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ON SOME ASSUMPTIONS OF THE THEOREM ON THE CONTINUOUS DEPENDENCE ON A PARAMETER

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The assumption of the convergence of the sum (1) in the theorem of J. KURZWEIL [1] is proved to be essential.

In the paper [1] of J. KURZWEIL, there is proved a theorem on the continuous dependence on a parameter for generalized differential equations (l. c., p. 442, theorem 4,2,1). For classical differential equations we obtain from this theorem the following result:

Let us denote by $\mathbf{F}(G, \omega_1, \omega_2, \sigma)$ a class of functions $F(x, t)$ which fulfil the conditions:

$F(x, t)$ is defined and continuous on an open set $G \subset E_{n+1}$, $F(x, t) \in E_n$;

$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) \quad \text{for} \quad |t_2 - t_1| \leqq \sigma ;$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\| \, \omega_2(|t_2 - t_1|)$$
$$\text{for} \quad \|x_2 - x_1\| \leqq 2\omega_1(\sigma), \quad |t_2 - t_1| \leqq \sigma .$$

Let $\omega_1(\eta)$, $\omega_2(\eta)$ be continuous increasing functions on an interval $\langle 0, \sigma \rangle$, $\omega_i(0) = 0$, $\omega_i(\eta) \geqq c\eta$ for $i = 1, 2$, where σ and c are positive constants.

Let the function $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ fulfil these two conditions:

i) the function $\eta^{-1}\,\psi(\eta)$ is non-decreasing;

ii) the sum

$$\sum_{i=1}^{\infty} 2^i \psi\left(\frac{\sigma}{2^i}\right) \tag{1}$$

converges.

For $k = 0, 1, 2, \ldots$, let us denote by $x_k(t)$ the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_k(x, t), \quad x_k(0) = \xi \tag{2}$$

where the functions f_k fulfil the Carathéodory conditions. Let $F_k(x, t)$ be primitive functions of $f_k(x, t)$, *i. e.* $F_k(x, t) = \int_0^t f_k(x, \tau)\, \mathrm{d}\tau$. Further, let $F_k \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$,

$F_k(x, t) \to F_0(x, t)$ uniformly with $k \to \infty$. Let us suppose that for $k = 0$ there exists a unique solution of (2). Then $x_k(t) \to x_0(t)$ uniformly with $k \to \infty$.

Later on it was proved by Z. VOREL that i) may be weakened to

i') there exists an $\alpha > 0$ such that the function $\eta^{-\alpha} \psi(\eta)$ is non-decreasing.

The paper [2] starts with this assumption and investigates assumption ii). The convergence of the sum (1) is proved to be essential in the following sense:

Let two functions ω_1, ω_2 fulfil all the conditions of the theorem except assumption ii). Further, let there exist two positive constants α_1, d such that $\omega_2 \leqq d\omega_1$ and that the functions $\eta^{-\alpha_1} \omega_i(\eta)$ $(i = 1, 2)$ are non-decreasing. Then there exists a sequence of ordinary differential equations (linear, in fact), which fulfil (together with the functions ω_1, ω_2) all the assumptions of our theorem except ii), but such that the sequence of solutions $x_k(t)$ diverges.

In the paper [3], J. Kurzweil proved that the assumption i) may be left out entirely.

The aim of the present paper is to show that even then the assumption ii) is essential in the sense mentioned above when $\omega_1 = \omega_2$. This will solve this question completely *e. g.* for linear homogenous differential equations.

In fact, it is easy to show that if $F(x, t) = A(t)\, x$, then $F \in \mathbf{F}(G, K_1\omega, K_1\omega, \sigma)$, where $\omega(\eta)$ is the modulus of continuity of the function $A(t)$ for $\eta \in \langle 0, \sigma \rangle$, $K_1 = \operatorname{Max}(1, K)$ and $\|x\| \leqq K$ for all $(x, t) \in G$. This follows immediately from the evident inequalities

$$\|F(x, t_2) - F(x, t_1)\| = \|x\| \,.\, \|A(t_2) - A(t_1)\| \leqq K\omega(|t_2 - t_1|)\,,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| = \|x_2 - x_1\| \,.\, \|A(t_2) - A(t_1)\| \leqq$$
$$\leqq \|x_2 - x_1\|\, \omega(|t_2 - t_1|)\,.$$

Further, it is quite natural to consider only such functions $\omega(\eta)$ for which the inequality

$$\omega(\eta_1) + \omega(\eta_2) \geqq \omega(\eta_1 + \eta_2) \tag{3}$$

holds whenever $\eta_1, \eta_2, \eta_1 + \eta_2 \in \langle 0, \sigma \rangle$.

In fact, taking the class H of all functions $h(t)$ for which the inequality

$$|h(t_2) - h(t_1)| \leqq \Omega(|t_2 - t_1|)$$

holds for $|t_2 - t_1| \leqq \sigma$, let us define

$$\omega(\eta) = \sup_{h \in H, |t_2 - t_1| \leqq \eta} |h(t_2) - h(t_1)|\,.$$

Of course, for any $h(t) \in H$ and t_1, t_2 with $|t_2 - t_1| \leqq \sigma$, the inequality $|h(t_2) - h(t_1)| \leqq \omega(|t_2 - t_1|)$ holds.

Let now $\eta_1, \eta_2 \in \langle 0, \sigma \rangle$, $|t_2 - t_1| \leqq \eta_1 + \eta_2$. Then evidently there exists a point t_3 such that $|t_3 - t_1| \leqq \eta_1$, $|t_3 - t_2| \leqq \eta_2$. The inequalities

$$\omega(\eta_1) + \omega(\eta_2) \geqq |h(t_1) - h(t_3)| + |h(t_3) - h(t_2)| \geqq |h(t_1) - h(t_2)|$$

and

$$\omega(\eta_1 + \eta_2) = \sup_{h\in H, |t_2 - t_1| \leqq \eta_1 + \eta_2} |h(t_2) - h(t_1)| \leqq \omega(\eta_1) + \omega(\eta_2)$$

show that the same class of functions is characterized by both the functions Ω, ω and that in this case the inequality (3) holds.

1. On an interval $\langle 0, 1\rangle$ let $\omega(\eta)$ be a continuous increasing function such that $\omega(0) = 0$, $\omega(\eta) \geqq c\eta$,

$$\sum_{i=1}^{\infty} 2^i \omega^2 \left(\frac{1}{2^i}\right) = \infty \,, \tag{4}$$

$$\sum_{i=1}^{\infty} \omega \left(\frac{1}{2^i}\right) < \infty \,. \tag{5}$$

Further, let there exist an $\varepsilon > 0$ such that for infinitely many positive integers i the inequality

$$\omega \left(\frac{1}{2^{i-1}}\right) > (1 + \varepsilon)\, \omega \left(\frac{1}{2^i}\right) \tag{6}$$

holds.

We shall now construct a sequence of linear differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_k(t)\, x + b_k(t) \,, \tag{7}$$

$k = 1, 2, 3, \ldots$, such that $F_k \in \mathbf{F}(G, \omega, \omega, \sigma)$, $\sigma > 0$, $F_k(x, t) \to 0$ uniformly with $k \to \infty$, if we put

$$A_k(t) = \int_0^t a_k(\tau)\, \mathrm{d}\tau \,, \quad B_k(t) = \int_0^t b_k(\tau)\, \mathrm{d}\tau \,, \quad F_k(x, t) = A_k(t)\, x + B_k(t) \,.$$

Nevertheless, the sequence of solutions of (7) for which $x_k(0) = 0$, diverges.

Put

$$e^{-A_k(t)} = 1 + C_1 \sum_{i=k}^{m_k} e_i \omega \left(\frac{1}{2^i}\right) \sin 2^i t \,, \tag{8a}$$

$$B_k(t) = C_2 \sum_{i=k}^{m_k} e_i \omega \left(\frac{1}{2^i}\right) \cos 2^i t \,. \tag{8b}$$

Let us fix C_1, C_2 sufficiently small, and m_k such that

$$\lim_{k\to\infty} \sum_{i=k}^{m_k} 2^i \omega^2 \left(\frac{1}{2^i}\right) = \infty \,.$$

In the sums on the right sides of the equations (8a, b) we shall leave out some terms; namely, we shall choose $e_i = 1$, $e_i = 0$ respectively, so that the inequalities

$$|A_k(t_2) - A_k(t_1)| \leqq \tfrac{1}{2}\omega(|t_2 - t_1|) \,, \tag{9a}$$

$$|B_k(t_2) - B_k(t_1)| \leqq \tfrac{1}{2}\omega(|t_2 - t_1|) \tag{9b}$$

hold. If G is a set such that for all $(x, t) \in G$ it holds, $|x| \leqq 1$, then, evidently, from the inequalities (9a, b) it follows that $F_k \in \mathbf{F}(G, \omega, \omega, \sigma)$, where $\omega(\sigma) = \frac{1}{2}$.

For a fixed positive number C let us put (i, s are arbitrary positive integers)

$$(10) \quad L_s = \mathop{\mathscr{E}}_i \left[i > s,\ \omega\left(\frac{1}{2^i}\right) \leqq 2^{s-i} \omega\left(\frac{1}{2^s}\right) \frac{1}{C} \left(1 + \frac{1}{C}\right)^{i-s-1} \right], \quad L = \bigcup_{s=1}^{\infty} L_s .$$

Further let us denote by $\{p_i\}$ the increasing sequence of all indices for which

$$(11) \qquad \omega\left(\frac{1}{2^{p_i}}\right) > (1 + \varepsilon)\, \omega\left(\frac{1}{2^{p_i+1}}\right).$$

Denote the sequence $p_1, p_2 - 1, p_2, p_3 - 1, p_3, \ldots$[1]) by $\{l_j\}$, and put

$$e_i = 1 \text{ for } i \in \{l_j\} - L,$$
$$e_i = 0 \text{ for all other indices}.$$

Now we can prove *e. g.* the inequality (9b). Let

$$(12) \qquad \frac{1}{2^r} \leqq |t_2 - t_1| < \frac{1}{2^{r-1}}.$$

Then

$$(13) \quad |B_k(t_2) - B_k(t_1)| \leqq C_2 \left[\sum_{i=k}^{r-1} e_i \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| + \sum_{i=r}^{\infty} e_i \omega\left(\frac{1}{2^i}\right) \right].$$

Let us estimate the first sum. If $r \notin L$, then

$$(14) \qquad \sum_{i=k}^{r-1} e_i \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| \leqq \sum_{i=k}^{r-1} e_i \,.\, 2^{i-r} \omega\left(\frac{1}{2^i}\right) \leqq$$
$$\leqq \sum_{i=k}^{r-1} 2^{i-r} \omega\left(\frac{1}{2^r}\right) 2^{r-i} C \left(1 + \frac{1}{C}\right)^{i-r+1} \leqq C\omega\left(\frac{1}{2^r}\right) \sum_{i=0}^{\infty} \left(1 + \frac{1}{C}\right)^{-i} \leqq C(C+1)\,\omega\left(\frac{1}{2^r}\right).$$

If $r \in L$, then there exists an index $r_1 < r$, $r_1 \notin L$ such that for all the indices i, $r_1 < i \leqq r$, there is $i \in L$. As $e_i = 0$ for $i \in L$, the inequality

$$\sum_{i=k}^{r-1} e_i \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| \leqq 2^{r_1 - r} \sum_{i=k}^{r_1} e_i \omega\left(\frac{1}{2^i}\right) 2^{i-r_1}$$

holds. According to the preceding case, we have

$$\sum_{i=k}^{r-1} e_i \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| \leqq 2^{r_1 - r} \left[C(C+1)\,\omega\left(\frac{1}{2^{r_1}}\right) + \omega\left(\frac{1}{2^{r_1}}\right) \right] \leqq$$
$$\leqq 2^{r_1 - r} (C+1)^2\, \omega\left(\frac{1}{2^{r_1}}\right).$$

[1]) Every term of the sequence, even if repeated, is written once only.

From the inequality (3) it follows that

$$\omega\left(\frac{1}{2^{r_1}}\right) \leqq 2\omega\left(\frac{1}{2^{r_1+1}}\right) \leqq 2^{r-r_1}\omega\left(\frac{1}{2^r}\right).$$

According to (14), we have the estimate

$$\sum_{i=k}^{r-1} e_i\omega\left(\frac{1}{2^i}\right)|\sin 2^{i-1}(t_2 - t_1)| \leqq (C + 1)^2\,\omega\left(\frac{1}{2^r}\right)$$

for all r.

For the second sum on the right side of (13) we have the estimate

$$\sum_{i=r}^{\infty} e_i\omega\left(\frac{1}{2^i}\right) \leqq \sum_{p_j \geqq r} \omega\left(\frac{1}{2^{p_j}}\right) + \sum_{p_j > r} \omega\left(\frac{1}{2^{p_j-1}}\right) \leqq \omega\left(\frac{1}{2^r}\right) + 2\sum_{p_j \geqq r} \omega\left(\frac{1}{2^{p_j}}\right).$$

If $P = p_J$ is the least element of the sequence $\{p_j\}$ for which $P \geqq r$, then according to (11)

$$\omega\left(\frac{1}{2^r}\right) + 2\sum_{p_j \geqq r} \omega\left(\frac{1}{2^{p_j}}\right) \leqq \omega\left(\frac{1}{2^r}\right) + 2\omega\left(\frac{1}{2^P}\right)\sum_{j \geqq J}(1 + \varepsilon)^{J-j} \leqq \tag{15}$$

$$\leqq \left[1 + \frac{2(1 + \varepsilon)}{\varepsilon}\right]\omega\left(\frac{1}{2^r}\right).$$

From (14) and (15) we obtain immediately

$$|B_k(t_2) - B_k(t_1)| \leqq C_2\left[(C + 1)^2 + 1 + \frac{2(1 + \varepsilon)}{\varepsilon}\right]\omega(|t_2 - t_1|).$$

For arbitrarily fixed positive constants C, ε we can choose $C_2 > 0$ so small that (9b) holds.

The proof of the inequality

$$|e^{-A_k(t_2)} - e^{-A_k(t_1)}| \leqq \tfrac{1}{2}\omega(|t_2 - t_1|)$$

is quite analogous. By means of a further decrease of C_1 we obtain the inequality (9a).

Now we shall proceed to the proof of the divergence of the sequence of $x_k(t)$. First let us show that the elimination of some terms has no influence on the divergence of (4), *i. e.* that the sum

$$\sum_{i=1}^{\infty} e_i \,.\, 2^i\omega^2\left(\frac{1}{2^i}\right) \tag{16}$$

diverges for fixed sufficiently small $\varepsilon > 0$ and sufficiently large $C > 0$.

To this purpose we shall estimate the sum of the terms for which $e_i = 0$. As for $i < l_j - l_{j-1}$, it follows that

$$\omega\left(\frac{1}{2^{l_j-i}}\right) \leqq (1 + \varepsilon)\,\omega\left(\frac{1}{2^{l_j-i+1}}\right) \leqq (1 + \varepsilon)^i\,\omega\left(\frac{1}{2^{l_j}}\right);$$

from (11), we have the estimate

$$\sum_{i=p_{j-1}+1}^{p_j-2} 2^i\omega^2\left(\frac{1}{2^i}\right) \leqq \sum_{i=p_{j-1}+1}^{p_j-2} 2^i(1+\varepsilon)^{2(p_j-i)}\,\omega^2\left(\frac{1}{2^{p_j}}\right) \leqq$$

$$\leqq 2^{p_j}\omega^2\left(\frac{1}{2^{p_j}}\right)\sum_{i=2}^{\infty}\left[\frac{(1+\varepsilon)^2}{2}\right]^i = \tfrac{1}{2}\,2^{p_j}\,\omega^2\left(\frac{1}{2^{p_j}}\right)\frac{(1+\varepsilon)^4}{1-(1+\varepsilon)^2}\,.$$

By decreasing $\varepsilon > 0$ we can obtain

$$\sum_{i=p_j+1}^{p_j-2} 2^i\omega^2\left(\frac{1}{2^i}\right) \leqq \frac{3}{5}\,2^{p_j}\omega^2\left(\frac{1}{2^{p_j}}\right). \tag{17}$$

Further, from the definition (10) of L_s, it results that

$$\sum_{i\in L_s} 2^i\omega^2\left(\frac{1}{2^i}\right) \leqq \sum_{i>s} 2^i\left[2^{s-i}\,\omega\left(\frac{1}{2^s}\right)\frac{1}{C}\left(1+\frac{1}{C}\right)^{i-s-1}\right]^2 \leqq$$

$$\leqq 2^{2s}_{\mathbf{i}}\omega^2\left(\frac{1}{2^s}\right)\frac{1}{C^2}\left(1+\frac{1}{C}\right)^{-2(s+1)}\sum_{i=s+1}^{\infty}\left[\frac{\left(1+\frac{1}{C}\right)^2}{2}\right]^i = 2^s\omega^2\left(\frac{1}{2^s}\right)\frac{1}{C^2\left[2-\left(1+\frac{1}{C}\right)^2\right]}\,.$$

Evidently we can choose the constant C so large that

$$\sum_{i\in L_s} 2^i\omega^2\left(\frac{1}{2^i}\right) \leqq \frac{1}{5}\,2^s\omega^2\left(\frac{1}{2^s}\right). \tag{18}$$

From (17) and (18) if tollows that

$$\sum_{i=1}^{n} e_i\,.\,2^i\omega^2\left(\frac{1}{2^i}\right) \geqq \sum_{i=1}^{n} 2^i\omega^2\left(\frac{1}{2^i}\right) - \frac{3}{5}\sum_{p_j\leqq n} 2^{p_j}\omega^2\left(\frac{1}{2^{p_j}}\right) - \frac{1}{5}\sum_{i=1}^{n} 2^i\omega^2\left(\frac{1}{2^i}\right) \geqq \tag{19}$$

$$\geqq \frac{1}{5}\sum_{i=1}^{n} 2^i\omega^2\left(\frac{1}{2^i}\right).$$

Therefore the sum (16) diverges.[2])

Now we can estimate the absolute value of $x_k(t)$. Let us write

$$x_k(t) = e^{A_k(t)}\int_0^t e^{-A_k(\tau)}\,b_k(\tau)\,\mathrm{d}\tau\,.$$

Then

$$|x_k(t)| = \left|\left(1 + C_1\sum_{i=k}^{m_k} e_i\omega\left(\frac{1}{2^i}\right)\sin 2^i t\right)^{-1}\left[\int_0^t\left(C_2\sum_{i=k}^{m_k} e_i\,.\,2^i\omega\left(\frac{1}{2^i}\right)\sin 2^i\tau\right)\,.\right.\right.$$

$$\left.\left.\cdot\left(1 + C_1\sum_{i=k}^{m_k} e_i\omega\left(\frac{1}{2^i}\right)\sin 2^i\tau\right)\mathrm{d}\tau\right]\right| \geqq C_2C_3\left|\sum_{i=k}^{m_k} e_i\omega\left(\frac{1}{2^i}\right)\cos 2^i t\right. -$$

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[2]) Hence the set $\{n\} - L$ is infinite.

$$- \sum_{i=k}^{m_k} e_i\omega\left(\frac{1}{2^i}\right) + C_1 \int_0^t \left[\sum_{i=k}^{m_k} e_i \,.\, 2^i\omega^2\left(\frac{1}{2^i}\right)\sin^2 2^i\tau + \right.$$

$$\left. + \sum_{i,j=k,i\neq j}^{m_k} 2^i e_i e_j \omega\left(\frac{1}{2^i}\right)\omega\left(\frac{1}{2^j}\right)\sin 2^i\tau \sin 2^j\tau\right]\mathrm{d}\tau \Bigg| \geqq C_2C_3\left[\frac{C_1}{2}\,t\sum_{i=k}^{m_k} e_i \,.\, 2^i\omega^2\left(\frac{1}{2^i}\right) - \right.$$

$$\left. - 2\sum_{i=k}^{m_k}\omega\left(\frac{1}{2^i}\right) - \frac{C_1}{2}\sum_{i=k}^{m_k}\omega^2\left(\frac{1}{2^i}\right) - \frac{3C_1}{2}\sum_{i=k}^{m_k}\sum_{j=k,i\neq j}^{m_k}\omega\left(\frac{1}{2^i}\right)\omega\left(\frac{1}{2^j}\right)\right],$$

because

$$\left|2^i\int_0^t \sin 2^i\tau \sin 2^j\tau\,\mathrm{d}\tau\right| = \frac{1}{2}\left|\frac{2^i}{2^i - 2^j}\sin\left(2^i - 2^j\right)t - \frac{2^i}{2^i + 2^j}\sin\left(2^i + 2^j\right)t\right| \leqq \frac{3}{2}.$$

Besides, $C_3 > 0$ by choice of C_1.

According to (5) and (19), the sequence of $x_k(t)$ diverges. Thus the proof is finished; we can now turn to the case when (5) or (6) is not fulfilled.

2. Let us now suppose that for $\omega(\eta)$ (5) holds, but (6) does not hold for any $\varepsilon > 0$. Then, evidently,

$$\lim_{i\to\infty}\frac{\omega\left(\dfrac{1}{2^{i-1}}\right)}{\omega\left(\dfrac{1}{2^i}\right)} = 1\,. \tag{20}$$

Define the set L by (10) again, and denote by $\{k_j\}$ a sequence of indices $k_j \notin L$[3]) such that

$$\omega\left(\frac{1}{2^{k_j-1}}\right) > (1+\varepsilon)\,\omega\left(\frac{1}{2^{k_j}}\right).$$

Put $e_{k_j} = 1$ and $e_i = 0$ for all other indices i. As

$$\omega\left(\frac{1}{2^{k_j}}\right) > (1+\varepsilon)^s\,\omega\left(\frac{1}{2^{k_j+s}}\right),$$

the inequality

$$\sum_{k_j\geqq r}\omega\left(\frac{1}{2^{k_j}}\right) \leqq \frac{1+\varepsilon}{\varepsilon}\,\omega\left(\frac{1}{2^r}\right)$$

holds and we obtain (9a, b) analogously to the preceding case.

According to (20), there exists an index i_0 such that for $i \geqq i_0$

$$\frac{\omega\left(\dfrac{1}{2^{i-1}}\right)}{\omega\left(\dfrac{1}{2^i}\right)} < \left(\frac{4}{3}\right)^{\frac{1}{2}}.$$

[3]) Cf. footnote on p. 409. (The definition of L_s and L does not depend on the inequality (6).)

For $k_{j-1} \geqq i_0$ we obtain

$$\omega^2\left(\frac{1}{2^{k_j}}\right) \geqq \frac{3}{4}\,\omega^2\left(\frac{1}{2^{k_j-1}}\right) \geqq \left(\frac{3}{4}\right)^{k_j-k_{j-1}}\omega^2\left(\frac{1}{2^{k_{j-1}}}\right),$$

$$2^{k_j}\,\omega^2\left(\frac{1}{2^{k_j}}\right) \geqq \frac{3}{2}\,2^{k_{j-1}}\,\omega^2\left(\frac{1}{2^{k_{j-1}}}\right).$$

Thus evidently, the sum

$$\sum_{j=1}^{\infty} 2^{k_j}\omega^2\left(\frac{1}{2^{k_j}}\right) = \sum_{i=1}^{\infty} e_i\,.\,2^i\omega^2\left(\frac{1}{2^i}\right)$$

diverges. The remainder of the proof goes through as in the preceding case.

3. There remains the last possibility, that (5) does not hold, and that therefore

$$\sum_{i=1}^{\infty}\omega\left(\frac{1}{2^i}\right) = \infty\,.$$

Let us now define the functions $A_k(t)$, $B_k(t)$ by

(8'a) $$e^{-A_k(t)} = 1 + C_1\sum_{i=k}^{m_k} e_i\alpha^{-i}\omega\left(\frac{1}{2^i}\right)\sin 2^i t\,,$$

(8'b) $$B_k(t) = C_2\sum_{i=k}^{m_k} e_i\alpha^{-i}\omega\left(\frac{1}{2^i}\right)\cos 2^i t$$

where $1 < \alpha < \sqrt{2}$, $e_i = 0$ for $i \in L$ and $e_i = 1$ for all the other indices.

The functions $A_k(t)$, $B_k(t)$ converge uniformly to zero with $k \to \infty$, because

(21) $$\sum_{i=1}^{\infty}\alpha^{-i}\omega\left(\frac{1}{2^i}\right) \leqq \omega\left(\frac{1}{2}\right)\sum_{i=1}^{\infty}\alpha^{-i} < \infty\,.$$

Again choose the constants m_k such that

$$\lim_{k\to\infty}\sum_{i=k}^{m_k} 2^i\alpha^{-2i}\omega^2\left(\frac{1}{2^i}\right) = \infty\,.$$

This is possible, because

(4') $$\sum_{i=1}^{\infty} 2^i\alpha^{-2i}\omega^2\left(\frac{1}{2^i}\right) = \infty\,.$$

In fact, if (4') did not hold, then $\lim_{i\to\infty}\delta^i\omega^2(2^{-i}) = 0$ where $\delta = 2/\alpha^2 > 1$, and therefore $\lim_{i\to\infty}\delta^{\frac{1}{2}i}\omega(2^{-i}) = 0$. But then for a properly chosen constant K and for all i large enough, $\omega(2^{-i}) < K\delta^{-\frac{1}{2}i}$. Evidently from this (5) follows, in contradiction with our assumptions.

We shall estimate the difference $|B_k(t_2) - B_k(t_1)|$ again, supposing that (12) holds. Then

$$|B_k(t_2) - B_k(t_1)| \leqq C_2 \left[\sum_{i=k}^{r-1} e_i \alpha^{-i} \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| + \sum_{i=r}^{\infty} e_i \alpha^{-i} \omega\left(\frac{1}{2^i}\right)\right].$$

According to (14), we have

$$\sum_{i=k}^{r-1} e_i \alpha^{-i} \omega\left(\frac{1}{2^i}\right) |\sin 2^{i-1}(t_2 - t_1)| \leqq (C+1)^2 \omega\left(\frac{1}{2^r}\right).$$

We can estimate the second sum by

$$\sum_{i=r}^{\infty} e_i \alpha^{-i} \omega\left(\frac{1}{2^i}\right) \leqq \omega\left(\frac{1}{2^r}\right) \sum_{i=r}^{\infty} \alpha^{-i} \leqq C' \omega\left(\frac{1}{2^r}\right),$$

where C' is a constant. The inequalities (9a, b) are obtained by decreasing C_1, C_2 again.

Now we shall prove that the sum

$$\sum_{i=1}^{\infty} e_i \,.\, 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) \tag{16'}$$

diverges. In fact, according to (10), we have

$$\sum_{i \in L_s} 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) \leqq 2^{2s} \omega^2\left(\frac{1}{2^s}\right) \frac{1}{C^2} \left(1 + \frac{1}{C}\right)^{-2(s+1)} \sum_{i>s} \left[\frac{\left(1 + \frac{1}{C}\right)^2}{2\alpha^2}\right]^i =$$

$$= 2^s \alpha^{-2s} \omega^2\left(\frac{1}{2^s}\right) \left\{C^2 \left[2\alpha^2 - \left(1 + \frac{1}{C}\right)^2\right]\right\}^{-1} \leqq 2^s \alpha^{-2s} \omega^2\left(\frac{1}{2^s}\right) (C^2 - 2C - 1)^{-1}.$$

Evidently it is possible to choose C so large that

$$\sum_{i \in L_s} 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) \leqq \frac{1}{2}\left[2^s \alpha^{-2s} \omega^2\left(\frac{1}{2^s}\right)\right].$$

Then

$$\sum_{i=1}^{n} e_i \,.\, 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) \geqq \frac{1}{2} \sum_{i=1}^{n} 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) \tag{22}$$

and the sum (16') diverges.

The proof of the divergence of the sequence is then performed analogously to the preceding cases. We obtain

$$|x_k(t)| \geqq C_2 C_3 \left[\frac{C_1}{2} t \sum_{i=k}^{m_k} e_i 2^i \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) - 2 \sum_{i=k}^{m_k} \alpha^{-i} \omega\left(\frac{1}{2^i}\right) - \right.$$

$$\left. - \frac{C_1}{2} \sum_{i=k}^{m_k} \alpha^{-2i} \omega^2\left(\frac{1}{2^i}\right) - \frac{3C_1}{2} \sum_{i=k}^{m_k} \sum_{j=k, i \neq j}^{m_k} \alpha^{-(i+j)} \omega\left(\frac{1}{2^i}\right) \omega\left(\frac{1}{2^j}\right)\right].$$

According to (21), (22), the sequence $x_k(t)$ diverges. Thus the proof is complete.

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Výtah

O NĚKTERÝCH PŘEDPOKLADECH VĚTY O SPOJITÉ ZÁVISLOSTI NA PARAMETRU

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Buď $G \subset E_{n+1}$. Nechť $\mathbf{F}(G; \omega_1, \omega_2, \sigma)$ označuje množinu všech funkcí $F(x, t)$, majících vlastnosti:

$$F(x, t) \in E_n \quad \text{pro} \quad (x, t) \in G\,;$$
$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|)\,,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\| \, \omega_2(|t_2 - t_1|)$$
$$\text{pro} \quad |t_2 - t_1| \leqq \sigma\,, \quad \|x_2 - x_1\| \leqq 2\omega_1(\sigma)\,.$$

Buď dána funkce $\omega(\eta)$ spojitá a rostoucí v $\langle 0, 1 \rangle$, $\omega(0) = 0$, splňující nerovnosti $\omega(\eta) \geqq c\eta$, $\omega(\eta_1) + \omega(\eta_2) \geqq \omega(\eta_1 + \eta_2)$ pro všechna $\eta, \eta_1, \eta_2, \eta_1 + \eta_2 \in \langle 0, 1 \rangle$. Nechť řada (1) diverguje.

V článku se konstruuje posloupnost diferenciálních rovnic (7) těchto vlastností: Označíme-li $A_k(t) = \int_0^t a_k(\tau)\,d\tau$, $B_k(t) = \int_0^t b_k(\tau)\,d\tau$, $F_k(x, t) = A_k(t)\,x + B_k(t)$, je $F_k \in \mathbf{F}(G, \omega, \omega, \sigma)$ pro jistou oblast G a $\sigma > 0$. Posloupnost funkcí $F_k(x, t)$ konverguje stejnoměrně k nule. Posloupnost řešení $x_k(t)$ rovnic (7), pro něž je $x_k(0) = 0$, však diverguje.

Резюме

О НЕКОТОРЫХ ПРЕДПОЛОЖЕНИЯХ ТЕОРЕМЫ О НЕПРЕРЫВНОЙ ЗАВИСИМОСТИ ОТ ПАРАМЕТРА

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Пусть $G \subset E_{n+1}$. Пусть $\mathbf{F}(G, \omega_1, \omega_2, \sigma)$ обозначает множество всех функций $F(x, t)$ со следующими свойствами:

$$F(x, t) \in E_n \quad \text{для} \quad (x, t) \in G ;$$
$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) ,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\| \, \omega_2(|t_2 - t_1|)$$

для $|t_2 - t_1| \leqq \sigma$, $\|x_2 - x_1\| \leqq 2\omega_1(\sigma)$.

Пусть дана функция $\omega(\eta)$, непрерывная и возрастающая в $\langle 0, 1 \rangle$, $\omega(0) = 0$, выполняющая неравенства $\omega(\eta) \geqq c\eta$, $\omega(\eta_1) + \omega(\eta_2) \geqq \omega(\eta_1 + \eta_2)$ для всех $\eta, \eta_1, \eta_2, \eta_1 + \eta_2 \in \langle 0, 1 \rangle$. Пусть ряд (1) расходится.

В работе строится последовательность дифференциальных уравнений (7), обладающих следующими свойствами:

Если мы обозначим $A_k(t) = \int_0^t a_k(\tau) \, d\tau$, $B_k(t) = \int_0^t b_k(\tau) \, d\tau$, $F_k(x, t) = A_k(t) \, x + B_k(t)$, то $F_k \in \mathbf{F}(G, \omega, \omega, \sigma)$ для некоторой области G и $\sigma > 0$. Последовательность функций $F_k(x, t)$ сходится равномерно к нулю. Последовательность решений $x_k(t)$ уравнений (7), для которых $x_k(0) = 0$, все-таки расходится.