a 1-complex, one can write $K = c'h$, where $c' = [c_1, \ldots, c_r]$.

From the foregoing definitions it follows:

Lemma 1,2. *Let $K = c'h$; then K is a cycle if and only if $a'c = 0$.*

The proof is evident.

Let now the rank of the matrix a be smaller than r, and let X be the matrix from Theorem 1,1 where we put $A = a'$. It can be readily seen that the following statement is true: *The elements of the vector $X'h$ form a complete set of linearly independent* *cycles.* Indeed, the elements of $X'h$ are cycles, for $\partial X'h = X'\partial h = X'au = 0$. Suppose further that there is a vector α over $\boldsymbol{T}$ such that $\alpha'X'h = 0$. By definition of the zero 1-complex, the latter equation means that $\alpha'X' = 0$, *i. e.* $X\alpha = 0$. As X has maximal rank, it follows that $\alpha = 0$ and, consequently, the elements of $X'h$ are linearly independent. The completeness follows immediately from the maximality of the rank of X.

Let $K = \sum_{i=1}^{r} c_i h_i$ be a 1-complex with the c_i assuming only three values from $\boldsymbol{T}$, namely 0, 1, -1. Let G_k be a subgraph of $G = (H, U, \Gamma)$ with the following property: $h_i \in H$ is a branch of G_k if and only if $c_i \neq 0$. Let G_k^* be a graph which is obtained from G_k by changing the orientation of h_i whenever $c_i = -1$.

A 1-complex $K = \sum_{i=1}^{r} c_i h_i$, $c_i \in \{0, 1, -1\}$ is called a chain if G_k^* has the following property: The branches $h_i \in H$ for which $c_i \neq 0$ may be ordered to form a sequence $\{h_{i_1}, \ldots, h_{i_p}\}$, the terminal node of h_{i_j} being the initial node of $h_{i_{j+1}}$, $j = 1, 2, \ldots, p - 1$.

A node u_j of $G = (H, U, \Gamma)$ will be said to be of order n if u_j is incident with exactly n branches from H. A chain K is called elementary if each node of the corresponding graph G_k^* is at most of order 2, *i. e.* if each node of K is incident with at most two branches of K.

An elementary chain K which is simultaneously a cycle will be called a loop. From the above definitions it follows that the initial node of the first branch of G_k^* coincides with the terminal node of its last branch.

In a similar way as in [2] one can prove the following theorem:

Theorem 1,2. *Let* $K = \sum_{i=1}^{r} c_i h_i$ *be a cycle. Then there exist loops* $K_i = \sum_{j=1}^{r} e_{ij} h_j$, $i = 1, 2, \ldots, l$ *such that* $K = \sum_{i=1}^{l} d_i K_i$, $d_i \in \boldsymbol{T}$. *Moreover,* $e_{ij} \neq 0 \Rightarrow c_j \neq 0$ *for* $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, r$.

Now let us state the definition of a network.

Let G be an oriented graph and Z a function on $H \times H$ into some field $\boldsymbol{T}$; the ordered pair $N = (G, Z)$ will be called an abstract network.

Let further E be a function on H into $\boldsymbol{P}$, where $\boldsymbol{P}$ is a linear space with respect to $\boldsymbol{T}$. For the sake of brevity let us denote by E' the vector $E' = [E(h_1), E(h_2), \ldots, E(h_r)]$ and by $Z = [Z_{ik}] = [Z(h_i, h_k)]$ the matrix of type $r \times r$.

The statement the network N has a solution for a given function E will mean that there exists a function J on H into $\boldsymbol{P}$ such that the following conditions are fulfilled:

K1: $c'E = c'ZJ$ for every cycle $c'h$,

K2: $a'J = 0$ (a is the incidence matrix)

where $J' = [J(h_1), J(h_2), \ldots, J(h_r)]$.

Note 1,1. From Theorem 1,2 it follows easily that N possesses a solution J for a given E, if and only if K1 is fulfilled for every loop. Indeed, if K1 holds for every loop K_i, and K is a cycle, it is sufficient to multiply each of the l systems of equations K1 for $K_1, \ldots, K_l$ by the corresponding coefficient d_i and sum all the l systems.

Finally, we state that N is regular if N possesses a unique solution for every E, *i. e.* if for every E the corresponding solution J is determined uniquely.

The network N will be called non-trivial if the rank of the matrix a is smaller than r, trivial if rank $(a) = r$.

It is obvious that $N = (G, Z)$ *is non-trivial if and only if G contains at least one non-zero cycle.* Therefore *every trivial network is regular.*

Theorem 1,3. *Let N be a non-trivial network and X any matrix the columns of which form a complete set of linearly independent solutions of* $a'x = 0$ *in* $\mathbf{T}$ *(a is the incidence matrix); then N is regular if and only if*

$$\det X'ZX \neq 0 . \tag{1,4}$$

Moreover, the solution J corresponding to E is given by

$$J = X(X'ZX)^{-1} X'E . \tag{1,4a}$$

Proof. According to Theorem 1,1, K2 is equivalent to $J = Xy$; $c'h$ is a cycle if and only if $a'c = 0$. Consequently, K1 and K2 are equivalent to $X'E = X'ZXy$. From this the theorem follows.

A sufficient condition for the regularity of a network is given by

Theorem 1,4. *Let* $N = (G, Z)$ *be an abstract network; if* $c'Zc \neq 0$ *for every cycle* $c'h \neq 0$ *then N is regular.*

Proof. If N is trivial, the theorem obviously holds. Let N be non-trivial; according to Theorem 1,1 for every cycle $c'h \neq 0$ there exists an $y \neq 0$ over $\mathbf{T}$ such that $c = Xy$, X being fixed. By hypothesis $y'X'ZXy \neq 0$ for every $y \neq 0$, whence $\det X'ZX \neq 0$. This together with Theorem 1,3 proves our statement.

Note 1,2. It can be shown easily, that the condition in Theorem 1,4 is not necessary. Moreover, the condition "$c'Zc \neq 0$ for every cycle $c'h \neq 0$" cannot be replaced by the weaker condition "$c'Zc \neq 0$ for every cycle belonging to any set of linearly independent cycles", as shown in the following example:

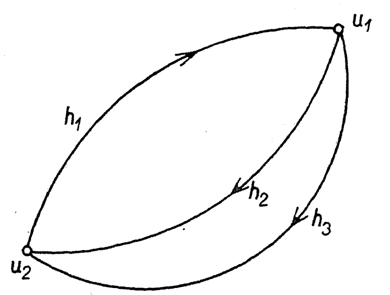

Fig. 1.

Example 1,1. Let us consider the network $N = (G, Z)$, where the graph G is schematically sketched on Fig. 1 and where $Z = \operatorname{diag}[0, 1, 0]$. It can be readily seen that $c_1'h$, $c_2'h$, where $c_1' = [1, 1, 0]$, $c_2' = [0, 1, -1]$, form a complete set of linearly independent cycles. Obviously $c_1'Zc_1 = c_2'Zc_2 = 1$; on the other hand, putting $X' = \begin{bmatrix} 1, 1, & 0 \\ 0, 1, & -1 \end{bmatrix}$ one obtains $X'ZX = \begin{bmatrix} 1, 1 \\ 1, 1 \end{bmatrix}$ which is a singular matrix.

Another necessary and sufficient condition for the regularity of a network is represented by the following theorem:

Theorem 1,5. *Let N be a non-trivial abstract network; then N is non-regular if and only if for $E = 0$ there exist a solution J and a cycle $c`h \neq 0$ such that $J_i \neq 0 \Leftrightarrow \Leftrightarrow c_i \neq 0$, where $c` = [c_1, \ldots, c_r]$, $J` = [J_1, \ldots, J_r]$.*

Proof. The sufficiency is trivial; let us therefore prove the necessity only. If N is non-regular, then a vector $y \neq 0$ over $\mathbf{T}$ exists such that

$$X`ZXy = 0\,. \tag{1,5}$$

By Theorem 1,1, $Xy \neq 0$; chooosing arbitrary $\alpha \in \mathbf{P}$, $\alpha \neq 0$, it is evident that $J = Xy\alpha \neq 0$. From (1,5) it follows that $X`ZJ = 0$; also $a`J = a`Xy\alpha = 0$. Hence $J$ represents the solution of $N$ for $E = 0$. But $c`h$ for $c = Xy$ is a non-zero cycle possessing the property stated in Theorem 1,5, *q. e. d.*

Theorem 1,6. *Let A be a matrix over $\mathbf{T}$ of type $\varrho \times \sigma$, with rank $\geqq 1$, y a ϱ-dimensional vector over $\mathbf{P}$; then there exists a σ-dimensional vector x over $\mathbf{P}$ satisfying the equation*

$$Ax = y \tag{1,6}$$

if and only if for every ϱ-dimensional vector z over $\mathbf{T}$, which satisfies the equation

$$z`A = 0 \tag{1,7}$$

the relation

$$z`y = 0 \tag{1,8}$$

is valid.

Proof is obvious.

With the aid of Theorem 1,6 the network problem can be formulated in another, equivalent manner.

Theorem 1,7. *Let $N = (G, Z)$ be an abstract network, E some vector over $\mathbf{P}$. Let us construct the system*

$$ZJ + aV = E\,, \tag{A}$$

$$a`J = 0\,, \tag{B}$$

where a is the incidence matrix, V an s-dimensional vector over $\mathbf{P}$. Then the following assertions are true:

1. *If N has a solution for a given E, then there exists a V such that (A), (B) are fulfilled.*

2. *If the system (A), (B) has a solution for a given E (i. e. if vectors J, V satisfying (A), (B) exist), then J is the solution of N.*

Proof. 1. Let J be a solution of N for a given E, *i. e.* let $c`(E - ZJ) = 0$, $a`J = 0$ hold for every cycle $c`h$. It can be readily seen that a V exists such that $aV = E - ZJ$

holds. As a matter of fact, for every cycle $c`h$ one has $a`c = 0$, *i. e.* $c`a = 0$, so that the assumptions of Theorem 1,6 are satisfied and V exists.

2. If conversely (A), (B) hold, then multiplying (A) by a $c`$ which corresponds to a cycle $c`h$, one obtains $c`ZJ = c`E$ and the theorem is proved.

Theorem 1,8. *Let $N = (G, Z)$ be an abstract network and let a be the incidence matrix. Let further d be a matrix the columns of which form a complete set of linearly independent columns of a; then N is regular if and only if*

$$\det \left[\begin{array}{c|c} Z & d \\ \hline d` & 0 \end{array}\right] \neq 0 . \tag{1,9}$$

Proof. Without loss of generality suppose that the first h columns of a from a complete set of linearly independent columns. (According to the definition of a it is obvious that $h \leqq s - 1$.)

1. Let N be regular and let J be its solution for a given E. By Theorem 1,7 a vector V exists such that (A), (B) are satisfied; simultaneously J is the unique vector fulfilling (A), (B). From (A) it follows that for any other vector $\bar{V}$ fulfilling (A), the equation

$$a(V - \bar{V}) = 0 \tag{1,10}$$

is valid. Now let V be fixed and consider the equation $ax = 0$; from the assumption on the columns of a it follows that choosing the components $x_{h+1}, x_{h+2}, \ldots, x_s$ of the vector x arbitrarily, the components $x_1, x_2, \ldots, x_h$ are determined uniquely. Let $\bar{x}_0$ be the unique solution of $ax = 0$, where $x_i = -V_i$ for $i = h + 1, h + 2, \ldots, s$. Adding the equation $a\bar{x}_0 = 0$ to (A), one obtains

$$ZJ + aV^* = E , \tag{1,11}$$

where $V^* = V + \bar{x}_0$; evidently, the last $s - h$ components of V^* are zero. It is obvious that there cannot exist any other vector V^{**} with the latter property and satisfying (1,11). Indeed, in the opposite case we would have $a(V^* - V^{**}) = 0$, so that $V^* - V^{**} = 0$ as above.

Consequently, putting $\tilde{V}_` = [V_1, V_2, \ldots, V_h]$, the system

$$ZJ + d\tilde{V} = E , \quad a`J = 0 \tag{1,12}$$

has the unique solution $J, \tilde{V}$. Finally, as the second equation of (1,12) is equivalent to $d`J = 0$, (1,12) is equivalent to a system with the coefficient-matrix $U = \left[\begin{array}{c|c} Z & d \\ \hline d` & 0 \end{array}\right]$; the uniqueness of $J, \tilde{V}$ proves our statement.

Conversely, let (1,9) be satisfied; then (1,12) possesses a unique solution, so that (A), (B) have a unique solution in J; this together with Theorem 1,7 proves the second assertion.

Two different abstract networks $N_1 = (G_1, Z_1)$, $N_2 = (G_2, Z_2)$ will be called equivalent (we write $N_1 \approx N_2$), if N_1 can be transformed into N_2 by a successive application of the following operations:

(i) Change of orientation of a branch, say h_i. In this case Z_{ii} remains unchanged, whereas every Z_{ij}, $i \neq j$, changes sign;

(ii) Replacing a pair of equally oriented adjacent branches h_i, h_j whose common node is of order two, by a single branch h_i for which

$$(1,13) \qquad Z^*_{ik} = Z_{ik} + Z_{jk}, \quad k \neq i, \; k \neq j, \quad Z^*_{ii} = Z_{ii} + Z_{ij} + Z_{ji} + Z_{jj},$$

the matrix without astric corresponding to the original network and that with an astric corresponding to the new one;

(iii) Replacing a single branch h_i by a pair of equally oriented adjacent branches h_i, h_j whose common node is of order two, (1,13) being fulfilled.

Evidently, the equivalence defined above is symmetrical, transitive and reflexive. Thus each set of networks may be decomposed into classes of equivalent networks.

Note 1,3. Let be given a network $N = (G, Z)$, a vector $E' = [E_1, \ldots, E_r]$ over $\mathbf{P}$ and the corresponding solution $J' = [J_1, J_2, \ldots, J_r]$. From eqs. K1, K2 there follow these statements:

a) Operation (i) applied to $h_i \in H$ implies that the resulting network $\bar{N}$ has a solution $\bar{J}' = [J_1, \ldots, -J_i, \ldots, J_r]$ if E' is replaced by $\bar{E}' = [E_1, \ldots, -E_i, \ldots, E_r]$;

b) Operation (ii) applied to h_i, h_j implies that $\bar{N}$ has a solution $\bar{J}' = [J_1, \ldots, J_i, \ldots, J_{j-1}, J_{j+1}, \ldots, J_r]$ if E' is replaced by

$$\bar{E}' = [E_1, \ldots, E_i + E_j, \ldots, E_{j-1}, E_{j+1}, \ldots, E_r].$$

c) Operation (iii) applied to h_i implies that $\bar{N}$ has a solution $\bar{J}' = [J_1, \ldots, J_i, \ldots, J_r, J_j]$, if E' is replaced by $\bar{E}' = [E_1, \ldots, E_r, E_j]$.

Further, it is evident that to each cycle of N there corresponds exactly one cycle of $\bar{N}$, and to each node of N which is at least of order three there corresponds exactly one node of $\bar{N}$ which is of the same order.

Theorem 1,9. *Let $N_1 = (G_1, Z_1)$, $N_2 = (G_2, Z_2)$ be two equivalent networks. Then N_1 is regular if and only if N_2 is regular.*

The proof follows from Theorem 1,5 and Note 1,3.

2. SOME SPECIAL FIELDS

Let $\mathbf{E}$ be the field of complex numbers, $\mathbf{E}_r$ the field of real numbers; let $\mathbf{R}$ be the field of rational functions of variable p the coefficients of which belong to $\mathbf{E}$, and $\mathbf{R}_r$ the field of rational functions in p with coefficients from $\mathbf{E}_r$.

Let us now construct the field of Heaviside's operators.

Let $\mathbf{D}$ be the set of all Schwartz' distributions vanishing in $(-\infty, 0)$ (see [3]). It can be easily shown that the following statements are true:

1) $f, g \in \mathbf{D} \Rightarrow f + g \in \mathbf{D}$,

2) $f \in \mathbf{D} \Rightarrow f' \in \mathbf{D}$,

3) $f \in \mathbf{D} \Rightarrow af \in \mathbf{D}$ *for every function* $a(t)$ *indefinitely differentiable in* $(-\infty, \infty)$,

4) *if* $f \in \mathbf{D}$ *is regular, then* $f(t) = 0$ *almost everywhere in* $(-\infty, 0)$.

Let us now define the notion of a primitive distribution; if $f \in \mathbf{D}$, let the functional $f^{(-1)}$ be given by

$$(2,1)\qquad (f^{(-1)}, \varphi) = \left(f, -\int_{-\infty}^{t} \varphi(\tau)\,\mathrm{d}\tau + \left(\int_{-\infty}^{\infty} \varphi(\tau)\,\mathrm{d}\tau\right)\int_{-\infty}^{t} \varphi_0(\tau)\,\mathrm{d}\tau\right), \quad \varphi \in \mathbf{K},$$

where $\varphi_0 \in \mathbf{K}$ ($\mathbf{K}$ denoting the set of all indefinitely differentiable functions with compact support), the support of φ_0 being a subset of $(-\infty, 0)$, and $\int_{-\infty}^{\infty} \varphi_0(\tau)\,\mathrm{d}\tau = 1$.

It can be readily proved that the following assertions hold:

1) $f^{(-1)} \in \mathbf{D}$ *and does not depend on the choice of* φ_0,

2) *if* $f \in \mathbf{D}$ *is regular, then* $f^{(-1)}$ *is also regular and equals* $\int_0^t f(\tau)\,\mathrm{d}\tau$ *almost everywhere*,

3) $(f + g)^{(-1)} = f^{(-1)} + g^{(-1)}$,

4) $(f')^{(-1)} = (f^{(-1)})' = f$.

Let us now define on $\mathbf{D}$ the operator D by the relation $Dx = x'$ and the operators $\lambda, 0$ by $\lambda x = (\lambda x)$, $0x = 0$ for $\lambda, 0 \in \mathbf{E}$.

If A, B are any operators defined on $\mathbf{D}$, let us define the sum $A + B$ and the product AB in the usual manner.

For any integer $n \geqq 2$ let $D^n = DD^{n-1}$, $D^1 = D$.

The operator A on $\mathbf{D}$ will be called regular if there exists an operator A^{-1} on $\mathbf{D}$ such that the equations $A^{-1}A = AA^{-1} = 1 \in \mathbf{E}$ are valid.

By assertion 4) it is obvious that D is regular and $D^{-1}x = x^{(-1)}$ holds.

Let now $\mathfrak{R}$ be the set of operators A which are defined on $\mathbf{D}$ by the relation

$$(2,2)\qquad A = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_0\,, \quad n \geqq 0\,, \quad a_i \in \mathbf{E}\,.$$

Then the following obvious statements are true:

Lemma 2,1. *If* $A, B \in \mathfrak{R}$, *then* $A + B \in \mathfrak{R}$, $AB \in \mathfrak{R}$ *and* $AB = BA$.

Lemma 2,2. *If* $A \in \mathfrak{R}$, *then* $A = 0$ *if and only if* $a_n = a_{n-1} = \ldots = a_0 = 0$.

Lemma 2,3. *Every operator* $D - \alpha \in \mathfrak{R}$, $\alpha \in \mathbf{E}$, *is regular and*

$$(2,3)\qquad (D - \alpha)^{-1}\, x = e^{\alpha t}(e^{-\alpha t}x)^{(-1)}\,.$$

Proof. One has

$$(D - \alpha)\,\{e^{\alpha t}(e^{-\alpha t}x)^{(-1)}\} = \alpha e^{\alpha t}(e^{-\alpha t}x)^{(-1)} + x - \alpha e^{\alpha t}(e^{-\alpha t}x)^{(-1)} = x\,;$$

similarly one obtains $(D - \alpha)^{-1}\,(D - \alpha) = 1$, *q. e. d.*

Lemma 2,4. *Each* $A \in \mathfrak{R}$, $A \neq 0$, *is regular.*

The proof follows from Lemma 2,3 and Lemma 2,1.

Let us now define Heaviside's operators. Let $\mathbf{H}$ be the set of all operators defined on $\mathbf{D}$ with the following property: If $A \in \mathbf{H}$ then there are operators $C_1, C_2 \in \mathfrak{R}$,

$C_1 \neq 0$, such that $A = C_1^{-1}C_2$. Let $A \in \mathbf{H}$. Then obviously $A = 0$ if and only if $C_2 = 0$.

Theorem 2,1. a) *Each $A \in \mathbf{H}$, $A \neq 0$ is regular and $A^{-1} \in \mathbf{H}$.* b) *If $A, B \in \mathbf{H}$, then $A + B \in \mathbf{H}$, $AB \in \mathbf{H}$ and $AB = BA$.*

The proof is evident.

Summarizing the foregoing results,one can state the following important assertion:

Theorem 2,2. a) *$\mathbf{H}$ is a field,* b) *$\mathbf{D}$ is a linear space with respect to $\mathbf{H}$,* c) *$\mathbf{H}$ is isomorphic with $\mathbf{R}$.*

It is obvious that this isomorphism can be represented by

$$\frac{a_n D^n + \dots + a_0}{b_m D^m + \dots + b_0} \leftrightarrow \frac{a_n p^n + \dots + a_0}{b_m p^m + \dots + b_0}\,.$$

Observe also that $\mathbf{R}$ contains $\mathbf{E}$ as a subfield, so that $\mathbf{H}$ contains a subfield which is isomorphic with $\mathbf{E}$.

An important consequence of this ismorphism is the fact that the partial fraction expansion theorem remains true, *i. e.* for each $A \in \mathbf{H}$ there exist $a_i, \alpha_i, \lambda_{ik} \in \mathbf{E}$ such that

$$A = \sum_j a_j D^j + \sum_{i,k} \lambda_{ik}/(D - \alpha_i)^k\,, \quad k \geqq 1\,, \quad j \geqq 0\,. \tag{2,3}$$

In addition, for further purposes let us derive some special properties of Heaviside's operators. If $A \in \mathbf{H}$, $A \neq 0$, let $A^{-n} = (A^{-1})^n$ for $n = 1, 2, \dots$ A number α will be called a pole of the operator $A \in \mathbf{H}$ if α is a pole of the isomorphic element $A(p)$, $A \leftrightarrow A(p) \in \mathbf{R}$.

Lemma 2,5. *Let $\alpha, p \in \mathbf{E}$, n an integer; then*

$$(D - \alpha)^{-n}(H_0 e^{pt}) = H_0(p - \alpha)^{-n}\{e^{pt} - e^{\alpha t}\sum_{k=0}^{n-1} t^k(p - \alpha)^k/k!\} \tag{2,4}$$

for $\alpha \neq p$, and

$$(D - \alpha)^{-n}(H_0 e^{\alpha t}) = H_0 e^{\alpha t} t^n/n!\,, \tag{2,5}$$

where $H_0(t) = 1$ for $t \geqq 0$, $H_0(t) = 0$ for $t < 0$.

Lemma 2,6. *Let $k \geqq 1$ be an integer, $\alpha \in \mathbf{E}$; then*

$$(D - \alpha)^{-k}\delta_0 = H_0 e^{\alpha t} t^{k-1}/(k - 1)! \tag{2,6}$$

The proofs of Lemmas 2,5 and 2,6 follow immediately from Lemma 2,3 by induction.

Theorem 2,3 *Let $A \in \mathbf{H}$; 1) $f = A\delta_0$ is regular, 2) $f(0+)$ exists and $f(0+) = c$, if and only if*

$$\lim_{p \to \infty} p\,A(p) = c \tag{2,7}$$

exists, where $A \leftrightarrow A(p) \in \mathbf{R}$.

Proof. Suppose that (2,7) holds; then $\lim_{p\to\infty} A(p) = 0$ and, consequently, $A(p) = \sum_{i=1}^{n} \sum_{k=1}^{q} \lambda_{ik}/(p - \alpha_i)^k$, where $n, q \geqq 1$ are fixed integers. From this it follows $\lim_{p\to\infty} p \,.\, A(p) = \sum_{i=1}^{n} \lambda_{i1} = c$.

On the other hand from Lemma 2,6 it follows

$$(2,8) \qquad f = A\delta_0 = \sum_{i=1}^{n} \sum_{k=1}^{q} H_0 \lambda_{ik} e^{\alpha_i t} t^{k-1}/(k-1)!\,;$$

hence f is regular and $f(t)$ is continuous in $\langle 0, \infty)$. But from (2,8) one obtains $f(0+) = \sum_{i=1}^{n} \lambda_{i1} = c$, *q. e. d.*

Conversely, let 1), 2) hold; using the partial fraction expansion (2,3) it is evident that $a_j = 0$ is necessary for $A\delta_0$ to be regular. Consequently, (2,8) holds, whence our statement immediately follows.

3. KIRCHHOFF'S NETWORKS

Let us now consider the main problem — the theory of Kirchhoff's networks. Let G be an oriented graph. The ordered pair $(G, \mathfrak{Z})$, where $\mathfrak{Z}$ is a function on $H \times H$ into $\mathbf{E} \times \mathbf{E} \times \mathbf{E}$ will be called a Kirchhoff network (K-network). A K-network will be called passive if the following conditions are satisfied: If $\mathfrak{Z}(h_i, h_k) = (R_{ik}, L_{ik}, S_{ik})$, then $R_{ik}, L_{ik}, S_{ik} \in \mathbf{E}_r$, and

1) $\mathfrak{Z}(h_i, h_k) = \mathfrak{Z}(h_k, h_i)$ for each pair i, k,

2) the square matrices $R = [R_{ik}]$, $L = [L_{ik}]$, $S = [S_{ik}]$ are positive semidefinite. For the sake of brevity we will write $\mathfrak{Z} = (R, L, S)$.

Let N be a K-network and let G be its graph; let further a be the incidence matrix over $\mathbf{E}$, formed in the same manner as for the abstract network, and let c be any solution of the equation $a'x = 0$ in $\mathbf{E}$. In agreement with notions of abstract network theory, $c'h$ will be called a cycle.

The K-network N will be called regular in the time domain (or briefly t-regular), if for every r-dimensional vector E over $\mathbf{D}$ and for any r-dimensional vectors J_0, q_0 over $\mathbf{E}$ (initial condition vectors) a unique r-dimensional vector J over $\mathbf{D}$ exists such that the following conditions are satisfied:

T1: $c'\{RJ + L(J' - J_0\delta_0) + S(J^{(-1)} + q_0 H_0)\} = c'E$

for every cycle $c'h$,

T2: $a'J = 0$.

Note 3,1. The foregoing definition is a generalization of the classical formulation using the notion of function, where T1 is stated as follows:

$$\text{T1*: } c' \left\{ R\, J(t) + L \frac{\mathrm{d}J(t)}{\mathrm{d}t} + S \left(\int_0^t J(\tau)\, \mathrm{d}\tau + q_0 \right) \right\} = c'\, E(t)\,, \quad t \geqq 0\,,$$

$$J(0) = J_0\,.$$

(*Cfr.* [4].)

It is readily seen that with the aid of Heaviside's operators the conditions T1, T2 can be rewritten as

$\overline{\text{T}}1$: $c'\{R + LD + SD^{-1}\}\, J = c'(E + LJ_0\delta_0 - Sq_0H_0)$,

$\overline{\text{T}}2$: $a'J = 0$.

In this manner the problem of t-regularity is reduced to the problem of an abstract network, if one puts $\mathbf{T} = \mathbf{H}$, $\mathbf{P} = \mathbf{D}$.

Note 3,2. In $\overline{\text{T}}1$ the vector c has the meaning of any solution of $a'x = 0$ in $\mathbf{E}$, but in K1 c was a solution of $a'x = 0$ in $\mathbf{T}$, *i. e.* in $\mathbf{H}$ in our case. But it is obvious that if $\overline{\text{T}}1$ holds for any c over $\mathbf{E}$ that it also holds for any c over $\mathbf{H}$, and *vice versa*. (Recall statement c) of Theorem 1,1 and the well-known theorem which states that if some equation $Ax = 0$ has a solution in a field $\mathbf{T}$ and the elements of A belong to some subfield $\mathbf{T}' \subset \mathbf{T}$, that then the equation also has a solution in $\mathbf{T}'$.)

Let us now state the frequency domain problem. Let N be a K-network; N will be called regular in the frequency domain (or briefly p-regular), if for every r-dimensional vector $\bar{E}$ over $\mathbf{R}$ a unique r-dimensional vector $\bar{J}$ over $\mathbf{R}$ exists such that

F1: $c'(R + pL + p^{-1}S)\, \bar{J} = c'\bar{E}$

for every cycle $c'h$ (c being a solution of $a'x = 0$ in $\mathbf{R}$),

F2: $a'J = 0$.

Note 3,3. If we replace the term "cycle" by "loop" in T1, F1, we obtain new definitions of t-regularity and p-regularity, respectively, which are equivalent to the presented ones (cfr. Note 1,1).

Theorem 3,1. *A* K-*network* N *is* t-*regular, if and only if it is* p-*regular.*

Proof. If N is trivial, the theorem is obviously true; thus let N be non-trivial. By Theorem 1,3 and $\overline{\text{T}}1$, $\overline{\text{T}}2$, N is t-regular if and only if $\det X'ZX \neq 0$, where $Z = R + LD + SD^{-1}$, X being taken over $\mathbf{H}$. Likewise, N is p-regular if and only if $\det \bar{X}'\bar{Z}\bar{X} \neq 0$, where $\bar{Z} = R + Lp + Sp^{-1}$, $\bar{X}$ being taken over $\mathbf{R}$. Choosing $\bar{X}$ such that $\bar{X} \leftrightarrow X$, we have $\det \bar{X}'\bar{Z}\bar{X} \leftrightarrow \det X'ZX$ which proves the statement. (See also assertion a) of Theorem 1,3.)

In view of Theorem 3,1, the expression "regular" only will be used. The following two theorems concern the relations between the time and frequency domains.

Theorem 3,2. *Let* N *be a regular* K-*network; let further* $\bar{E}$ *be an* r-*dimensional vector over* $\mathbf{E}$ *and let* $\bar{I}(p)$ (*over* $\mathbf{R}$) *be the corresponding solution of* N *in the frequency* *domain. Then for every* $\tilde{p} \in \mathbf{E}$ *except of the roots of* $\det X'(R + Lp + Sp^{-1}) X$, *the vector* $J = \bar{I}(\tilde{p}) H_0 \exp(\tilde{p}t)$ ($\bar{I}(\tilde{p})$ *is over* $\mathbf{E}$) *is the solution of* N *in time domain, corresponding to* $E = \bar{E} H_0 \exp(\tilde{p}t)$ *and to the initial conditions vectors* $J_0 = \bar{I}(\tilde{p})$, $q_0 = \bar{I}(\tilde{p})/\tilde{p}$.

Proof. By definition, the equations

$$c'(R + Lp + Sp^{-1}) \bar{I}(p) = c'\bar{E}, \quad a' \bar{I}(p) = 0 \tag{3,1}$$

are true for every solution c of $a'x = 0$ in $\mathbf{E}$. As $\bar{I}(p)$ is given by (1,4a), the vector $\bar{I}(\tilde{p})$ over $\mathbf{E}$ is defined for every number $\tilde{p} \in \mathbf{E}$ excluding the roots of $\det X'(R + Lp + Sp^{-1}) X$. Therefore, putting such $\tilde{p}$ for p in (3,1), one obtains an identity in $\mathbf{E}$, *i. e.*

$$c'(R + L\tilde{p} + S\tilde{p}^{-1}) \bar{I}(\tilde{p}) = c'\bar{E}, \quad a' \bar{I}(\tilde{p}) = 0. \tag{3,2}$$

Multiplying (3,2) by $H_0 \exp(\tilde{p}t)$ and adding $c'(L\bar{I}(\tilde{p})\, \delta_0 - S\, \bar{I}(\tilde{p})\, H_0/\tilde{p})$ to both sides, one gets

$$\begin{aligned} c'\{R\, \bar{I}(\tilde{p})\, H_0 e^{\tilde{p}t} + L\bar{I}(\tilde{p})\, (H_0 \tilde{p} e^{\tilde{p}t} + \delta_0) + S\, \bar{I}(\tilde{p})\, \tilde{p}^{-1} H_0(e^{\tilde{p}t} - 1)\} = \\ = c'(\bar{E} H_0 e^{\tilde{p}t} + L\bar{I}(\tilde{p})\, \delta_0 - S\, \bar{I}(\tilde{p})\, \tilde{p}^{-1} H_0). \end{aligned} \tag{3,3}$$

Comparing (3,3) with $\bar{\text{T}}1$, it is readily seen that $\bar{I}(\tilde{p}) H_0 \exp(\tilde{p}t)$ is the solution of N in the time domain and the theorem is proved.

Theorem 3,3. *Let* N *be a regular* K-*network; let further* E_k *be an* r-*dimensional vector over* $\mathbf{E}$, $E_k' = [0, 0, \ldots, 0, 1, 0, \ldots, 0]$ (*the unit standing in the* k-*th place*) *and* $I_k(p)$ *the corresponding solution of* N *in the frequency domain. Then the following statement holds: If* $e \in \mathbf{D}$, *then* $I_k(D)\, e$ (*where* $I_k(D) \leftrightarrow I_k(p)$) *is the solution of* N *in the time domain for zero initial condition vectors and* $E' = [0, 0, \ldots, 0, e, 0, \ldots, 0]$.

The proof is obvious.

The sufficient and necessary conditions for a general K-network to be regular can be obtained as a simple consequence of Theorems 1,3; 1,4; 1,8. In order to be able to give more effective conditions for passive networks, let us first consider some special properties of matrices over $\mathbf{R}$.

4. FUNCTION-THEORETICAL PROPERTIES OF A CLASS OF MATRICES OVER R

Let Z be a matrix over $\mathbf{R}$; the statement "Z has the pole α of m-th order" will mean that each element of Z has a pole of at most m-th order in α and at least one element has in α a pole of exactly m-th order. Let further G be the set of all complex numbers with positive real parts, $\bar{G}$ its closure (∞ belongs to $\bar{G}$).

Let $\mathfrak{S}_n$ be the set of all symmetrical matrices Z over $\mathbf{R}_r$ of type $n \times n$ which fulfil the following condition:

$$\operatorname{Re} x'Zx \geqq 0 \tag{4,1}$$

for every real n-dimensional vector x and for any $p \in G$ which is not a pole of Z. Let $\mathfrak{P}_n$ be the set of all matrices belonging to $\mathfrak{S}_n$ which fulfil the condition

$$\text{(4,2)} \qquad \operatorname{Re} x'Zx > 0$$

for every real n-dimensional non-zero vector x and for every $p \in G$ which is not a pole of Z.

Obviously: a) $Z_1, Z_2 \in \mathfrak{S}_n \Rightarrow \alpha_1 Z_1 + \alpha_2 Z_2 \in \mathfrak{S}_n$ provided $\alpha_1, \alpha_2 \geqq 0$, b) $Z_1 \in \mathfrak{S}_n$, $Z_2 \in \mathfrak{P}_n \Rightarrow Z_1 + Z_2 \in \mathfrak{P}_n$.

Observe that in particular *every positive (semi-) definite matrix belongs to* $(\mathfrak{S}_n)$, $\mathfrak{P}_n$.

Let us first consider the case $n = 1$. It can be easily shown that the following statement is true (see [5]):

Lemma 4,1. a) *If $Z \in \mathfrak{S}_1$, then either $Z \in \mathfrak{P}_1$ or $Z \equiv 0$.*

b) *If $Z \in \mathfrak{P}_1$ then Z has no poles in G and every pole on the imaginary axis (if it exists) is simple with real positive residuum; the same is true for the pole at infinity.*[2])

c) *If $Z \in \mathfrak{S}_1$, then there are real numbers $\omega_1, \omega_2, \ldots, \omega_m, \lambda_0, \lambda_1, \ldots, \lambda_m \geqq 0$ such that*

$$\text{(4,3)} \qquad Z = \tilde{Z} + \lambda_0 p + \sum_{k=1}^{m} \frac{\lambda_k p}{p^2 + \omega_k^2},$$

where $\tilde{Z}$ has no pole in $\bar{G}$ and $\tilde{Z} \in \mathfrak{S}_1$.

For $n \geqq 1$ the following statement is true:

Theorem 4,1. *If $Z \in \mathfrak{S}_n$, then there exist real numbers $\omega_1, \omega_2, \ldots, \omega_m$ and constant matrices $H_k \in \mathfrak{S}_n$, $k = 0, 1, 2, \ldots, m$, such that*

$$\text{(4,4)} \qquad Z = \tilde{Z} + H_0 p + \sum_{k=1}^{m} H_k \frac{p}{p^2 + \omega_k^2},$$

where $\tilde{Z} \in \mathfrak{S}_n$ has no poles in $\bar{G}$.

Proof. Let us first show that Z has no poles in G and that the poles occurring on the imaginary axis or at infinity are simple with real residues. By definition, $x'Zx \in \mathfrak{S}_1$ for every real vector x. From this it follows in particular that $Z_{ii} \in \mathfrak{S}_1$ for $i = 1, 2, \ldots, n$. Choosing indices r, q, $1 \leqq r, q \leqq n$ and putting $x_r = x_q = 1$, $x_i = 0$ for $i \neq r, q$ as components of x, one obtains $\Phi = x'Zx = Z_{rr} + Z_{qq} + 2Z_{rq} \in \mathfrak{S}_1$. Consequently, by Lemma 4,1, the elementes Z_{rr}, Z_{qq}, Z_{qr} have the mentioned property, which proves our statement. Expanding each element of Z into partial fractions, one can write

$$\text{(4,5)} \qquad Z = \tilde{Z} + H_0 p + \sum_{k=1}^{m} H_k \frac{p}{p^2 + \omega_k^2},$$

[2]) By "residuum at ∞" the coefficient c_1 in the expansion $Z = c_1 p + c_0 + c_{-1} p^{-1} + \ldots$ will be meant.

where H_0, H_k are real matrices and $\tilde{Z}$ has no poles in $\bar{G}$. (Note that to each pole p_0 of Z there corresponds the conjugate pole $\bar{p}_0$, as the elements of Z belong to $\mathbf{R}_r$; therefore the expansion of Z necessarily has the form (4,5).) Let now x be any real vector; by (4,5)

$$x'Zx = x'\tilde{Z}x + px'H_0x + \sum_{k=1}^{m} \frac{p}{p^2 + \omega_k^2} x'H_kx \in \mathfrak{S}_1 .$$

But by Lemma 4,1c), $x'\tilde{Z}x \in \mathfrak{S}_1$, $x'H_kx \geqq 0$ for $k = 0, 1, 2, \ldots, m$, which proves the Theorem.

Theorem 4,2. *Let* $Z \in \mathfrak{S}_n$; *then* $Z \in \mathfrak{P}_n$ *if and only if* $\det Z \neq 0$ *for every* $p \in G$.

Proof. Let $Z \in \mathfrak{S}_n$ and let $\det Z \neq 0$ for every $p \in G$; suppose that there exist a number $p_0 \in G$ and a non-zero real vector $\tilde{x}$ such that

$$\text{(4,6)} \qquad \operatorname{Re} \tilde{x}' Z(p_0) \tilde{x} = 0 .$$

As $\tilde{x}' Z(p) \tilde{x} \in \mathfrak{S}_1$, it follows by Lemma 4,1a) that $\tilde{x}' Z(p) \tilde{x} \equiv 0$. Therefore

$$\text{(4,7)} \qquad \tilde{x}' \operatorname{Re} Z(p) \tilde{x} = 0$$

for every $p \in G$. From the well-known theorem on quadratic forms and from (4,7) it follows that $\det [\operatorname{Re} Z_{ik}(p)] = 0$ for every $p \in G$; but $\det [\operatorname{Re} Z_{ik}(p)] = \det [Z_{ik}(p)]$ for every real p belonging to G. However, $\det [Z_{ik}(p)]$ is a regular function in G; consequently, $\det [Z_{ik}(p)] \equiv 0$, which is a contradiction. Hence $Z \in \mathfrak{P}_n$.

Conversely, let $Z \in \mathfrak{P}_n$ and suppose that there is a number $p_0 \in G$ such that $\det Z(p_0) = 0$. Then a non-zero vector $z = x + iy$ exists such that $Z(p_0) z = 0$; consequently, $\varphi = \bar{z}' Z(p_0) z = 0$. But

$$\text{(4,8)} \qquad \varphi = x' Z(p_0) x + y' Z(p_0) y + i(x' Z(p_0) y - y' Z(p_0) x) = 0 .$$

As the last term of the right hand side of (4,8) vanishes, one obtains $\operatorname{Re} \{x' Z(p_0) x + y' Z(p_0) y\} = 0$, which is in contradiction with the assumption $Z \in \mathfrak{P}_n$. Therefore $\det Z(p) \neq 0$ for every $p \in G$ and the Theorem is proved.

Using the proof of Theorem 4,2 again, one can state:

Theorem 4,3. *Let* $Z \in \mathfrak{S}_n$; *then* $Z \in \mathfrak{P}_n$ *if and only if* $\det Z \neq 0$ *in* $\mathbf{R}$ (*i. e. if it does not vanish identically*).

Actually, by Theorem 4,2 we have: $Z \in \mathfrak{P}_n \Rightarrow \det Z \neq 0$ for $p \in G \Rightarrow \det Z \not\equiv 0$. Conversely, if $Z \in \mathfrak{S}_n$ and $\det Z \not\equiv 0$, suppose that there exist a number $p_0 \in G$ and a non-zero real vector $\tilde{x}$ such that (4,6) holds; from this it follows in the same way as before that $\det [Z_{ik}] \equiv 0$, which is a contradiction.

Moreover, one can state:

Theorem 4,4. *If* $Z \in \mathfrak{P}_n$ *then* Z^{-1} *exists and* $Z^{-1} \in \mathfrak{P}_n$.

Proof. The existence of Z^{-1} is a simple consequence of Theorem 4,3. It is obvious that Z^{-1} is symmetrical and by Theorem 4,2 $\det Z(p) \neq 0$ for every $p \in G$. Choosing arbitrarily a real non-zero vector z and a number $p_0 \in G$, let us find the vector ω satisfying the equation $z = Z(p_0)\,\omega$. Evidently, ω is determined uniquely and $\omega \neq 0$. Putting $\omega = x + iy$, one can write

$$\begin{aligned} \operatorname{Re} z' Z^{-1}(p_0)\, z &= \operatorname{Re} \bar{z}' Z^{-1}(p_0)\, z = \\ &= \operatorname{Re} \bar{\omega}' \overline{Z(p_0)}\, Z^{-1}(p_0)\, Z(p_0)\, \omega = \operatorname{Re} \bar{\omega}' \overline{Z(p_0)}\, \omega = \\ &= \operatorname{Re} \{x' \overline{Z(p_0)}\, x + y' \overline{Z(p_0)}\, y + i(x' \overline{Z(p_0)}\, y - y' \overline{Z(p_0)}\, x)\} = \\ &= x' \operatorname{Re} Z(p_0)\, x + y' \operatorname{Re} Z(p_0)\, y > 0\,. \end{aligned}$$

Hence $Z^{-1} \in \mathfrak{P}_n$, *q. e. d.*

Lemma 4,2. *Let $Z \in \mathfrak{S}_n$; then for every imaginary $i\omega_0$ different from all poles of Z, the constant matrix* $\operatorname{Re} Z(i\omega_0)$ *belongs to $\mathfrak{S}_n$.*

The proof follows immediately from the continuity of elements of Z at the point $i\omega_0$.

Lemma 4,3. a) *If $Z \in \mathfrak{S}_n$ and C is any real constant $n \times k$ matrix, then $C'ZC \in \mathfrak{S}_k$.* b) *If $Z \in \mathfrak{P}_n$ and C is any real constant $n \times k$ matrix with rank k, then $C'ZC \in \mathfrak{P}_k$.*

Proof. If x is any real k-dimensional vector, then $x'(C'ZC)\,x = (Cx)'\,Z(Cx)$, from which a) follows. Moreover, if C has rank k, then from $x \neq 0$ it follows that $Cx \neq 0$ and therefore b) holds.

Theorem 4,5. *Let R be a real constant matrix and let $R \in \mathfrak{P}_n$, $Z \in \mathfrak{S}_n$; then $(R + Z)^{-1}$ has no poles in $\bar{G}$.*

Proof. Evidently $W = R + Z \in \mathfrak{P}_n$, so that by Theorem 4,4, W^{-1} exists and belongs to $\mathfrak{P}_n$; moreover, by Theorem 4,1, W^{-1} has no poles in G and the poles on the imaginary axis and at infinity are simple. Suppose that $i\omega_0$ is a pole of W^{-1}. From Theorem 4,1 it follows that

$$W^{-1} = A \frac{1}{p - i\omega_0} + B(p)\,, \tag{4,9}$$

where A is a real constant matrix belonging to $\mathfrak{S}_n$ and the elements of $B(p)$ are regular at $i\omega_0$. Similarly one can write

$$W = H \frac{1}{p - i\omega_0} + Q(p)\,, \tag{4,10}$$

where by Theorem 4,1

$$Q(p) = R + Z^* + H \frac{1}{p + i\omega_0} + H_0 p + \sum_{i=1}^{m} \frac{p}{p^2 + \omega_i^2} H_i\,, \tag{4,11}$$
$$\omega_i \neq \omega_0\,, \quad i = 1, 2, \ldots, m\,,$$

H, H_0, H_i being real constant matrices belonging to $\mathfrak{S}_n$, $Z^* \in \mathfrak{S}_n$ having no poles in $\bar{G}$. (H may be the zero matrix.) Let further $Q_0 = Q(i\omega_0)$, $B_0 = B(i\omega_0)$. From the assumption $R \in \mathfrak{P}_n$ and from Lemma 4,2 it follows immediately that $\operatorname{Re} Q_0 = R + \operatorname{Re} Z^*(i\omega_0) \in \mathfrak{P}_n$.

The identity $W^{-1}W = I$ (I is the unit matrix) then yields

$$(4,12) \qquad AH \frac{1}{(p - i\omega_0)^2} + [A\, Q(p) + B(p)\, H] \frac{1}{p - i\omega_0} + B(p)\, Q(p) = I.$$

From this it follows that $AH = 0$. Substituting this into (4,12), multiplying by $p - i\omega_0$ and letting $p \to i\omega_0$, one obtains

$$(4,13) \qquad AQ_0 + B_0 H = 0\,.$$

As $HA = AH = 0$ (the matrices A, H are symmetrical), then multiplying (4,13) by A one obtains the equation $AQ_0A = 0$. Therefore

$$(4,14) \qquad A(\operatorname{Re} Q_0)\, A = 0\,.$$

If now some column ξ of A were different from the zero vector, then according to the inequality $\xi' \operatorname{Re} Q_0 \xi > 0$ there would be $A(\operatorname{Re} Q_0)\, A \neq 0$; therefore $A = 0$.

The same is true for the pole of W^{-1} at infinity, which proves our statement.

5. PASSIVE K-NETWORKS

Let us now consider the passive K-networks in greater detail.

Theorem 5,1. *Let N be a passive* K-*network; let C be the set of all real solutions of $a'x = 0$, a being the incidence matrix; let Π_R, Π_L, Π_S be the sets of all real vectors x such that $x'Rx = 0$, $x'Lx = 0$, $x'Sx = 0$ respectively. Then N is regular if and only if the intersection $\Pi = C \cap \Pi_R \cap \Pi_L \cap \Pi_S$ contains only the zero vector.*

Proof. Suppose that there exist an $x \neq 0$, $x \in \Pi$. Then $x = Xy$, $y \neq 0$, and simultaneously $x'Rx = 0$, $x'Lx = 0$, $x'Sx = 0$. Consequently, for any $p \in G$ it holds: $\operatorname{Re}\{x'Rx + px'Lx + p^{-1}x'Sx\} = \operatorname{Re} x'\{R + pL + p^{-1}S\}\, x = 0$. Therefore $\operatorname{Re} y'X'ZXy = 0$, where $Z = R + pL + p^{-1}S \in \mathfrak{S}_r$, so that $X'ZX \notin \mathfrak{P}_n$. Then $\det X'ZX = 0$ in $\mathbf{R}$ by Theorem 4,2 and N is not regular.

Conversely, suppose that N is not regular, *i. e.* that $\det X'ZX = 0$ in $\mathbf{R}$; as $Z \in \mathfrak{S}_r$, then $X'ZX \in \mathfrak{S}_n$, $X'ZX \notin \mathfrak{P}_n$ by Theorem 4,2. Consequently, there exist a real vector $u \neq 0$ and a number $p \in G$ such that $\operatorname{Re} u'X'ZXu = 0$, *i. e.*

$$\operatorname{Re} u'X'RXu + \operatorname{Re} u'pX'LXu + \operatorname{Re} u'p^{-1}X'SXu = 0\,.$$

Putting $y = Xu$, one obtains $y \neq 0$ and $y \in C$. Then the latter equation can be rewritten as

$$\operatorname{Re} y'Ry + \operatorname{Re} py'Ly + \operatorname{Re} p^{-1}y'Sy = 0\,;$$

since for $p \in G$ we have

$$\operatorname{Re} p > 0\,, \quad \operatorname{Re} p^{-1} > 0$$

and simultaneously $y'Ry \geqq 0, \ldots$, it follows that $y'Ry = 0, \ldots$; hence $y \in \Pi$ and the Theorem is proved.

A simple consequence of Theorem 5,1 is the following:

Theorem 5,2. *A passive K-network is regular if and only if*

$$(5,1) \qquad c'(R + L + S)\, c > 0$$

for every real non-zero cycle $c'h$ (i. e. for every real non-zero solution of $a'x = 0$).

Proof. Let (5,1) be satisfied for every $c \in C$, $c \neq 0$; then for every $c \in C$, $c \neq 0$ either $c'Rc > 0$ or $c'Lc > 0$ or $c'Sc > 0$; consequently $c \notin \Pi_R \cap \Pi_L \cap \Pi_S$, so that $\Pi = \{0\}$, and by Theorem 5,1 the network is regular.

Conversely, let the network be regular; then by Theorem 5,1 $\Pi = \{0\}$, *i. e.* for $c \in C$, $c \neq 0$ there is $c \notin \Pi_R \cap \Pi_L \cap \Pi_S$; hence at least one of the numbers $c'Rc$, $c'Lc$, $c'Sc$ is positive and (5,1) holds, *q. e. d.*

Lemma 5,1. *Let $T = \operatorname{diag}(T_1, T_2, \ldots, T_r)$, $T_i \geqq 0$ for $i = 1, 2, \ldots, r$. If*

$$(5,2) \qquad c'Tc > 0$$

for every loop $c'h$, then (5,2) holds for every real non-zero cycle.

Proof. Let $\tilde{c}'h$ be a real non-zero cycle. By Theorem 1,2, where we put $\boldsymbol{T} = \boldsymbol{E}_r$, we have $c'h = \sum_{j=1}^{l} d_j \sum_{i=1}^{r} e_{ji}h_i$, $\sum_{i=1}^{r} e_{ji}h_i$ being loops and

$$(5,3) \qquad e_{ji} \neq 0 \text{ for some } j \Rightarrow \tilde{c}_i \neq 0,\ i = 1, 2, \ldots, r\,.$$

By hypothesis $\sum_{i=1}^{r} T_i e_{ji}^2 > 0$ for each $j = 1, 2, \ldots, l$, and, consequently, $T_i e_{ji}^2 > 0$ for at least one pair (j^*, i^*). Hence and from (5,3) it follows that $T_{i^*}\tilde{c}_{i^*}^2 > 0$ which implies (5,2) for $c = \tilde{c}$.

Note 5,1. Let R, L, S be diagonal; from Lemma 5,1 and from Theorem 5,2 it follows that a passive K-network is regular if and only if for every loop $c'h$ any one of the following conditions is satisfied: (i) $c'Rc > 0$, (ii) $c'Lc > 0$, (iii) $c'Sc > 0$. The condition that R, L, S be diagonal cannot be omitted, since the assertion of Lemma 5,1 need not hold if the condition that T is diagonal with non-negative elements is replaced by the condition that T is positive semidefinite. This is shown in the following example:

Example 5,1. *Let the graph G_1 be defined as follows:*

$$H = \{h_1, h_2, h_3\}\,, \quad U = \{u_1, u_2\}\,, \quad \Gamma(h_1) = \Gamma(h_2) = \Gamma(h_3) = (u_1, u_2)\,.$$

Let

$$M = \begin{bmatrix} 1, & -1, & 0 \\ -1, & 1, & 0 \\ 0, & 0, & 0 \end{bmatrix};$$

M is obviously positive semidefinite. Evidently, for every loop $c'h$ of G_1 we have $c'Mc > 0$. However, $d'Md = 0$ for the cycle $d'h = h_1 + h_2 - 2h_3$.

Let M be a constant $n \times n$ matrix, $M \in \mathfrak{S}_n$. Let Π_M be the set of all real vectors x fulfilling $x'Mx = 0$.

Lemma 5,2. *Let $x \in \Pi_M$, y an n-dimensional vector over $\mathbf{E}_r$. Then $y'Mx = 0$.*

Proof. For every real number ε there holds

$$(y + \varepsilon x)' M(y + \varepsilon x) = y'My + 2\varepsilon y'Mx \geqq 0 .$$

If $y'Mx \neq 0$, ε could be chosen so that $y'My + 2\varepsilon y'Mx < 0$, which is impossible.

Lemma 5,3. *Let $\mathfrak{M}$ be the system of all real solutions of $Mx = 0$. Then $\mathfrak{M} = \Pi_M$.*

Proof. Evidently $\mathfrak{M} \subset \Pi_M$. On the other hand, let $x \in \Pi_M$. As by Lemma 5,2 $y'Mx = 0$ for every vector y over $\mathbf{E}_r$, one has $Mx = 0$. Thus $x \in \mathfrak{M}$ *q. e. d.*

Theorem 5,3. *Let N be a passive* K*-network. N is regular if and only if the rank of the matrix $Y = [a \mid R \mid L \mid S]$ is r.*

Proof. $Y'x = 0$ has a solution $c \neq 0$ if and only if the rank of Y is smaller than r. By Lemma 5,3, $Y'c = 0$ if and only if $c \in C \cap \Pi_R \cap \Pi_L \cap \Pi_S$. From this and from Theorem 5,1 the proof follows.

Let us now examine some properties of passive K-networks in the time-domain. An r-dimensional vector x over $\mathbf{D}$ will be called a C-vector if

$$x = \sum_{i=0}^{k} a_i \delta_0^{(i)} + \bar{x}, \quad k \geqq 0, \tag{5,4}$$

where a_i, $i = 0, 1, \ldots, k$ are constant vectors, $\bar{x}$ is a vector the elements $\bar{x}_k$ of which are regular distributions, the corresponding functions $\bar{x}_k(t)$ being continuous in $\langle 0, \infty)$. The vector x will be called a regular C-vector if x is a C-vector and if the elements of x are regular distributions, *i. e.* if in the expansion (5,4) all the a_i vanish. If x is a C-vector, let $\|x\|_t = \max_{1 \leqq i \leqq r} [|\bar{x}_i(t)|]$ for $t \geqq 0$. Likewise, if c is a constant vector, let $\|c\| = \max_{1 \leqq i \leqq r} [|c_i|]$, c_i being its components.

Let now N be a regular passive K-network and let J be its solution in the time-domain corresponding to the vector E over $\mathbf{D}$ and to the initial conditions J_0, q_0. The solution J will be called stable with respect to the initial conditions, if the following conditions are satisfied: For every $\varepsilon > 0$ a $\delta > 0$ exists such that for every solution J^* of N corresponding to the vector E and to initial conditions J_0^*, q_0^* which satisfy $\|J_0 - J_0^*\| < \delta$, $\|q_0 - q_0^*\| < \delta$, the vector $J - J^*$ is a C-vector and $\|J - J^*\|_t < \varepsilon$ for every $t \geqq 0$. If in addition to this $\|J - J^*\|_t \to 0$ for $t \to \infty$, then the solution J will be called asymptotically stable.

Let N be a passive K-network; if $c'Rc > 0$ for every non-zero real cycle $c'h$, then N will be called a dissipative network.

Theorem 5,4. *Every solution of a regular passive* K*-network N is stable with respect to the initial conditions.*

Theorem 5,5. *Every dissipative network is regular and every its solution is asymptotically stable.*

Proof of Theorem 5,4. Choose arbitrarily an E over $\mathbf{D}$ and initial condition vectors

J_0, q_0. Let further X be a constant matrix over $\boldsymbol{H}$, which will be fixed in the subsequent considerations. From (1,4a) and $\overline{T}_1$, $\overline{T}_2$ it follows that

$$(5,5) \qquad J = X(X'ZX)^{-1} X'(E + LJ_0\delta_0 - Sq_0H_0)\,,$$

where $Z = R + LD + SD^{-1}$. By Theorems 4,3 and 4,4, $(X' Z(p) X)^{-1} \in \mathfrak{P}_n$ so that, by Lemma 4,3, $A(p) = X(X' Z(p) X)^{-1} X' \in \mathfrak{S}_r$. Consequently, by Theorem 4,1,

$$(5,6) \qquad A(p) = A_1 p + A_0 + B(p)\,,$$

where A_1, A_0 are constant matrices and $B(p)$ vanishes at infinity. Moreover, $B(p)$ can be expanded as follows (see Theorem 4,1):

$$(5,7) \qquad B(p) = \sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}(p - \alpha_i)^{-k} + \sum_{j=1}^{s} \Gamma_j(p - \beta_j)^{-1} + \Gamma_0 p^{-1}\,,$$

where A_{ik}, Γ_j, Γ_0 are constant matrices and where $\operatorname{Re} \alpha_i < 0$ for $i = 1, 2, \ldots, m$; $\operatorname{Re} \beta_j = 0$, $\beta_j \neq 0$ for $j = 1, 2, \ldots, s$.

Let us now choose any constant vectors J_0^*, q_0^*. For the solution J^* of N corresponding to E, J_0^*, q_0^* one obtains by (5,5)

$$(5,8) \qquad J - J^* = A[L(J_0 - J_0^*)\,\delta_0 - S(q_0 - q_0^*)\,H_0]\,.$$

From (5,6) and (5,8) it follows that

$$(5,9) \quad J - J^* = f - A_0S(q_0 - q_0^*)\,H_0 + BL(J_0 - J_0^*)\,\delta_0 - BS(q_0 - q_0^*)\,H_0\,,$$

where

$$(5,10) \quad f = A_1L(J_0 - J_0^*)\,\delta_0' - A_1S(q_0 - q_0^*)\,\delta_0 + A_0L(J_0 - J_0^*)\,\delta_0\,.$$

By (5,7) and Lemma 2,6 one has

$$(5,11) \quad BL(J_0 - J_0^*)\,\delta_0 = \Big(\sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}e^{\alpha_i t}t^{k-1}/(k-1)!\Big)\,L(J_0 - J_0^*)\,H_0 + \\ + \sum_{j=1}^{s} \Gamma_j e^{\beta_j t}\Big)\,L(J_0 - J_0^*)\,H_0 + \Gamma_0 L(J_0 - J_0^*)\,H_0\,.$$

Similarly, by Lemma 2,5

$$(5,12) \quad BS(q_0 - q_0^*)\,H_0 = \Big\{\sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}(-\alpha_i)^{-k}\Big[1 - e^{\alpha_i t}\sum_{\nu=0}^{k-1} t^\nu(-\alpha_i)^\nu/\nu!\Big]\Big\}\,. \\ .\,S(q_0 - q_0^*)\,H_0 + \Big\{\sum_{j=1}^{s} \Gamma_j(-\beta_j)^{-1}\big[1 - e^{\beta_j t}\big]\Big\}\,S(q_0 - q_0^*)\,H_0 + \\ + \Gamma_0 S(q_0 - q_0^*)\,H_0 t\,.$$

Consider now the obvious identity

$$(5,13) \qquad A(p)\,Z(p)\,X = X\,;$$

it is readily seen that $p\,A(p) \to \Gamma_0$, $p\,Z(p) \to S$ for $p \to 0$. Consequently, multiplying (5,13) by p^2 and letting $p \to 0$ it follows that $\Gamma_0 S = 0$.

From eqs. (5,9), (5,10), (5,11), (5,12) it follows that $J - J^*$ is a C-vector; moreover, it is obvious that there exist positive numbers μ, ν depending on N only, such that

$$(5,14) \qquad \|J - J^*\|_t \leqq \mu\|J_0 - J_0^*\| + \nu\|q_0 - q_0^*\|$$

for every $t \geqq 0$. But (5,14) proves the Theorem.

Proof of Theorem 5,5: The regularity of a dissipative network follows immediately from Theorem 5,2. From the assumption $c`Rc > 0$ for every non-zero cycle $c`h$ it follows that $X`RX \in \mathfrak{P}_n$; simultaneously, by Lemma 4,3 $X`(Lp + Sp^{-1}) X \in \mathfrak{S}_n$. Consequently, by Theorem 4,5, $[X`(R + Lp + Sp^{-1}) X]^{-1}$ has no poles in $\bar{G}$, from which it follows that the matrix $A(p)$ also has no poles in $\bar{G}$. Using the results of the proof of Theorem 5,4, it follows that in (5,6) $A_1 = 0$ and in (5,11) (5,12) $\Gamma_0 = 0$, $\Gamma_j = 0$ for $j = 1, 2, \ldots, s$.

From (5,6) and (5,7) it follows that $A(p) \to A_0 + \sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}(-\alpha_i)^{-k}$ for $p \to 0$; consequently, multiplying the identity (5,13) by p and letting $p \to 0$ it follows that

$$(5,15) \qquad (A_0 + \sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}(-\alpha_i)^{-k}) S = 0 .$$

On the other hand, from (5,9), (5,11), (5,12) it follows that

$$J - J^* - f \to - A_0 S(q_0 - q_0^*) - (\sum_{i=1}^{m} \sum_{k=1}^{q} A_{ik}(-\alpha_i)^{-k}) S(q_0 - q_0^*)$$

for $t \to \infty$; consequently, by (5,15) $\|J - J^*\|_t \to 0$ for $t \to \infty$, which proves the Theorem.

Let us now solve the compatibility problem. Let N be a regular passive K-network; the ordered triple of vectors (E, J_0, q_0) (E over **D**, J_0, q_0 over **E**) will be called compatible, if the corresponding solution J on N is a regular C-vector and if $J(0+) = J_0$. Then the following statement holds:

Theorem 5,6. *Let N be a regular passive* K*-network; let further E be a regular C-vector, E' be a C-vector; denote $E_0 = E(0+)$, $E_1 = F(0+)$, where $F = E' - E_0\delta_0$. Then the triple (E, J_0, q_0) is compatible if and only if*

$$(5,16) \qquad \lim_{p \to \infty} A(p) \{E_0 + E_1 p^{-1} + LJ_0 p - Sq_0\} = J_0 ,$$

where $A(p) = X(X` Z(p) X)^{-1} X`$.

Moreover, if $A(p)$ has no pole at infinity, then the assumption "E' is a C-vector" as well as the term $E_1 p^{-1}$ in (5,16) may be omitted.

Proof. Let us define the vector E^* by the relation $E^* = E - E_0 H_0 - E_1 H_0 t$. It is obvious that E^* is a regular C-vector for which $E^*(0+) = E^*(0-) = 0$; simultaneously, $E^{*\prime} = E' - E_0\delta_0 - E_1 H_0 = F - E_1 H_0$; consequently, $E^{*\prime}$ is a regular C-vector and $E^{*\prime}(0+) = E^{*\prime}(0-) = 0$.

By the foregoing, for the solution J corresponding to E, J_0, q_0 one can write

$$(5,17) \qquad J = X(X`ZX)^{-1} X`(E + LJ_0\delta_0 - Sq_0 H_0) .$$

According to Lemma 4,3 $A(p) = X(X'ZX)^{-1} X' \in \mathfrak{S}_r$; consequently, $A(p)$ has a pole of at most first order at ∞. Therefore $A = A_{-1}D + A_0 + B$, where the matrix $B(p) \leftrightarrow B$ vanishes at infinity. Rewriting (5,17), one obtains

$$(5,18) \quad J = A(E_0 H_0 + E_1 H_0 t + LJ_0 \delta_0 - Sq_0 H_0) + (A_{-1}D + A_0 + B) E^* .$$

It is obvious that $\bar{J} = (A_{-1}D + A_0 + B) E^*$ is a regular C-vector satisfying the equation $\bar{J}(0+) = 0$. Consequently, the first term on the right hand side of (5,18) will decide whether (E, J_0, q_0) is compatible or not. But $A(E_0 H_0 + \ldots) = A(E_0 D^{-1} + E_1 D^{-2} + LJ_0 - Sq_0 D^{-1}) \delta_0$. According to Theorem 2,3, the necessary and sufficient condition for $\bar{J}(0+) = J_0$ is that $\lim_{p\to\infty} p\, A(p)(E_0 p^{-1} + E_1 p^{-2} + LJ_0 - Sq_0 p^{-1}) = J_0$, *q. e. d.*

The proof of the second statement is obvious.

The assertion of the preceding Theorem can be completed by the following.

Theorem 5,7. *Let N be a passive* K*-network and let C, Π_R, Π_L, Π_S have the meaning mentioned above (see Theorem 5,1). Let $A(p) = X(X' Z(p) X)^{-1} X'$ (provided that it exists), where $Z(p) = R + Lp + Sp^{-1}$. Then the following statements are true:*

1. *If $C \cap \Pi_L = \{0\}$, then N is regular and every element of the matrix $A(p)$ has a zero at infinity; simultaneously $\lim_{p\to\infty} p\, A(p)\, LX = X$.*

2. *If $C \cap \Pi_L \cap \Pi_R = \{0\}$, then N is regular and $A(p)$ has no pole at infinity.*

Conversely, if N is regular and

1*. *each element of $A(p)$ has a zero at infinity, then $C \cap \Pi_L = \{0\}$ and $\lim_{p\to\infty} p\,.\,A(p)\, LX = X$;*

2*. *if $A(p)$ has no pole at infinity, then $C \cap \Pi_L \cap \Pi_R = \{0\}$.*

For the proof the following obvious Lemma will be necessary:

Lemma 5,4. *Let Y be a real constant $r \times n$ matrix with rank n; then there is a real constant $n \times r$ matrix q such that $qY = I$, where I is the $n \times n$ unit matrix.*

Proof of Theorem 5,7. Let $\bar{L} = X'LX$, $\bar{R} = X'RX$, $\bar{S} = X'SX$, $\bar{Z}(p) = \bar{L}p + \bar{R} + \bar{S}p^{-1}$; if N is regular, then $\bar{Z}^{-1}(p) \in \mathfrak{P}_n$ exists. By Theorem (4,1), one can write

$$(5,19) \qquad \bar{Z}^{-1}(p) = Hp + Q(p) ,$$

where H is a symmetrical real constant matrix belonging to $\mathfrak{S}_n$ and where $Q(p) \in \mathfrak{S}_n$ has no pole at infinity. The identity $Z^{-1}Z = I$ yields

$$(5,20) \qquad H\bar{L}p^2 + Q(p)\, \bar{L}p + H\bar{R}p + Q(p)\, \bar{R} + H\bar{S} + Q(p)\, Sp^{-1} = I .$$

Let now the assumption 1 be satisfied. Then obviously $\bar{L} \in \mathfrak{P}_n$ and N is regular. (In particular, det $\bar{L} \neq 0$.) From the identity (5,20) it follows that $H\bar{L} = 0$, therefore $H = 0$. Substituting in (5,20) one gets

$$(5,21) \qquad Q(p)\, \bar{L}p + Q(p)\, \bar{R} + Q(p)\, \bar{S}p^{-1} = I .$$

Dividing by p and letting $p \to \infty$, one obtains $\lim_{p\to\infty} Q(p) \cdot \bar{L} = 0$, so that $Q(\infty) = 0$. From (5,21) it follows that $\lim_{p\to\infty} Q(p)\, \bar{L} p = I$. Therefore, the elements of $\bar{Z}^{-1}(p)$ have a zero at infinity and, consequently, the same is true for $A(p) = X\, \bar{Z}^{-1}(p)\, X'$. From the foregoing equation it follows that

$$X = X(\lim_{p\to\infty} Q(p)\, p)\, \bar{L} = X(\lim_{p\to\infty} \bar{Z}^{-1}(p)\, p)\, X' L X = \\ = \lim_{p\to\infty} p(X\, \bar{Z}^{-1}(p)\, X')\, LX = \lim_{p\to\infty} p\, A(p)\, LX\,,$$

q. e. d.

Let now the assumption 2 be satisfied. Then evidently $\bar{L} + \bar{R} \in \mathfrak{P}_n$ and N is regular by Theorem 5,1. The identity (5,20) yields the equation $H\bar{L} = 0$. Putting $Q_\infty = \lim_{p\to\infty} Q(p)$, and dividing (5,20) by p, one can write

$$Q(p)\, \bar{L} + H\bar{R} + p^{-1}\, Q(p)\, \bar{R} + p^{-1} H\bar{S} + p^{-2}\, Q(p)\, \bar{S} = p^{-1} I\,.$$

Hence $Q_\infty \bar{L} + H\bar{R} = 0$. Adding this to the equation $H\bar{L} = 0$, one gets

(5,22) $$H(\bar{L} + \bar{R}) + Q_\infty \bar{L} = 0\,.$$

As H, $\bar{L}$ are symmetrical, it follows that $0 = (H\bar{L})' = \bar{L}H$. Multiplying (5,22) by H, one can write

(5,23) $$H(\bar{L} + \bar{R})\, H = 0\,.$$

From (5,23) it follows immediately that $H = 0$. Therefore $\bar{Z}^{-1}(p)$ has no pole at infinity and, consequently, the same is true for $A(p)$, *q. e. d.*

Let us now prove 1*. By the assumption $A(p) = X\, \bar{Z}^{-1}(p)\, X'$ exists and each of its elements has a zero at infinity. According to Lemma 5,4, $\bar{Z}^{-1}(p) = q\, A(p)\, q'$, so that each element of $\bar{Z}^{-1}(p)$ has a zero at infinity. Using the previous notation, one can write $H = 0$, $\lim_{p\to\infty} Q(p) = 0$; also (5,21) holds. From (5,21) it follows that

(5,24) $$(\lim_{p\to\infty} p\, Q(p))\, \bar{L} = I\,.$$

But (5,24) shows that $\bar{L}$ is a regular matrix; consequently $\bar{L} \in \mathfrak{P}_n$. Therefore $C \cap \Pi_L = \{0\}$. Moreover, (5,24) yields $\lim_{p\to\infty} p\bar{Z}^{-1} X' L X = I$, so that $\lim_{p\to\infty} p\, A(p)\, LX = X$, *q. e. d.*

P r o o f of 2*. In the same way as before it follows that $\bar{Z}^{-1}(p)$ has no pole at infinity. Suppose that a non-zero vector $c \in C \cap \Pi_L \cap \Pi_R$ exists, *i. e.* that $c'Lc = 0$, $c'Rc = 0$. Let us choose the matrix X such that its first column equals c. Then the elements standing in the first column and row of the matrices $\bar{L} = X'LX$ and $\bar{R} = X'RX$ are zero; as $\bar{L}, \bar{R} \in \mathfrak{S}_n$, it follows that the first column and the first row of $\bar{L}$ and $\bar{R}$ are zero vectors. Consequently, the elements of the first column and row of $\bar{Z}(p)$ have the form $p^{-1}\bar{S}_{1k}$, where $\bar{S}_{1k}$ are the elements of $\bar{S}$.

If now $M(p)$ is a rational function, then there is an integer k such that a finite non-zero $\lim_{p\to\infty} p^{-k} M(p)$ exists; let us denote $\mathrm{r}(M) = k$. (It is obvious that for $k > 0$ the order of pole of $M(p)$ at infinity is k.) Evidently $\mathrm{r}(M + N) \leqq \max[\mathrm{r}(M), \mathrm{r}(N)]$, $\mathrm{r}(MN) = \mathrm{r}(M) + \mathrm{r}(N)$, $\mathrm{r}(M^{-1}) = -\mathrm{r}(M)$, $M \not\equiv 0$.

Let now $\Delta_{ik}(p)$ be the cofactors of the matrix $\bar{Z}(p)$; let $\Delta_{jl}(p)$ be the cofactor such that $\mathrm{r}(\Delta_{jl}) = h = \max_{i,k}[\mathrm{r}(\Delta_{ik})]$. By the assumption on regularity of N, we have $\det \bar{Z}(p) \not\equiv 0$; expanding this determinant by the first column it is readily seen that

$$\det \bar{Z}(p) = p^{-1} \sum_{k=1}^{n} S_{k1}\, \Delta_{k1}(p)\,. \tag{5,25}$$

Therefore $\mathrm{r}(\det \bar{Z}(p)) \leqq h - 1$, so that $\mathrm{r}(\Delta_{jl}/\det \bar{Z}) \geqq 1$. Hence $Z^{-1}(p)$ has a pole at infinity, which is a contradiction.

The following statement is a simple consequence of Theorem 5,6 and 5,7.

Theorem 5,8. *Let N be a passive* K-*network satisfying the condition $C \cap \Pi_L = \{0\}$ and let E be any regular C-vector; then the triple (E, J_0, q_0) is compatible for any J_0 fulfilling the condition $a' J_0 = 0$.*

For proof it suffices to note that there exists a vector y such $J_0 = Xy$.

Finally let us state some results for the admittance matrix of some special classes of passive K-networks. Let N be a passive K-network, R, L, S being the corresponding matrices; the matrix $X(X' Z(p) X)^{-1} X'$ (if it exists), where $Z(p) = R + Lp + Sp^{-1}$, will be called the admittance matrix of N. Further, N will be called of type

a) LC, if $R = 0$,
b) RC, if $L = 0$,
c) RL, if $S = 0$.

Then the following assertion is true:

Theorem 5,9. *Let N be a regular* K-*network, $A(p)$ the corresponding admittance matrix; then*

$$\text{a) } A(p) = pH_0 + \sum_{i=1}^{m} \frac{p}{p^2 + \omega_i^2} H_i\,, \quad \omega_i \geqq 0\,, \quad H_i \in \mathfrak{S}_r\,, \quad i = 0, 1, 2, \ldots, m \tag{5,26}$$

if N is of type LC,

$$\text{b) } A(p) = pH_0 + \sum_{i=1}^{m} \frac{p}{p + k_i} H_i\,, \quad k_i \geqq 0\,, \quad H_i \in \mathfrak{S}_r\,, \quad i = 0, 1, 2, \ldots, m \tag{5,27}$$

if N is of type RC,

$$\text{c) } A(p) = H_0 + \sum_{i=1}^{m} \frac{1}{p + k_i} H_i\,, \quad k_i \geqq 0\,, \quad H_i \in \mathfrak{S}_r\,, \quad i = 0, 1, 2, \ldots, m \tag{5,28}$$

if N is of type RL. (H_i are constant matrices.)

Proof. Let N be of type LC; then

$$A(p) = X[X'(Lp + Sp^{-1}) X]^{-1} X' , \tag{5,29}$$

where $\bar{Z}(p) = X'(Lp + Sp^{-1}) X \in \mathfrak{P}_n$; by Lemma 4,3 we have $A(p) \in \mathfrak{S}_r$, so that by Theorem 4,1

$$A(p) = pH_0 + \sum_{i=1}^{m} \frac{p}{p^2 + \omega_i^2} H_i + Q(p) , \tag{5,30}$$

where $H_0, H_i \in \mathfrak{S}_r$ are constant matrices and $Q(p) \in \mathfrak{S}_r$ has no poles in $\bar{G}$. If now $p = i\omega$, $\omega \neq 0, \omega_i$, then $\operatorname{Re} \bar{Z}(i\omega) = 0$; consequently, by (5,29), $\operatorname{Re} A(i\omega) = 0$. From (5,30) it follows that $\operatorname{Re} Q(i\omega) = 0$ and therefore $Q(p) \equiv 0$. The statement a) is proved.

Let now N be of type RC; then $A(p) = X[X'(R + Sp^{-1}) X]^{-1} X'$; putting $p = z^2$ and multiplying both sides of the latter equation by $1/z$, one obtains

$$A(z^2)/z = X[X'(R + Sz^{-2}) X]^{-1} X'/z = X[X'(Rz + Sz^{-1}) X]^{-1} X' .$$

Evidently $X'(Rz + Sz^{-1}) X \in \mathfrak{P}_n$, and it corresponds to a network of type LC. Therefore, by a) one can write

$$z^{-1} A(z^2) = zH_0 + \sum_{i=1}^{m} \frac{z}{z^2 + k_i} H_i , \quad k_i \geqq 0 , \quad H_i \in \mathfrak{S}_r ,$$
$$i = 0, 1, 2, \ldots, m ,$$

whence the statement b) follows. The assertion c) can be proved in a similar manner.

6. SINUSOIDAL SOLUTIONS

In this paragraph periodic solutions of a passive K-network with the diagonal matrix R of the form $J_0 \exp(i\omega t)$, J_0 a vector over $\mathbf{E}$, will be examined. The following considerations are closely connected with those of [4]. In [4] the existence of a unique solution of the form $J_0 \exp(i\omega t)$ was proved under the assumption that all of the diagonal elements of the diagonal matrix R are different from zero. In the following theorem this assumption will be weakened.

Theorem 6,1. *Let N be a passive* K*-network, R a diagonal matrix, and let for every loop $d'h$ of N*

$$d'Rd > 0 . \tag{6,1}$$

Then to every r-dimensional vector $\bar{E}$ over $\mathbf{E}$ *and to every $\omega > 0$ there exists a unique vector $\bar{J}$ (r-dimensional over* $\mathbf{E}$*) such that $J = \bar{J}H_0 \exp(i\omega t)$ is the solution of N in the time-domain corresponding to $E = \bar{E}H_0 \exp(i\omega t)$ and to initial condition vectors $J_0 = \bar{J}$, $q_0 = (i\omega)^{-1} \bar{J}$. Moreover, J is asymptotically stable with respect to the initial conditions.*

Proof. By Lemma 5,1, N is a dissipative network, *i. e.* (6,1) is fulfilled for every non-zero real cycle $c'h$. Hence $X'RX \in \mathfrak{P}_n$, the meaning of X, $\mathfrak{P}_n$ being the same as in paragraph 4. By definition of $\mathfrak{S}_n$ (paragraph 4), $Lp + Sp^{-1} \in \mathfrak{S}_r$ and by Lemma 4,3, $X'(Lp + Sp^{-1}) X \in \mathfrak{S}_n$. By Theorem 4,5, $[X'(R + Lp + Sp^{-1}) X]^{-1}$ has no poles in $\bar{G}$, G being the set of all complex numbers with positive real parts, $\bar{G}$ its closure. Thus

$$\det X'[R + Li\omega + S(i\omega)^{-1}] X \neq 0 \tag{6,2}$$

for every positive ω, and by Theorem 3,2 where we put $\tilde{p} = i\omega$, there exists a unique solution J with the desired properties. The asymptotical stability follows from Theorem 5,5.

Let $\omega > 0$. A passive K-network N will be called ω-regular if for every r-dimensional complex vector E and for every non-zero real cycle $c'h$ there exists a unique solution J (over **E**) of the system

(Ω_1) $c'(R + i\omega L - i\omega^{-1}S) J = c'E$,

(Ω_2) $a'J = 0$,

a' being the transposed incidence matrix of G (graph of N).

Note 6,1. From Theorem 1,8, where we put $\mathbf{P} = \mathbf{E} = \mathbf{T}$, it follows that N is ω-regular if and only if (1,9) holds with $Z = R + i\omega L - i\omega^{-1}S$.

Note 6,2. Evidently, if N is ω-regular, then to every r-dimensional vector $\bar{E}$ over **E** there exists a unique vector $\bar{J}$, which is r-dimensional over **E**, such that $J = \bar{J}H_0 . \exp(i\omega t)$ is the solution of N in the time-domain corresponding to $E = \bar{E}H_0 . \exp(i\omega t)$ and to initial condition vectors $J_0 = \bar{J}$, $q_0 = (i\omega)^{-1} \bar{J}$, and *vice versa*. By Theorem 6,1, condition (6,1) is sufficient for N to be ω-regular for every $\omega > 0$. The folllowing theorem shows that this condition cannot be substantially weakened.

Theorem 6,2. *Let be given an oriented graph G, which contains at least one loop, a matrix $R = \operatorname{diag}(R_{11}, R_{22}, \ldots, R_{rr})$ with non-negative elements and a positive number ω_1. If $d'Rd = 0$ for some non-zero loop $d'h$, then there exist non-zero $r \times r$ positive semidefinite diagonal matrices L, S such that $N = \{G, [R, L, S]\}$ is not ω_1-regular, $L_{ii} > 0$, $S_{ii} > 0$ for $d_i \neq 0$.*

Proof. By hypothesis there exists a non-zero loop $d'h$ such that $R_{ii}d_i = 0$ for $i = 1, 2, \ldots, r$. If $d_i \neq 0$, let us choose $\omega_1^2 L_{ii} = S_{ii}$, if $d_i = 0$, let L_{ii}, S_{ii} be arbitrary. As by Theorem 1,1 and Lemma 1,2 $d = Xy$, where the columns of X form a complete set of linearly independent solutions of $a'x = 0$ where a is the incidence matrix of G and y is a non-zero complex vector, we have

$$X'(R + i\omega L + (i\omega)^{-1} S) Xy = 0 .$$

From this, Note 6,1, Theorems 1,3 and 1,8 the proof follows immediately.

A graph G is called connected if each pair of nodes $u_i, u_j \in U$ can be connected by a chain K, *i. e.* if there exists a chain K such that u_i is the initial node of the first branch of K and u_j is the terminal node of its last branch. A subgraph $\bar{G}$ of G is called a component of G if $\bar{G}$ is connected and if for every connected subgraph $\bar{\bar{G}}$ of G with $\bar{\bar{G}} \supset \bar{G}$ there holds $\bar{\bar{G}} = \bar{G}$.

A connected subgraph $\bar{G}$ of G is called a tree if for every loop $\sum_{i=1}^{r} c_i h_i$ there is at least one index α, $\alpha = 1, 2, \ldots, r$, $c_\alpha \neq 0$, for which h_α is not a branch of $\bar{G}$. A maximal tree $\bar{G}_m$ is a tree of G with the following property: If $\bar{G}$ is a tree of G, $\bar{G}_m \subset \bar{G}$, then $\bar{G}_m = \bar{G}$. A subgraph of G whose all components are (maximal) trees, is called a (maximal) forest.

Theorem 6,3. *Let be given a non-trivial passive* K*-network* $N = (G, \mathfrak{Z})$, $G = (H, U, \Gamma)$, $\mathfrak{Z} = (R, L, S)$, *R diagonal. Then* (6,1) *holds for each non-zero loop $d'h$ if and only if the following condition* (A) *is fulfilled*:

(A): *There exists a forest $\bar{G}^*$ of G such that $R_{ii} = 0$ implies that h_i is a branch of $\bar{G}^*$ for $i = 1, 2, \ldots, r$.*

Proof. If (A) is fulfilled, let us choose a loop $d'h$. From the definition of a forest it follows that there exists at least one branch $h_i \in H$ such that h_i is not a branch of $\bar{G}^*$, $d_i \neq 0$. Hence $R_{ii} d_i^2 > 0$.

Conversely, let (6,1) be fulfilled for every non-zero loop $d'h$. In each loop $d'h$ of G choose a branch h_i for which $d_i \neq 0$, $R_{ii} > 0$ and denote this set of branches by H_1. Evidently, the subgraph F_1 of G which results by deleting all branches of H_1 from G is a forest.

In what follows we shall give a necessary and sufficient condition for N to be ω-regular. This condition will be derived from the determinant in (1,9), where we put $Z = R + i\omega L - i\omega^{-1} S$.

We shall make use of the following

Note 6,3. If h is the number of all linearly independent columns of the incidence matrix a, then G has $s - h$ components. This follows easily from the fact that in the equation $ax = 0$, $x' = [x_1, \ldots, x_s]$, there is $x_i = x_j$ whenever the nodes u_i and u_j are incident with the same branch.

Let there be given a passive K-network $N = (G, \mathfrak{Z})$, $G = (H, U, \Gamma)$, $\mathfrak{Z} = (R, L, S,)$ R, S diagonal. Let a be the incidence matrix of G. Moreover, we shall assume that if N^* with the incidence matrix a^* is a network equivalent to N, then the number of rows of a is not larger than that of a^*. In other words, from a class of equivalent networks we choose that which has the smallest possible number of branches. From Theorem 1,9 it follows that this assumption is no limitation of generality.

Let G_1 be a subgraph of G with the following property: $h_i \in H$, $i = 1, 2, \ldots, r$, is a branch of G_1 if and only if $R_{ii} > 0$, $L_{ij} = 0$ for $j \neq i$. Let G_{1j}, $j = 1, 2, \ldots, r_1$ be the components of G_1. For each $j = 1, 2, \ldots, r_1$ let us sum all columns of a for which the corresponding nodes are contained in G_{1j}. Denote these sums by b_j, $j = 1, 2, \ldots, r_1$. Let $b = [b_1, \ldots, b_{r_1}, b_{r_1+1}, \ldots, b_{r_1+r_2}]$, where b_{r_1+j}, $j = 1, 2, \ldots, r_2$, are the columns of a corresponding to the nodes of G not belonging to G_1.

Further, in each component $\bar{G}_j$ of G choose a node, say u_{t_j}, $j = 1, 2, \ldots, r_3$. If we delete in b all columns corresponding to u_{t_j}, $j = 1, 2, \ldots, r_3$, we obtain a matrix $\bar{b} = [\bar{b}_1, \ldots, \bar{b}_{r_4}]$.

Let $\bar{a}$ be the matrix obtained from a by omitting the columns corresponding to the nodes u_{t_j}, $j = 1, 2, \ldots, r_3$.

Finally, let us delete in

$$\begin{bmatrix} Z & \bar{b} \\ \bar{a}' & 0 \end{bmatrix}$$

where $Z = R + i\omega L - i\omega^{-1}S$, the i-th row and column, $i = 1, 2, \ldots, r$, whenever $R_{ii} > 0$, $L_{ij} = 0$ for $j \neq i$, and the i-th column, whenever $R_{ii} > 0$. The $n_1 \times n_2$ matrix thus obtained will be denoted by B.

Theorem 6,4. *Let $Z = R + i\omega L - i\omega^{-1}S$, $\omega > 0$. Then N is ω-regular if and only if the columns of B are linearly independent.*

Lemma 6,1. *Let J be a solution of (Ω_1), (Ω_2), where $E = 0$. Then $R_{ii} > 0$ implies $J_i = 0$ for $i = 1, 2, \ldots, r$.*

Proof. As the incidence matrix a is real, $a'J = 0$ implies $a'J^* = 0$ where J^* is complex conjugate to J. From (Ω_1), where we put $E = 0$, $c = J^*$, it follows that $J^{*\prime}(R + i(\omega L - \omega^{-1}S))\,J = 0$. As by symmetry of matrices L, S the number $J^{*\prime}(\omega L - \omega^{-1}S)\,J$ is real and as $J^{*\prime}RJ$ is non-negative, we have $J^{*\prime}RJ = 0$. From this and from the fact that that R is diagonal with non-negative elements the assertion of the lemma follows immediately.

Proof of Theorem 6,4. From Theorem 1,8 it follows easily that N is ω-regular if and only if (1,9) holds, where the matrix d is obtained from a by deleting the columns corresponding to the nodes u_{t_j}, $j = 1, 2, \ldots, r_3$. Consequently, N is ω-regular if and only if the system (1,12), where we put $E = 0$, $Z = R + i\omega L - i\omega^{-1}S$, has no non-trivial solution. In the next part of the proof we are going to show that the system

$$\begin{bmatrix} Z & a \\ a' & 0 \end{bmatrix} \begin{bmatrix} J \\ V \end{bmatrix} = 0 \tag{6,3}$$

may be written as

$$By = 0\,, \tag{6,4}$$

where y is a complex n_2-dimensional vector.

By Lemma 6,1, $J_i = 0$ whenever $R_{ii} > 0$, $i = 1, 2, \ldots, r$ and, consequently, in the system (6,3), $V_i = V_j$ whenever for some $l = 1, 2, \ldots, r$ there is $a_{li} = -a_{lj} \neq 0$, $R_{ll} > 0$, $L_{lj} = 0$ for $j \neq l$. Thus the components of V form r_1 groups, each of these groups corresponding to some of the components G_{1j}, $j = 1, 2, \ldots, r_1$ of G_1, all V_k from one group being equal. Obviously (6,3) remains unchanged if the columns of the incidence matrix a and the corresponding components of V are rearrangend so that the components V_j corresponding to the nodes u_{s_j}, $j = 1, 2, \ldots, r_1$ stand in the first places (from each G_{1j}, $j = 1, 2, \ldots, r_1$ one node u_{s_j} had been chosen).

Further, as the difference $V - \bar{V}$, where $\begin{bmatrix} J \\ V \end{bmatrix}$, $\begin{bmatrix} J \\ \bar{V} \end{bmatrix}$ are two solutions of (6,3), satisfies the equation

(6,5) $$az = 0$$

and as by Note 6,3 the number of linearly independent solutions of (6,5) is equal to r_3, r_3 being the number of components of G, we may choose $V_i = 0$ whenever V_i corresponds to a node u_{tj}, $j = 1, 2, \ldots, r_3$. By Theorem 1,8, N is ω-regular if and only if $\begin{bmatrix} J \\ V \end{bmatrix} = 0$ is the unique solution of (6,3) satisfying the condition $V_i = 0$ for $j = 1, 2, \ldots, r_3$.

If $R_{ii} > 0$, then the i-th column of the matrix $\left[\begin{array}{c|c} Z & a \\ \hline a' & 0 \end{array}\right]$ in (6,3) is multiplied by zero in view of Lemma 6,1.

If for some $i = 1, 2, \ldots, r$, $R_{ii} > 0$, $L_{ij} = 0$ for $j \neq i$, then all terms are zero in the i-th equation of (6,3).

Finally, the equations of system (6,3) corresponding to the nodes u_{tj}, $j = 1, 2, \ldots, r_3$ are linear combinations of the rest of equation (6,3) and, therefore, may be deleted.

Hence we obtain that if (6,3) has no non-trivial solution in J, then the system (6,4) has the unique solution $y = 0$, and *vice versa*.

7. NORMAL SYSTEMS OF DIFFERENTIAL EQUATIONS

In this paragraph the problem of passive K-networks is examined from the point of view of the theory of ordinary differential equations.

Let $\bar{D}$ be the system of all Schwarz's distributions on $(-\infty, \infty)$. Let $e' = [e_1, e_2, \ldots, e_r]$, $e_i \in \bar{D}$ for $i = 1, 2, \ldots, r$. Let N be a passive K-network, a its incidence matrix. Let $q' = [q_1, q_2, \ldots, q_r]$, $q_i \in \bar{D}$ for $i = 1, 2, \ldots, r$ be a solution of the system of ordinary differential equations

(7,1) $$c'(Lq'' + Rq' + Sq - e) = 0\,, \quad a'q = 0$$

for every real cycle $c'h$ of G, q' being the derivative of q in the distributional sense. If X is a matrix formed of a complete system of linearly independent solutions of the equation

(7,2) $$a'x = 0$$

in the field of real numbers, then the system (7,1) is equivalent to the system

(7,3) $$X'LXw'' + X'RXw' + X'SXw = X'e\,,$$

where

$$w' = [w_1, w_2, \ldots, w_n]\,, \quad w_i \in \bar{D}\,, \quad q = Xw\,.$$

Let C be the set of all real vectors x satisfying (7,2). Further, let P_L be the set of all $x \in C$ satisfying

(7,4) $$x'Lx = 0$$

and P_R the set of all $x \in C$ which fulfil

(7,5) $$x'Rx = 0 .$$

Finally, denote the set of all $x \in C$ satisfying

(7,6) $$x'Sx = 0$$

by P_S.

Evidently, C, P_L, P_R, P_S are linear subspaces of E_r, E_r being the r-dimensional Euclidean space.

In what follows it will be shown that X may be chosen so that (7,3) is solved with respect to the derivatives of the highest order of some components of w.

Lemma 7,1. *Let T be a symmetrical real $r \times r$ matrix. Let P be a linear subspace of E_r,* $\dim P = p$. *Then there exists a basis $\tilde{X} = [\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_p]$ of P, $\tilde{X}_i \in E_r$, $i = 1, 2, \ldots, p$, such that* $\tilde{X}'T\tilde{X} = \operatorname{diag}(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_p)$.

Proof. Let $\tilde{Y} = [\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_p]$ be a basis of P in E_r. The matrix $\tilde{Y}'T\tilde{Y}$ is a symmetrical $p \times p$ matrix with the corresponding quadratic form $y'\tilde{Y}'T\tilde{Y}y$, $y \in E_r$. There exists a regular matrix A such that

$$\tilde{y}'A'\tilde{Y}'T\tilde{Y}A\tilde{y} = \sum_{i=1}^{p} \tilde{T}_i \tilde{y}_i^2 .$$

Hence we obtain that $\tilde{X} = \tilde{Y}A$ is the basis with the desired property.

Let T be a symmetrical positive semidefinite $r \times r$ matrix. By Π_T denote the set of all $x \in E_r$ which fulfil the equation

(7,7) $$x'Tx = 0 .$$

Lemma 7,2. *Let C_1 be a linear subspace of E_r and let C_1 be the direct sum of the subspaces $C_1 \cap \Pi_T$ and K_T, i. e. $C_1 = C_1 \cap \Pi_T \dotplus K_T$. Let $\bar{X} = [\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_p]$, $\bar{X}_i \in E_r$ for $i = 1, 2, \ldots, n - p$ be a basis in K_T such that* $\bar{X}'T\bar{X} = \operatorname{diag}(T_1, T_2, \ldots, T_p)$. *Let $\bar{Y} = [\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_q]$ be a basis in $C_1 \cap \Pi_T$. Then $T_i > 0$ for $i = 1, 2, \ldots, p$ and $\bar{Y}'T\bar{Y} = 0$.*

Proof. Suppose $T_{i_*} = 0$ for some $i^* = 1, 2, \ldots, p$. Choose $\bar{x} \in E_p$ such that $\bar{x}_{i_*} = 1$, $\bar{x}_j = 0$ for $j \neq i^*$. Then $\bar{x}_{i_*}\bar{X}'T\bar{X}x_{i_*} = \bar{X}_{i_*}'T\bar{X}_{i_*} = 0$, which contradicts the fact that $\bar{X}_{i_*} \in K_T$. Thus $T_i > 0$ for $i = 1, 2, \ldots, p$.

Further, if $y, z \in C_1 \cap \Pi_T$, then $y'Tz = 0$ by Lemma 5,2. Hence $\bar{Y}'TY = 0$, as the columns $\bar{Y}_i$ of $\bar{Y}$ form a basis in $C_1 \cap \Pi_T$.

Lemma 7,3. *Let $T = [t_{ij}]$ be a symmetrical positive semidefinite $r \times r$ matrix. Then* a) $t_{ii} \geqq 0$; b) *if $t_{ii} = 0$ for some $i = 1, 2, \ldots, r$, then $t_{ij} = t_{ji} = 0$ for each $j = 1, 2, \ldots, r$.*

The proof is obvious.

Lemma 7,4. *Let B be an $r \times r$ matrix; let A be a matrix with r columns,* $A = \begin{bmatrix} A_1 \\ \vdots \\ A_l \end{bmatrix}$, *F* *a matrix with r rows,* $F = [F_1, \ldots, F_m]$. *Then*

$$ABF = \left[\begin{array}{c|c|c} A_1BF_1 & \ldots & A_1BF_m \\ \hline \vdots & & \vdots \\ \hline A_lBF_1 & \ldots & A_lBF_m \end{array}\right].$$

The proof is obvious.

Let C be a direct sum of the subspaces P_L and K_L, *i. e.* $C = P_L \dotplus K_L$. Similarly, let $P_L = H_L \dotplus P_L \cap P_R$. Let the dimension of C be n, *i. e.* $\dim C = n$, further let $\dim K_L = k_1$, $\dim H_L = k_2$, $\dim P_L \cap P_R = k_3$, $k_i \geqq 0$; evidently, $k_1 + k_2 + k_3 = n$.

Theorem 7,1. *Let* $N = (G, \mathfrak{Z})$ *be a regular passive* K-*network,* $\mathfrak{Z} = (L, R, S)$. *Then there exist a basis* $U = [u_1, u_2, \ldots, u_{k_1}]$ *in* K_L, $u_i \in E_r$, *a basis* $V = [v_1, v_2, \ldots, v_{k_2}]$ *in* H_L, $v_i \in E_r$, *and a basis* $Y = [y_1, y_2, \ldots, y_{k_3}]$ *in* $P_L \cap P_R$ *such that* $X = [u_1, u_2, \ldots, u_{k_1}, v_1, v_2, \ldots, v_{k_2}, y_1, y_2, \ldots, y_{k_3}] = [U, V, Y]$ *is a basis in C with the following properties*:

a) $X`LX = [l_{ij}] = \operatorname{diag}(l_{11}, l_{22}, \ldots, l_{k_1k_1}, 0, 0, \ldots, 0)$, $l_{ii} > 0$ *for* $i = 1, 2, \ldots, k_1$,

b) $X`RX = [r_{ij}]$, $r_{ij} = 0$ *for* $i > k_1 + k_2$, $j > k_1 + k_2$; $r_{ii} > 0$ *for* $k_1 < i \leqq k_2$; $r_{ij} = 0$ *for* $k_1 < i \leqq k_2$, $k_1 < j \leqq k_2$, $i \neq j$,

c) $X`SX = [s_{ij}]$, $s_{ii} > 0$ *for* $i > k_1 + k_2$; $s_{ij} = 0$ *for* $i > k_1 + k_2$, $j > k_1 + k_2$, $i \neq j$.

Proof. From Lemma 7,2, where we put $C_1 = C$, $T = L$, we obtain that there exists a basis U in K_L such that $U`LU$ is a diagonal $k_1 \times k_1$ matrix with positive elements in the diagonal. In Lemma 7,2 put $T = R$, $C_1 = P_L$, $K_T = H_L$. We obtain that there exists a basis $V$ in $H_L$ such that $V`RV$ is diagonal with positive elements in the diagonal.

As N is regular, $P_L \cap P_R \cap P_S = \{0\}$ by Theorem 5,1. If we put $T = S$, $C_1 = P_L \cap P_R$ in Lemma 7,2, it follows that $C_1 \cap \Pi_T = P_L \cap P_R \cap \Pi_S = P_L \cap P_R \cap P_S = \{0\}$ and, consequently, $\bar{Y}$ is a zero vector. Thus there exists a basis Y in $P_L \cap P_R$ such that $Y`SY$ is diagonal with positive diagonal elements. As $C = K_L \dotplus H_L \dotplus P_L \cap P_R$, $X = [U, V, Y]$ is a basis in C.

Now putting $\bar{Y} = [V, Y]$ in Lemma 7,2 one obtains that $[V, Y]` L[V, Y] = 0$. Using Lemma 7,3 and 7,4, where we put $B = L$, $F = [U, V, Y]$, $A = F`$, we prove assertion a).

To prove assertion b), put $B = R$, $F = [U, V, Y]$, $A = F`$ in Lemma 7,4. By Lemma 7,2 $Y`RY = 0$ and from Lemma 7,3 b) follows immediately.

Assertion c) is an obvious consequence of Lemma 7,4.

Theorem 7,2. *Let* $N = (G, \mathfrak{Z})$ *be a regular passive* K-*network. Then a complete system X of linearly independent cycles may be chosen so that* (7,3) *is a normal system, i. e. it may be written as*

$$\xi' = \Xi\xi + \eta\,, \tag{7,9}$$

where $\xi^{`} = [\xi_1, \xi_2, \ldots, \xi_{2k_1+k_2}]$, $\xi_i \in \bar{\mathbf{D}}$, $\Xi$ *is a constant* $(2k_1 + k_2) \times (2k_1 + k_2)$ *real matrix*, $\eta^{`} = [\eta_1, \eta_2, \ldots, \eta_{2k_1+k_2}]$, $\eta_i \in \bar{\mathbf{D}}$.

Proof. Let $w^{`} = [u^{`}, v^{`}, y^{`}]$, $u^{`} = [u_1, u_2, \ldots, u_{k_1}]$, $u_i \in \bar{\mathbf{D}}$; $v^{`} = [v_1, v_2, \ldots, v_{k_2}]$, $v_i \in \bar{\mathbf{D}}$; $y^{`} = [y_1, y_2, \ldots, y_{k_3}]$, $y_i \in \bar{\mathbf{D}}$. For X in (7,3) choose $X = [U, V, Y]$ from Theorem 7,1. Then one can easily verify that (7,3) may be written as

$$U^{`}LUu'' + U^{`}RUu' + U^{`}RVv' + U^{`}SXw = U^{`}e\,, \tag{7,10}$$

$$V^{`}RUu' + V^{`}RVv' + V^{`}SXw = V^{`}e\,, \tag{7,11}$$

$$Y^{`}SUu + Y^{`}SVv + Y^{`}SYy = Y^{`}e\,. \tag{7,12}$$

If dim $P_L \cap P_R = k_3 = 0$, equation (7,12) vanishes and y does not occur in (7,10) and (7,11). If $k_3 \geqq 1$, y can be expressed from (7,12) and substituted into (7,10) and (7,11) since $Y^{`}SY$ is a regular diagonal matrix. If $k_2 = 0$, equation (7,11) vahishes and $v$, $v'$ do not occur in (7,10). If $k_2 \geqq 1$, $v'$ can be expressed from (7,11) and substituted into (7,10) since $V^{`}RV$ is a regular diagonal matrix.

If we now put $u' = z$, $\xi^{`} = [u^{`}, (U^{`}LUz)^{`}, (V^{`}RVv)^{`}]$, the proof is completed readily.

Note 7,1. A distribution $f \in \bar{\mathbf{D}}$ is said to be of order k, k a positive integer, if there exists a function $F(t)$ locally integrable in $(-\infty, \infty)$ such that $F^{(k)} = f$ and $F^{(1)}$ is not a function ($F^{(k)}$ denotes the derivative of F in the distributional sense). The order of a vector over $\bar{\mathbf{D}}$ is the maximal order of its components.

It is evident that if e in (7,3) is a distribution of order $k \geqq 3$, then y is a distribution of at most the same order, whereas the order of v cannot exceed $k - 1$ and that $k - 2$ of u. Analogous assertions hold for $k = 2$ and $k = 1$. If e is a regular distribution, then v is differentiable almost everywhere and u has a derivative which is absolutely continuous in every closed interval.

Note 7,2. In (7,1) put $e = e^* + LJ_0\delta_0 - Sq_0H_0$, $(e^*)^{`} = [e_1^*, e_2^*, \ldots, e_r^*]$, $e_i^* \in \mathbf{D}$, J_0 a real r-dimensional vector, H_0 the Heaviside function, δ_0 its derivative in the distributional sense. Then it is readily seen that if $q \in \mathbf{D}$ is a solution of (7,1) (vanishing in $(-\infty, 0)$), then its derivative q' is a solution of T_1, T_2 and conversely, if $i \in \mathbf{D}$ is a solution of T_1, T_2, then its primitive distribution $q = i^{(-1)}$ which vanishes on $(-\infty, 0)$ is a solution of (7,1) where we put $e = e^* + LJ_0\delta_0 - Sq_0H_0$.

Lemma 7,5. *Let* $N = (G, \mathfrak{Z})$ *be a regular passive* K-*network. Let* e *in* (7,1) *be a locally integrable r-dimensional vector function defined in* $\langle 0, \infty)$ *such that* $d^{`}\,e(t)$ *is continuous in* $\langle 0, \infty)$ *for every* $d \in H_L$ *and* $\dfrac{\mathrm{d}}{\mathrm{d}t}(c^{`}\,e(t))$ *exists and is continuous in* $\langle 0, \infty)$ *for every* $c \in P_L \cap P_R$. *Then there exists a unique vector function* $q(t)$ *with these properties*: $q(t)$ *is a solution of* (7,1), $q(0) = 0$, $q'(t)$ *is continuous in* $\langle 0, \infty)$, $q'(0) = J_0$, *if and only if the following conditions are fulfilled*: 1) $a^{`}J_0 = 0$, 2) $(Y^{`}e)(0) = 0$, 3) $Y^{`}SJ_0 = (Y^{`}e')(0)$, 4) $V^{`}R(UJ_u + VJ_v) = (V^{`}e)(0)$, *where* U, V, Y *are matrices from Theorem* 7,1, $J_0 = UJ_u + VJ_v + YJ_y$.

Proof. The existence of $q(t)$ with the properties stated above is equivalent to the existence of functions $u(t)$, $v(t)$, $y(t)$ from (7,10), (7,11), (7,12) which have continuous derivatives in $\langle 0, \infty)$ and fulfil the conditions $u(0) = v(0) = y(0) = 0$, $u'(0) = J_u$, $v'(0) = J_v$, $y'(0) = J_y$.

Sufficiency. From Theorem 7,1 it follows that there exist functions $u(t)$, $v(t)$ with continuous derivatives in $\langle 0, \infty)$ satisfying the initial conditions $u(0) = v(0) = 0$, $u'(0) = J_u$. From 2) and (7,12) one obtains that there exists a $y(t)$ which together with $u(t)$ and $v(t)$ satisfies the system (7,10), (7,11), (7,12), and such that $y'(t)$ is continuous in $\langle 0, \infty)$ and $y(0) = 0$. From the conditions 3) and 4) it follows that one can choose $v'(0) = J_v$, $y'(0) = J_y$.

Necessity follows immediately from (7,10), (7,11), (7,12).

Lemma 7,6. *Let N be a regular passive* K*-network. Let J_0, q_0 be r-dimensional vectors over $\mathbf{E}_r$, e^* over $\mathbf{D}$. Then the triple (e^*, J_0, q_0) is compatible (cfr. paragraph 5), if and only if there exists a solution q over $\mathbf{D}$ of (7,1) where we put $e = e^* + LJ_0\delta_0 - - Sq_0H_0$, such that its distributional derivative q' is a regular C-vector (cfr. p. 5), $q'(0) = J_0$.*

The proof is an obvious consequence of Note 7,2.

Theorem 7,3. *Let $N = (G, \mathfrak{Z})$, $\mathfrak{Z} = (L, R, S)$ be a regular passive* K*-network. Let $X = [U, V, Y]$ be the matrix from Th. 7,1. Let E in* T1 *have the following property: E is locally integrable, $E(t) = 0$ for $t < 0$, $V`E$ and $Y`E$ are regular C-vectors. Then the triple (E, J_0, q_0) is compatible, if and only if the conditions 1), 2), 3), 4) of Lemma 7,5 are fulfilled, where $e = E - Sq_0H_0$.*

Proof. a) Let the conditions 1), 2), 3), 4) of Lemma 7,5 be satisfied. Then there exists a function $\tilde{q}(t)$ defined and continuous on $(-\infty, \infty)$ which for $t \geqq 0$ fulfils (7,1) with $e = E - Sq_0H_0$ and such that $\tilde{q}(t) = 0$ for $t \leqq 0$, $\tilde{q}'(t)$ is continuous in $(-\infty, \infty)$ except at the origin where there only exists the derivative from the right $\tilde{q}'(0+) = J_0$. As (7,1) is equivalent to the system (7,10), (7,11), (7,12), by Lemma 7,6 it is sufficient to show that in $(-\infty, \infty)$ $\tilde{q}(t)$ fulfils (7,10), (7,11), (7,12) with $e = E + LJ_0\delta_0 - - Sq_0H_0$. (7,10) is fulfilled, as

$$U`L\tilde{q}'' = U`L\left(\frac{\mathrm{d}^2\,\tilde{q}(t)}{\mathrm{d}t^2} + J_0\delta_0\right)$$

where q'' denotes derivative in the distributional sense, while $\mathrm{d}^2\tilde{q}(t)/\mathrm{d}t^2$ denotes a function defined everywhere in $(-\infty, \infty)$ except the origin. Evidently, equations (7,11) and (7,12) are also fulfilled, as $U`LX = Y`LX = 0$ and J_0 fulfils condition 1).

b) The converse assertion follows immediately from Lemmas 7,6 and 7,5.

Note 7,3. From Theorem 7,3 it follows that if (E, J_0, q_0) is to be compatible, q_0 must be chosen so as to fulfil condition 2), J_u may be chosen arbitrarily, J_v is to be calculated from condition 4) and J_y from condition 3) (this is possible, as the matrices $V`RV$ and $Y`SY$ are regular). Specially, if the subspace $P_L \cap P_R$ contains only the zero vector, equation (7,12) vanishes and q_0 may be chosen arbitrarily. If P_L contains only the zero vector, (7,1) is equivalent to (7,10). In this case q_0 may be chosen arbitrarily and for J_0 one can take any solution of $a`x = 0$.

The fact that the derivatives of y do not occur in the equations (7,10), (7,11), (7,12) suggests a new definition of compatibility.

Let $N = (G, \mathfrak{Z})$, $\mathfrak{Z} = (L, R, S)$ be a regular K-network, J_0, q_0 r-dimensional constant vectors, E over **D**, $J_0 = UJ_u + VJ_v + YJ_y$ where $X = [U, V, Y]$ is the matrix from the Theorem 7,1. The triple (E, J_0, q_0) will be called weakly compatible, if the solution J of T1, T2 fulfils the following condition: $J = Uu' + Vv' + Yy'$, u', v' are regular C-vectors, $u'(0) = J_u$, $v'(0) = J_v$.

Theorem 7,4. *Let N be a regular* K-*network, E a regular C-vector. Then (E, J_0, q_0) is weakly compatible if and only if there exists an r-dimensional constant vector ξ such that $a`\xi = 0$ and for every cycle $c`h$ of G there holds*

$$c`(L\xi + RJ_0 + Sq_0 - E(0)) = 0\,. \tag{7,13}$$

The proof is analogous to that of Theorem 7,3.

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Výtah

TEORIE KIRCHHOFFOVÝCH SÍTÍ

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V práci je zaveden pojem abstraktní sítě, který je jistým zobecněním pojmu elektrické sítě se soustředěnými parametry. Abstraktní síť je definována pomocí grafu, který charakterisuje topologickou strukturu sítě, a zobecněné impedanční matice Z, jejíž prvky jsou z nějakého tělesa $\boldsymbol{T}$. Buď E vektor s prvky z nějakého modulu $\boldsymbol{P}$. Řekneme, že vektor J nad modulem $\boldsymbol{P}$ je řešením sítě, když jsou splněny rovnice

K1: $c`(ZJ - E) = 0$

pro každý vektor c, odpovídající nějakému cyklu grafu,

K2: $a'J = 0$,

kde a je incidenční matice grafu, jejíž prvky nabývají pouze hodnot 0, 1, $-1 \in \boldsymbol{T}$. Rovnice K1, K2 vyjadřují zobecněné Kirchhoffovy zákony.

Jsou uvedeny některé podmínky pro to, aby abstraktní síť měla jediné řešení J pro každé E. Bereme-li za $\boldsymbol{P}$ modul všech Schwartzových distribucí, které jsou rovny nule na $(-\infty, 0)$ a za $\boldsymbol{T}$ těleso Heavisideových operátorů, popisují rovnice K1, K2 chování proudů v síti při libovolných počátečních podmínkách, když elektromotorické síly jsou distribuce. Tak je popsáno chování sítě v časové oblasti. Dosadíme-li za $\boldsymbol{P}$ i $\boldsymbol{T}$ těleso všech racionálních funkcí s komplexními koeficienty, vedou rovnice K1, K2 na vyšetřování sítě ve frekvenční oblasti. Je ukázáno společné matematické pozadí obou metod. Zvolíme-li speciálním způsobem impedanční matici

$$Z = Lp + R + Sp^{-1}, \tag{1}$$

kde L, R, S jsou symetrické positivně semidefinitní matice, p komplexní proměnná, dostaneme pasivní Kirchhoffovu síť. Jsou uvedeny podmínky pro to, aby existovalo jediné řešení pasivní Kirchhoffovy sítě, aby bylo stabilní a dále aby dané počáteční podmínky byly kompatibilní.

Položíme-li v rovnici (1) $p = i\omega$, $\omega =$ konst, a bereme-li vektory E, J nad tělesem komplexních čísel, dostaneme problém sítě se sinusově proměnnými elektromotorickými silami a proudy. Pro tento speciální případ jsou opět uvedeny některé podmínky pro existenci jediného řešení sítě.

Konečně je ukázána transformace proměnných, která převede úlohu řešení sítě v časové oblasti na soustavu občejných diferenciálních rovnic vyřešených vzhledem k nejvyšším derivacím. Odtud jsou odvozeny některé výsledky týkající se kompatibility počátečních podmínek.

Резюме

ТЕОРИЯ СЕТЕЙ КИРХГОФФА

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В настоящей работе вводится понятие абстрактной сети, являющейся обобщением электрической сети со сосредоточенными параметрами. Абстрактная сеть определяется с помощью графа, характеризирующего топологическую структуру сети, и импедансной матрицей Z, элементами которой являются элементы некоторого поля $\boldsymbol{T}$.

Пусть E — вектор, элементы которого принадлежат некоторому линейному

пространству **P**. По определению, сеть имеет решение, если существует вектор J с элементами из **P** так, что удовлетворены уравнения

K1. $$c'(ZJ - E) = 0$$

для каждого вектора c, соответствующего некоторому циклу графа,

K2. $$a'J = 0\,,$$

где a — матрица инцидентности графа, элементами которой являются только $1, -1, 0 \in \boldsymbol{T}$. Уравнения K1, K2 представляют обобщенные законы Кирхгоффа.

В статье приведены некоторые условия для того, чтобы абстрактная сеть имела единственное решение J для каждого E. Если подставить вместо **P** множество всех обобщенных функций, исчезающих на $(-\infty, 0)$, и в место **T** поле операторов Хевисайда, уравнения K1, K2 описывают поведение токов в сети при любых начальных условиях, когда электродвижущие силы являются обобщенными функциями (рассматривание сети в области времени). Если подставить вместо **P** и **T** поле всех рациональных функций с комплексными коэффициентами, получится задача решения сети в частотной области. В статье показано общее математическое основание обоих методов.

Если выбрать, в частности, матрицу

(1) $$Z = Lp + R + Sp^{-1}\,,$$

где L, R, S — симметрические неотрицательно-определенные матрицы, p — комплексное переменное, получится пассивная сеть Кирхгоффа. Приведены условия для того, чтобы существовало единственное решение пассивной сети Кирхгоффа, условия его устойчивости и совместности его начальных условий.

Если положить в (1) $p = i\omega$, ω — постоянная, и если элементы векторов J, E — комплексные числа, получится проблема сети со синусоидальными электродействующими силами и токами. Для этого частного случая опять показаны некоторые условия для существования и единственности решения.

В заключение показано преобразование переменных, с помощью которого проблема решения в области времени сводится к системе обыкновенных дифференциальных уравнений, которые решены по отношению к старшим производным. На основании этого получены некоторые результаты, касающиеся совместности начальных условий.