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MEDIAL SUBCARTESIAN PRODUCTS OF FIELDS

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Let us consider a Cartesian product $A = \bigtimes_{i \in J} F_i$ of a given system $\{F_i\}_{i \in J}$ of fields. A is a commutative ring with a unity-element 1 and the zero-element 0 such that

$\forall i \in J: \quad pr_i 1 = f_i \quad \text{and} \quad pr_i 0 = n_i$,

where f_i and n_i are by order the unity-element and the zero-element of the field F_i.

For any $i \in J$ we have a natural isomorphic embedding u_i: $F_i \longrightarrow A$ given by

$\forall a \in F_i: \ pr_i u_i(a) = a, \quad pr_j u_i(a) = n_j \quad (j \in J, \ j \neq i)$.

For any $i \in J$ let us denote by E_i the Im $u_i = u_i(F_i)$. E_i is of course a field, moreover it is an ideal of the ring A and finally, it may be described by

$$E_i = \{\bar{x} \in A \mid \forall j \in J, \ j \neq i : pr_j \bar{x} = n_j\} .$$

For any $i \in J$ the element $e_i = u_i(f_i)$ is the unity-element of the field E_i while all E_i have the common zero-element 0.

The Cartesian product A contains as an ideal and consequently as a subring the (exterior) direct sum $B = \bigoplus_{i \in J} F_i$. The B is obviously a subcartesian product of the system $\{F_i\}_{i \in J}$. As a subring of A, in generally, it does not contain the unity-element 1. It is the goal of our article to describe all rings R for that $B \subset R \subset A$ and $1 \in R$. For the purpose of this paper we will call all such rings medial subcartesian products of the system $\{F_i\}_{i \in J}$ (medial - "between" B and A).

Examples

1. As a trivial example of the medial subcartesian product (of the system $\{F_i\}_{i \in J}$ of fields) we may take the Cartesian product A itself.
2. Let $M = \{n \times 1 + a \mid n \in Z,\ a \in B\}$. M is obviously the medial subcartesian product of the system $\{F_i\}_{i \in J}$ which is minimal in the sense of being contained in any other one.
3. Let J = N be the set of natural numbers and let for any $i \in J$ the F_i be the field of rational numbers (=> the Cartesian product $A = \bigtimes_{i \in J} F_i$ is the ring of all sequences of rational numbers). Then the set R of all convergent sequences of rational numbers is a medial subcartesian product of the system $\{F_i\}_{i \in J}$ different from A as well as from the minimal medial subcartesian product.

Theorem 1. Let M be the minimal subcartesian product of the system $\{F_i\}_{i \in J}$ of fields. Then M = A if and only if the set J is finite.

P r o o f: It is sufficient to prove that the infinity of J implies $M \neq A$. For this reason we need to construct an element x of A whose projections are not almost the constant multiples of unity-elements. We may see without difficulty that the following two cases are possible, only. 1. There exists an infinite subset K of J such that all F_i, $i \in K$ have the same characteristic. 2. There exists an infinite subset K of J such that for any two distinct indices $i, j \in K$ the F_i, F_j have different characteristics. In both cases we may assume without loss of generality that K is countable: $K = \{k(1), k(2), k(3), \ldots\}$. In the first case, let $x \in A$ be an element for that $pr_{k(1)} = 1 \times f_{k(1)}$, $pr_{k(2)} = 2 \times f_{k(2)}$, $pr_{k(3)} = 3 \times f_{k(3)}$, ... In the second case, let us denote by p_1, p_2, p_3, ... the characteristics of the fields $F_{k(1)}$, $F_{k(2)}$, $F_{k(3)}$ - the eventuality of the zero-characteristics may be ommited. Now, let $x \in A$ be an element for that $pr_{k(1)}x = (p_1 - 1) \times f_{k(1)}$, $pr_{k(2)}x = (p_2 - 1) \times f_{k(2)}$, $pr_{k(3)}x = (p_3 - 1) \times f_{k(3)}$, ... The proof is completed.

Now, let us consider an arbitrary medial subcartesian product R of the system $\{F_i\}_{i \in J}$ of fields. The ring R contains any field E_i as an ideal, especially it contains any element e_i - the generator of the ideal $E_i = e_i . R$. Let us put $U_i = (1 - e_i) . R$. The system $\{e_i\}_{i \in J}$ consists of orthogonal idempotenties and has following properties:

(i) For any $i \in J$ the ideal $U_i = (1 - e_i) . R$ is maximal.

(ii) If for any $i \in J$ and for some $x \in R$ the $e_i . x = 0$ is true, then $x = 0$.

The (ii) is evident. To prove (i) we use the fact that

R as R -module is the direct sum of its ideals E_i and U_i: $R = E_i \oplus U_i$ allowing the unique expression

$$x = e_i . x + (1 - e_i) . x \qquad (1)$$

for any $x \in R$ and summands in order of E_i and U_i. In such a way, it follows from (1) that the mapping $R \to E_i$ given by $x \mapsto e_i . x$ is an epimorphism with the kernel U_i. Thus we have proved:

Theorem 2. Any medial subcartesian product R of the system $\{F_i\}_{i \in J}$ of fields possesses a system $\{e_i\}_{i \in J}$ of orthogonal idempotenties satisfying the conditions (i) and (ii) above.

Conversely, let us suppose that a commutative ring R with a unity-element 1 is endowed by a system $\{e_i\}_{i \in J}$ of orthogonal idempotent elements fulfilling (i) and (ii). Evidently, for any $i \in J$ the elements e_i and $1 - e_i$ are orthogonal idempotenties. Consequently, putting $E_i = e_i . R$ we get

$$R = E_i \oplus U_i .$$

As U_i is a maximal ideal the E_i is a field. Let us denote by A the Cartesian product $\times_{i \in J} E_i$ of the system $\{E_i\}_{i \in J}$ and let us define a mapping $f : R \to A$ by virtue of

$$\forall x \in R : \qquad pr_i f(x) = e_i . x \quad .$$

Evidently, f is a homomorphism carrying the unity-element 1 of R onto the element I of A for which $pr_i I = e_i . 1 = e_i$. Hence, I is the unity-element of the Cartesian product A.

According to the condition (ii) the kernel of f is the zero-ideal of the ring R. Consequently, f is an isomorphic embedding $R \to A$.

Let us denote by S the image of the ring R under the embedding f. As we have seen, the ring S contains the unity-element I of A. The fields A_i defined by

$$A_i = \{\bar{x} \in A \mid \forall j \in J, \quad j \neq i : pr_j \bar{x} = 0\}$$

are the images of the fields E_i under the isomorphic embedding f. It follows from this that S contains the (interior) direct sum $\bigoplus_{i \in J} A_i$ as well as the (exteriour) direct sum $\bigoplus_{i \in J} E_i$. Therefore S is a medial subcartesian product of the system $\{E_i\}_{i \in J}$.
We conclude our consideration by formulating:

Theorem 3. *Let a commutative ring R with a unity-element 1 possess a system $\{e_i\}_{i \in J}$ of orthogonal idempotenties fulfilling the conditions (i) and (ii) above. Then R is isomorphic to some medial subcartesian product of a system $\{E_i\}_{i \in J}$ of fields.*

Remark. We may replace the system of fields by a system of integral domains in simultaneous replacing (i) by the condition:

(i) For any $i \in J$ the ideal $U_i = (1 - e_i). R$ is a prime-ideal.

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SOUHRN

Mediální subkartézské součiny těles

Dalibor K l u c k ý, Libuše M a r k o v á

V článku jsou studovány subkartézské součiny systému těles $\{F_i\}_{i\in J}$ obsahující jednotkový prvek okruhu $\times_{i\in J} F_i$ a současně jeho ideál $\oplus_{i\in J} F_i$ (vnější direktní součet těles systému $\{F_i\}_{i\in J}$).

РЕЗЮМЕ

Медиальные подпрямые произведения полей

Д а л и б о р К л у ц к и , Л и б у ш е М а р к о в а

В статье изучаются подпрямые произведения системы полей $\{F_i\}_{i\in J}$ содержающее единицу кольца $\times_{i\in J} F_i$ и в тоже время его идеал $\oplus_{i\in J} F_i$ (внешную прямую сумму полей системы $\{F_i\}_{i\in J}$).