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DOUBLE VECTOR SPACES

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In differential geometry of higher order, one deals with some interesting algebraical structures possessing partial operations. Here we shall investigate the simplest of them - double vector spaces and their morphisms. The category $\mathcal{DL}$ of double linear morphisms has been introduced by J.Pradines in [1]. Analogous investigations in double affine and affine-linear case have been studied by I.Kolář in [2]. In this paper we shall show a slightly more general point of view. The category $\mathcal{DL}$ will be described geometrically.

1. Preliminary notions

Definition 1.1. An *S-fibrations* is a triple $\phi = (Y,p,X)$ where X and Y are (non-empty) sets and $p: Y \longrightarrow X$ is a projection. The set X is called a basis of the S-fibration, Y is a *total space* of the S-fibration, and the set $Y_x = p^{-1}(x)$ for $x \in X$ will be called *a fibre* of the S-fibration ϕ over a point x.

Remark. Let $\mathcal{C}$ be a category. A morphism in a category $\mathcal{C}$ will be called a $\mathcal{C}$*-morphism*. A $\mathcal{C}$-isomorphism is an isomorphism in $\mathcal{C}$ in the sense of the theory of categories. The identical morphism on an object X will be denoted by 1_X.

Definition 1.2. Let $\Phi = (Y,p,X)$, $\bar{\Phi} = (\bar{Y},\bar{p},\bar{X})$ be two S-fibrations. A morphism of Φ to $\bar{\Phi}$ is a mapping $f:Y \longrightarrow \bar{Y}$ such that there exists a mapping $g:X \longrightarrow \bar{X}$ so that the following diagram is commutative:

$$\begin{array}{ccc} Y & \xrightarrow{f} & \bar{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow{g} & \bar{X} \end{array} .$$

Obviously, if g exists, then it is unique. We shall say that f induces g, or that f is over g. It can be easily seen that S-fibrations and their morphisms form a category. We shall denote it by $\mathcal{SF}$.

Definition 1.3. An S-fibration (Y,p,X) will be called a trivial fibration if there exists a set Z and an $\mathcal{SF}$-isomorphism $f:(X \times Z, pr_1, X) \longrightarrow (Y,p,X)$ such that the induced mapping $g = 1_X$.

Trivial fibrations constitute a complete subcategory $\mathcal{TF}$ in the category $\mathcal{SF}$.

Let K be a given field. Under a vector space we shall always understand a finite-dimensional vector space over K. Vector spaces and their homomorphisms form a category which will be denoted by $\mathcal{L}$.

Definition 1.4. An S-fibration $\Phi = (Y,p,X)$ will be called a vector S-fibration if for every $x \in X$, the fibre Y_x is endowed with a structure of the vector space. Given two vector S-fibrations Φ, $\bar{\Phi}$, a morphism of the vector S-fibration Φ to $\bar{\Phi}$ is a $\mathcal{SF}$- morphism $f: \Phi \longrightarrow \bar{\Phi}$ over g such that for every $x \in X$, the induced mapping $f_x: Y_x \longrightarrow Y_{g(x)}$ of vector spaces is an $\mathcal{L}$-morphism.

Vector S-fibrations together with their morphisms form a category denoted by $\mathcal{LSF}$.

Example 1.1. Let X be a set and let V be a vector space. Let us define a structure of the vector space on each set $\{x\} \times V$, $x \in X$ so that the natural bijection $\{x\} \times V \longrightarrow V$ is an $\mathcal{L}$-morphism. Then the triple $(X \times V, pr_1, X)$ is a vector S-fibration.

Definition 1.5. A vector S-fibration (Y,p,X) is a trivial vector fibration if there exists a vector space V and an $\mathcal{L}\mathcal{S}\mathcal{F}$-isomorphism over identity $f:(X \times V, pr_1, X) \longrightarrow (Y,p,X)$.

Again, trivial vector fibrations form a complete subcategory $\mathcal{T}\mathcal{L}\mathcal{F}$ in the category $\mathcal{L}\mathcal{S}\mathcal{F}$.

Let us recall that an affine space Z with associated vector space V is a set Z together with a free and transitive right action of the additive group of the vector space V on Z. We shall denote the operation $Z \times V \longrightarrow Z$ by $+$. That is, for $v \in V$ and $z \in Z$, the result of the action of the element v on z will be denoted by $z+v$.

Definition 1.6. Let $\Psi = (W,q,X)$ be a vector S-fibration. An S-fibration (Y,p,X) will be called an affine S-fibration with associated vector S-fibration Ψ if for each $x \in X$, the group W_x acts freely and transitively (or equivalently, 1-transitively) on Y_x. In other words, the fibre Y_x is an affine space with associated vector space W_x.

Definition 1.7. Let Φ and $\overline{\Phi}$ be affine S-fibrations associated with vector S-fibrations $\Psi = (W,q,X)$ and $\overline{\Psi} = (\overline{W},\overline{q},\overline{X})$ respectively. An $\mathcal{S}\mathcal{F}$-morphism $f: \Phi \longrightarrow \overline{\Phi}$ over $g:X \longrightarrow \overline{X}$ is an affine morphism if there is an $\mathcal{L}\mathcal{S}\mathcal{F}$-morphism $h: \Psi \longrightarrow \overline{\Psi}$ over g such that

$f(y+w) = f(y) + h(w)$ for every $y \in Y_x$, $w \in W_x$, and $x \in X$.

It can be easily seen that if the mapping h exists, then it is specified uniquely. We shall say that h is associated with f. The family of affine S-fibrations together with affine morphisms form a category denoted $\mathcal{A}\mathcal{S}\mathcal{F}$.

Example 1.2. Let Z be an affine space with associated vector space V. Further, let X be a set. Then $(X \times Z, pr_1, X)$ is clearly an S-fibration, and $(X \in V, pr_1, X)$ is a vector S-fibration. For every $x \in X$, we define an action $+$ of the group $\{x\} \times V$ on the fibre $\{x\} \times Z$ by the formula $(x+z) + (x+v) =: (x, z+v)$. Now $(X \times Z, pr_1, X)$ is an affine S-fibration with associated vector S-fibration $(X \times V, pr_1, X)$.

Definition 1.8. An affine S-fibration ϕ is a trivial affine fibration if there exists an affine space Z and an $\mathcal{AYF}$-isomorphism f: $(X \times Z, pr_1, X) \longrightarrow \phi$ over 1_X.

Remark 1.1. Let ϕ be an affine S-fibration with associated vector S-fibration ψ. It can be easily verified that ϕ is a trivial affine fibration if and only if ψ is a trivial vector fibration.

Remark 1.2. A vector (affine) S-fibration is a trivial vector (affine) fibration if and only if all fibres have the same dimension.

Trivial affine fibrations form a complete subcategory $\mathcal{TAF}$ in $\mathcal{AYF}$.

2. Double vector spaces

Now let A,B be two vector spaces, and let C be a set. Let $\pi : C \longrightarrow A \times B$ be a given mapping. Denote by $pr_1 : A \times B \longrightarrow A$ and $pr_2 : A \times B \longrightarrow B$ projections to the first and second component, respectively. Further, let us denote

$$\pi_1 = pr_1 \circ \pi : C \longrightarrow A, \quad \pi_2 = pr_2 \circ \pi : C \longrightarrow B$$

and for $a \in A$, $b \in B$ let

$$C_a = \pi_1^{-1}(a), \quad C_b = \pi_2^{-1}(b), \quad C_{a,b} = \pi^{-1}(a,b) .$$

Finally, denote

$$\pi_{1,b} = \pi_1 \mid C_b : C_b \longrightarrow A, \quad \pi_{2,a} = \pi_2 \mid C_a : C_a \longrightarrow B.$$

Definition 2.1. Let A,B,C π be as above. Let c_{ik}, $i=1, \ldots, r$, $k = 1, \ldots, s$ be elements of C. We shall say that $\{c_{ik}\}_{\substack{i=1,\ldots,r \\ k=1,\ldots,s}}$ is an (r,s)-net in C if the following is satisfied:

$$\pi_1(c_{ik}) = \pi_1(c_{il}), \quad \pi_2(c_{ik}) = \pi_2(c_{jk})$$

for all $i,j = 1, \ldots, r$, $k,l = 1, \ldots, s$.

For an (r,s)-net $\{c_{ik}\}$ we set $a_i = \pi_1(c_{ik})$, $b_k = \pi_2(c_{ik})$.

Definition 2.2. A double vector space is a set C provided with a mapping $\widetilde{\pi}: C \longrightarrow A \times B$, where A,B are vector spaces with zero elements 0_A and 0_B respectively, having the following properties:

(i) For every $a \in A$ ($b \in B$), a structure of the vector space is given on C_a (C_b).

(ii) $(C, \widetilde{\pi}_1, A)$ and $(C, \widetilde{\pi}_2, B)$ are trivial vector fibrations with addition and scalar multiplication in fibres denoted by $+_1$, $\cdot_1$ and $+_2$, $\cdot_2$ respectively.

(iii) $\widetilde{\pi}_{2,a} : C_a \longrightarrow B$ ($\widetilde{\pi}_{1,b} : C_b \longrightarrow A$) is an epimorphism of vector spaces for every $a \in A$ ($b \in B$).

(iv) On the set $V = C_{0_A, 0_B}$ which is a subspace in C_{0_A} as well as in C_{0_B}, both vector structures coincide. So on V, we may write merely +, ., and 0.

(v) If $\{c_{ik}\}_{\substack{i=1,2\\k=1,2}}$ is a (2,2)-net in C then the following condition is satisfied:

$$(c_{11} +_1 c_{12}) +_2 (c_{21} +_1 c_{22}) = (c_{11} +_2 c_{21}) +_1 (c_{12} +_2 c_{22}) .$$

(vi) For every $\lambda, \mu \in K$, $\lambda \cdot_1 (\mu \cdot_2 c) = \mu \cdot_2 (\lambda \cdot_1 c)$.

(vii) Let $\widetilde{\pi}_2(c) = \widetilde{\pi}_2(c')$, $\widetilde{\pi}_1(c) = \widetilde{\pi}_1(c'')$, $\lambda \in K$. Then

$$\lambda \cdot_1 (c +_2 c') = (\lambda \cdot_1 c) +_2 (\lambda \cdot_1 c')$$

and

$$\lambda \cdot_2 (c +_1 c'') = (\lambda \cdot_2 c) +_1 (\lambda \cdot_2 c'') .$$

The vector space V will be called the centre of C. By (ii), $\widetilde{\pi}$ is a projection.

Example 2.1. Let A,B,V be three vector spaces. Let us set $C = A \times B \times C$ and define $\widetilde{\pi} : C \longrightarrow A \times B$ as a natural projection. For every $a \in A$ ($b \in B$), assume a structure of the vector space on C_a (C_b) given as follows:

$$(a,b,v) +_1 (a,\bar{b},\bar{v}) = (a, b+\bar{b}, v+\bar{v}),$$

$$\lambda \cdot_1 (a,b,v) = (a, \lambda b, \lambda v),$$

$$(a,b,v) +_2 (\bar{a},b,\bar{v}) = (a+\bar{a},b,v+\bar{v}),$$

$$\lambda \cdot_2 (a,b,v) = (\lambda a, b, \lambda v).$$

The conditions (i) - (vii) required in Def.2.2. are satisfied. The proof is straightforward. The double vector space $A \times B \times C$ (with the above projection and partial linear operations) is called *a trivial double vector space*.

Definition 2.3. Let $C, \bar{C}$ be double vector spaces with corresponding projections $\pi: C \longrightarrow A \times B$ and $\bar{\pi}: \bar{C} \longrightarrow \bar{A} \times \bar{B}$ respectively. A mapping $f: C \longrightarrow \bar{C}$ is a morphism of double vector spaces if $f_1 = \bar{\pi}_1 \circ f$ and $f_2 = \bar{\pi}_2 \circ f$, $f_1: A \longrightarrow \bar{A}$, $f_2: B \longrightarrow \bar{B}$ are $\mathcal{L}$-morphism, $f:(C, \pi_1, A) \longrightarrow (\bar{C}, \bar{\pi}_1, \bar{A})$ is an $\mathcal{LYF}$-morphism and at the same time, $f:(C, \pi_2, B) \longrightarrow (\bar{C}, \bar{\pi}_2, \bar{B})$ is an $\mathcal{LYF}$-morphism.

We shall say that f_1, f_2 are underlying $\mathcal{L}$-morphisms of the $\mathcal{DL}$-morphism f.

Double vector spaces together with morphisms just defined form a category $\mathcal{DL}$.

Let $C \in \mathcal{DL}$ with the centre V.

Lemma 2.1. *Let $a \in A$ ($b \in B$) and let us assume the operations of the first (second) linear structure on $C_{a,0_B}$ (on $C_{0_A,b}$). Then the mapping $\ell_a: V \longrightarrow C_{a,0_B}$ ($\ell_b: V \longrightarrow C_{0_A,b}$) given by the formula*

$$\ell_a(v) = v +_2 0_a \qquad (\ell_b(v) = v +_1 0_b)$$

is an $\mathcal{L}$-isomorphism.

P r o o f. Let $v, v' \in V$. Then the elements $v, v', 0_a, 0_a$ form a (2,2)-net. By (v) of Def.2.2.,

$$(v +_1 v') +_2 (0_a +_1 0_a) = (v +_2 0_a) +_1 (v' +_2 0_a) .$$

Therefore

$$\iota_a(v+v') = \iota_a(v+_1v') = (v+_1v')+_2 0_a = (v+_1v')+_2(0_a+_1 0_a) =$$
$$= \iota_a(v)+_1 \iota_a(v') .$$

Further, let $\lambda \in K, v \in V$. The property (vii) implies

$$\iota_a(\lambda v) = \iota_a(\lambda \cdot_1 v) = (\lambda \cdot_1 v)+_2 0_a = (\lambda \cdot_1 v)+_2(\lambda \cdot_1 0_a) =$$
$$= \lambda \cdot_1 (v+_2 0_a) = \lambda \cdot_1 \iota_a(v).$$

Thus ι_a is an $\mathcal{L}$-morphism.

Now let $v \in V$ be such that $\iota_a(v) = 0_a$. That is, $v+_2 0_a = 0_a$. It follows $v = 0_a$ which proves that ι_a is a monomorphism. Finally, choose an arbitrary $c \in C_{a,0_b}$. Since c and 0_a are in C_{0_B} there exists a unique $v \in C_{0_B}$ such that $c = v+_2 0_a$. We have

$$a = \mathcal{T}_1(c) = \mathcal{T}_1(v+_2 0_a) = \mathcal{T}_1(v)+ \mathcal{T}_1(0_a) = \mathcal{T}_1(v)+a$$

which gives $\mathcal{T}_1(v) = 0$. Hence $v \in V$, and ι_a is an epimorphism. For the mapping ι_b, the proof is similar.

Lemma 2.2. *Let $v \in V$ and $c \in C_{a,b}$. Then the following is satisfied:*

$$(v+_1 0_b) +_2 c = (v+_2 0_a) +_1 c.$$

P r o o f. Clearly, the elements $v, 0_b, 0_a, c$ form a (2,2)-net. By (v),

$$(v+_1 0_b)+_2(0_a+_1 c) = (v+_2 0_a)+_1(0_b+_2 c)$$

and consequently

$$(v+_1 0_b)+_2 c = (v+_2 0_a)+_1 c.$$

Now we may give the following definition:

Definition 2.4. Let $c \in C$, $v \in V$. We define

$$c+v = (v+_1 0_b)+_2 c = (v+_2 0_a)+_1 c,$$

where $a = \mathcal{T}_1(c)$, $b = \mathcal{T}_2(c)$.

Theorem 2.1. A mapping $C \times V \longrightarrow C$ given by $(c,v) \longmapsto c+v$ is a right action of the group V on the set C. The group V acts on C freely and its orbits are just the sets $C_{a,b}$ with $a \in A$, $b \in B$.

P r o o f. Let $v,v' \in V$ and suppose $c \in C$ with $\mathcal{T}_1(c) = a$, $\mathcal{T}_2(c) = b$.

Then

(1) $\mathcal{T}_1(c+v) = \mathcal{T}_1((v+_1 0_b)+_2 c) = \mathcal{T}_1(v+_1 0_b)+\mathcal{T}_1(c) = 0_A+a = a$

and

(2) $\mathcal{T}_2(c+v) = \mathcal{T}_2((v+_2 0_a)+_1 c) = \mathcal{T}_2(v+_2 0_a)+\mathcal{T}_2(c) = 0_B+b = b.$

Hence

$$(c+v)+v' = (v'+_1 0_b)+_2((v+_1 0_b)+_2 c).$$

According to L.2.2. and (v) of Def.2.2. for a (2,2)-net $\{v, 0_b, v+_2 0_a, c\}$ we obtain

$$(c+v)+v' = (v'+_1 0_b)+_2((v+_2 0_a)+_1 c) = (v'+_2(v+_2 0_a))+_1$$

$$(0_b+_2 c) = ((v+_2 v')+_2 0_a)+_1 c = ((v+v')+_2 0_a)+_1 c = c+(v+v').$$

Further,

$$c+0 = (0+_1 0_b)+_2 c = 0_b+_2 c = c.$$

At the beginning of the proof we have seen that if $c \in C_{a,b}$ and $v \in V$ then $c+v \in C_{a,b}$ (see (1), (2)). We show that the group V acts transitively on $C_{a,b}$. Let $c,c' \in C_{a,b}$. Since $c,c' \in C_b$, there exists a unique element $d \in C_b$ such that $c' = d+_2 c$. Moreover, $\mathcal{T}_1(d) = 0_A$, that is, $d \in C_{0_A,b}$. Therefore by L.2.1., there exists $v \in V$ such that $v+_1 0_b = d$. We get

$$c' = d+_2c = (v+_1 0_b)+_2 c = c + v.$$

It remains to prove that V acts freely. So let $c \in C$, $v \in V$ be elements satisfying $c+v = c$. We have

$$(v+_1 0_b)+_2 c = c,$$

$$v+_1 0_b = 0_b,$$

$$v = 0.$$

Theorem 2.2. *Let C (with projection $\mathcal{T}: C \longrightarrow A \times B$) be a double vector space. Then $\phi = (C, \mathcal{T}, A \times B)$ is a trivial affine fibration with associated trivial vector fibration $\psi = (A \times B \times C, pr, A \times B)$.*

P r o o f. It follows immediately that ϕ is an affine S-fibration with associated trivial vector fibration ψ (Th.2.1.). By Remark 1.1., ϕ is also trivial.

3. Morphisms of double vector spaces

Theorem 3.1. *Let $C, \overline{C}$ be two double vector spaces with projections $\mathcal{T}: C \longrightarrow A \times B$ and $\overline{\mathcal{T}}: \overline{C} \longrightarrow \overline{A} \times \overline{B}$ respectively. Let $f: C \longrightarrow \overline{C}$ be a $\mathcal{DL}$-morphism. Then f is an $\mathcal{ASF}$-morphism over $f_1 \times f_2$: $: A \times B \longrightarrow \overline{A} \times \overline{B}$ with associated $\mathcal{LSF}$-morphism $f_1 \times f_2 \times (f|V)$: $: (A \times B \times V, pr, A \times B) \longrightarrow (\overline{A} \times \overline{B} \times \overline{V}, pr, \overline{A} \times \overline{B})$ over $f_1 \times f_2$.*

P r o o f. Since $\overline{\mathcal{T}}_1 \circ f = f_1 \circ \mathcal{T}_1$ and $\overline{\mathcal{T}}_2 \circ f = f_2 \circ \mathcal{T}_2$, it follows $\overline{\mathcal{T}} \circ f = (f_1 \times f_2) \circ \mathcal{T}$. Therefore f is a morphism of S-fibrations over $f_1 \times f_2$. Let $c \in C$, $v \in V$. Then

$$f(c+v) = f((v+_2 0_a)+_1 c) = f(v+0_a)+_1 f(c) =$$

$$= (f(v)+_2 0_{f_1(a)})+_1 f(c) = f(c) + f(v)$$

where $a = \mathcal{T}_1(c)$. This finishes the proof.

Definition 3.1. A basis of a double vector space C is an ordered couple $(\{c_{ik}\}, \{v_m\})$ where $\{c_{ik}\}_{\substack{i=1,\dots,r\\ K=1,\dots,s}}$ is an (r,s)-net such that $\{a_i\}_{i=1,\dots,r}$ is a basis for the vector space A, $\{b_k\}_{k=1,\dots,s}$ is a basis of B and $\{v_m\}_{m=1,\dots,t}$ is a basis of the centre V.

Theorem 3.2. Let $(\{c_{ik}\}, \{v_m\})$ be a basis of the given double vector space C with $\mathcal{T}: C \longrightarrow A \times B$. Then for every element $c \in C$, there are uniquely determined elements $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s, \nu_1, \dots, \nu_t \in K$ such that

$$c = \sum_{k=1}^{s}{}_1 \sum_{i=1}^{r}{}_2\, \mu_k \cdot_1 (\lambda_i \cdot_2 c_{ik}) + \sum_{m=1}^{t} \nu_m v_m =$$

$$= \sum_{i=1}^{r}{}_2 \sum_{k=1}^{s}{}_1\, \lambda_i \cdot_2 (\mu_k \cdot_1 c_{ik}) + \sum_{m=1}^{t} \nu_m v_m .$$

Here + means either $+_1$ or $+_2$, and $\sum_m \nu_m v_m$ is a linear combination of vectors in V, where both vector structures coincide.

P r o o f. Clearly, there exist elements $\lambda_1, \dots, \lambda_r \in K$ such that

$$\mathcal{T}_1(c) = \sum_{i=1}^{r} \lambda_i a_i .$$

Denote $c_k = \sum_{i=1}^{r}{}_2\, \lambda_i \cdot_2 c_{ik}$, $k = 1, \dots, s$. It is clear that $\mathcal{T}_1(c_k) = \mathcal{T}_1(c)$, $\mathcal{T}_2(c) = b_k$. There exist elements $\mu_1, \dots, \mu_s \in K$ such that

$$\mathcal{T}_2(c) = \sum_{k=1}^{s} \mu_k b_k .$$

We set

$$c' = \sum_{k=1}^{s}{}_1 \mu_k \cdot_1 c_k = \sum_{k=1}^{s}{}_1 \mu_k \cdot_1 (\sum_{i=1}^{r}{}_2 \lambda_i \cdot_2 c_{ik}) =$$

$$= \sum_{k=1}^{s}{}_1 \sum_{i=1}^{r}{}_2 \mu_k \cdot_1 (\lambda_i \cdot_2 c_{ik}).$$

We have $\mathcal{T}_1(c') = \mathcal{T}_1(c)$, $\mathcal{T}_2(c') = \mathcal{T}_2(c)$. By Th.2.1., there is $v \in V$ such that $c = c' + v$. Writing v in the form $v = \sum_{m=1}^{t} \nu_m v_m$ we get

$$(3) \quad c = \sum_{k=1}^{s}{}_1 \sum_{i=1}^{r}{}_2 \mu_k \cdot_1 (\lambda_i \cdot_2 c_{ik}) + \sum_{m=1}^{t} \nu_m v_m \, .$$

We show that the above expression is uniquely determined. So suppose $c = \sum_k{}_1 \sum_i{}_2 \mu'_k \cdot_1 (\lambda'_i \cdot_2 c_{ik}) + \sum_{m=1}^{t} \nu'_m v_m$.

Applying $\mathcal{T}_1$ on both sides of both sides of the previous equality we obtain

$$\sum_i \lambda_i \mathcal{T}_1(c_{ik}) = \sum_i \lambda'_i \mathcal{T}_1(c_{ik}) \text{ for } k=1,\dots,s$$

and further

$$\sum_{i=1}^{r} \lambda_i a_i = \sum_{i=1}^{r} \lambda'_i a_i .$$

Thus $\lambda'_i = \lambda_i$ for $i=1,\dots,r$. In a similar way, using $\mathcal{T}_2$ yields $\mu'_k = \mu_k$ for $k=1,\dots,s$. Both this results give $\sum_m \nu_m v_m = \sum_m \nu'_m v_m$. Therefore $\nu'_m = \nu_m$ for $m=1,\dots,t$ which proves uniqueness of the expresion (3). By symmetry, we deduce that there are uniquely determined elements $\bar{\lambda}_1, \dots, \bar{\lambda}_r$, $\bar{\mu}_1, \dots, \bar{\mu}_s$, $\bar{\nu}_1, \dots, \bar{\nu}_t \in K$ such that

(4) $$c = \sum_i {}_2 \sum_k {}_1 \bar{\lambda}_i \cdot_2 (\bar{\mu}_k \cdot_1 c_{ik}) + \sum_m \bar{\nu}_m v_m .$$

Application of $\mathcal{T}_1$ on (3) and (4) gives $\sum_i \lambda_i a_i = \sum_i \bar{\lambda}_i a_i$, $\lambda_i = \bar{\lambda}_i$, i=1,...,r. Similar using of $\mathcal{T}_2$ yields $\mu_k = \bar{\mu}_k$ for k=1,...,s.

<u>Theorem 3.3</u>. *Let $C, \mathcal{T}: C \longrightarrow A \times B$ and $\bar{C}, \bar{\mathcal{T}}: \bar{C} \longrightarrow \bar{A} \times \bar{B}$ be two double vector spaces. Let $(\{c_{ik}\}_{\substack{i=1,\dots,r \\ k=1,\dots,s}}, \{v_m\}_{m=1,\dots,t})$ be a basis in C. Further, let $\{\bar{c}_{ik}\}_{\substack{i=1,\dots,r \\ k=1,\dots,s}}$ be a (r,s)-net in C and let $\{\bar{v}_m\}_{m=1,\dots,t}$ be elements in $\bar{V}$. Then there exists a unique $\mathcal{DL}$-morphism $f: C \longrightarrow \bar{C}$ such that $f(c_{ik}) = \bar{c}_{ik}$, i=1,...,r, k=1,...,s, and $f(v_m) = \bar{v}_m$ for m=1,...,t.*

P r o o f. Suppose $c \in C$. According to the previous theorem, there are uniquely determined numers λ_i, μ_k, ν_m such that

$$c = \sum_{k=1}^{s} {}_1 \sum_{i=1}^{r} {}_2 \mu_k \cdot_1 (\lambda_i \cdot_2 c_{ik}) + \sum_{m=1}^{t} \nu_m v_m .$$

Define

$$f(c) = \sum_k {}_1 \sum_i {}_2 \mu_k \cdot_1 (\lambda_i \cdot_2 \bar{c}_{ik}) + \sum_m \nu_m v_m .$$

Clearly, f maps C on $\bar{C}$ and it satisfies the above conditions. Now let $a_i = \mathcal{T}_1(c_{ik})$, $b_k = \mathcal{T}_2(c_{ik})$, $\bar{a}_i = \mathcal{T}_1(\bar{c}_{ik})$, $\bar{b}_k = \mathcal{T}_2(\bar{c}_{ik})$.

There exists a unique $\mathcal{L}$-morphism $f_1: A \longrightarrow \bar{A}$ ($f_2: B \longrightarrow \bar{B}$) satisfying $f_1(a_i) = \bar{a}_i$, i=1,...,r ($f_2(b_k) = \bar{b}_k$, k=1,...,s). We shall show that f is a $\mathcal{DL}$-morphism of C onto $\bar{C}$ with the underlying $\mathcal{L}$-morphisms f_1 and f_2.

Since $\overline{\pi}_1(f(c)) = \sum_{i=1}^{r} \lambda_i \overline{a}_i = f_1(\pi_1(c))$ we have $\overline{\pi}_1 \circ f = f_1 \circ \pi_1$.

Consider $c, c' \in C$ with the property $\pi_1(c) = \pi_1(c')$. Let ϱ, $\varrho' \in K$. Let us write

$$c = \sum_k{}_1 \sum_i{}_2 \mu_k \cdot_1 (\lambda_i \cdot_2 c_{ik}) + \sum_m \nu_m v_m ,$$

$$c' = \sum_k{}_1 \sum_i{}_2 \mu'_k \cdot_1 (\lambda'_i \cdot_2 c_{ik}) + \sum_m \nu'_m v_m .$$

For $\pi_1(c) = \pi_1(c')$, we have $\lambda_i = \lambda'_i$, $i=1,\dots,r$. Moreover, $(\varrho \cdot_1 c) +_1 (\varrho' \cdot_1 c') = \sum_k{}_1 \sum_i{}_2 (\varrho\mu_k + \varrho'\mu'_k) \cdot_1 (\lambda_i \cdot_2 c_{ik}) +$

$$+ \sum_m (\varrho \nu_m + \varrho' \nu_m) v_m .$$

It follows that $f((\varrho \cdot_1 c) +_1 (\varrho' \cdot_1 c')) = (\varrho \cdot_1 f(c)) +_1 (\varrho' \cdot_1 f(c'))$. Hence $f:(C, \pi_1, A) \longrightarrow (\overline{C}, \overline{\pi}_1, \overline{A})$ is a $\mathcal{LYF}$-morphism over f_1. Similarly, $f:(C, \pi_2, B) \longrightarrow (\overline{C}, \overline{\pi}_2, \overline{B})$ is a $\mathcal{LYF}$-morphism. Thus f is a $\mathcal{DL}$-morphism. The prove of uniqueness of f involves no difficulties.

Corollary. *Two double vector spaces $C, \pi: C \to A \times B$, and $\overline{C}, \overline{\pi}: \overline{C} \longrightarrow \overline{A} \times \overline{B}$, are $\mathcal{DL}$-isomorphic if and only if* $\dim A = \dim \overline{A}$, $\dim B = \dim \overline{B}$ *and* $\dim V = \dim \overline{V}$.

So we may define dimension of C $\dim C =: (r,s,t)$, where $r = \dim A$, $s = \dim B$, $t = \dim V$. In this case, C is $\mathcal{DL}$-isomorphic to the trivial $\mathcal{DL}$-space $K(r,s,t) =: K^r \times K^s \times K^t$ with projection $\pi = pr: K^r \times K^s \times K^t \longrightarrow K^r \times K^s$.

Now we shall investigate morphisms of the trivial double vector space $K(r,s,t)$ to another trivial $\mathcal{DL}$-space $K(\overline{r}, \overline{s}, \overline{t})$. For simplicity, let us denote $A = K^r$, $B = K^s$, $V = K^t$, $C = A \times B \times V$ and similarly $\overline{A} = K^{\overline{r}}$ etc. Let $f: C \longrightarrow \overline{C}$ be a $\mathcal{DL}$-morphism with underlying $\mathcal{L}$-morphisms $f_1: A \longrightarrow \bar{A}$, $f_2: B \longrightarrow \bar{B}$. Let $c = (a,b,v) \in C$ and let us write $f(c) = (\bar{a},\bar{b},\bar{v})$. Since $f:(C,\mathcal{T}_1,A) \longrightarrow (\bar{C},\bar{\mathcal{T}}_1,\bar{A})$ is a $\mathcal{L}\mathcal{Y}\mathcal{F}$-morphism over f_1 and $f:(C,\mathcal{T}_2,B) \longrightarrow (\bar{C},\bar{\mathcal{T}}_2,\bar{B})$ is a $\mathcal{L}\mathcal{Y}\mathcal{F}$-morphism over f_2, we have $a = f_1(a), b = f_2(b)$. Define a mapping $\sigma: A \times B \longrightarrow \bar{V}$ by $f((a,b,0)) = (f_1(a),\ f_2(b),\ \sigma(a,b))$.

Lemma 3.1. *The mapping $\sigma: A \times B \longrightarrow \bar{V}$ is bilinear.*

P r o o f. Let $a,a' \in A, b \in B$. Then

$$f(a+a',b,0)) = (f_1(a+a'),f_2(b),\sigma(a+a',b)) =$$

$$= (f_1(a)+f_1(a'),f_2(b),\sigma(a+a',b))$$

and further

$$f((a+a',b,0)) = f((a,b,0)+_2(a',b,0)) = f((a,b,0))+_2f((a',b,0)) =$$

$$= (f_1(a),f_2(b),\sigma(a,b))+_2(f_1(a'),f_2(b),\sigma(a',b) =$$

$$= (f_1(a)+f_1(a'),f_2(b),\sigma(a,b)+\sigma(a',b)).$$

Therefore $\sigma(a+a',b)=\sigma(a,b)+\sigma(a',b)$. A proof of the equality $\sigma(a,b+b')=\sigma(a,b)+\sigma(a,b')$ is quite similar.

Let f_3 denote the restriction of f to the vector space V. Obviously, $f_3(V) \in \bar{V}$. We have $f((a,b,v)) = f((a,b,0)+v) = f((a,b,0))+f_3(V) = (f_1(a),f_2(b),\sigma(a,b))+f_3(v) = (f_1(a),f_2(b),\sigma(a,b)+f_3(v))$.

Further, denote by the symbol Hom $(C,\bar{C})$ the set of all $\mathcal{D}\mathcal{L}$-morphisms of the trivial $\mathcal{D}\mathcal{L}$-space C to the trivial $\mathcal{D}\mathcal{L}$-space $\bar{C}$. Let Hom $(A,\bar{A})$ be the vector space of all $\mathcal{L}$-morphisms of A to $\bar{A}$ (similarly for B,V) and let Hom $(A \times B,\bar{V})$ denote the vector space of all bilinear mappings of $A \times B$ to $\bar{V}$.

Theorem 3.4. *There exists a bijection*
χ: Hom $(C,\bar{C}) \longrightarrow$ Hom $(A,\bar{A}) \times$ Hom $(B,\bar{B}) \times$ Hom $(V,\bar{V}) \times$ Hom $(A \times B,\bar{V})$. *The mapping χ sends each $\mathcal{D}\mathcal{L}$-morphism $f \in$ Hom $(C,\bar{C})$ onto an ordered quadruple (f_1,f_2,f_3,σ). The inverse mapping χ^{-1} maps*

a quadruple (f_1, f_2, f_3, σ) on $f \in \mathrm{Hom}\,(C, \overline{C})$ given by

$$f((a,b,v)) = (f_1(a), f_2(b), \sigma(a,b)+f_3(v)).$$

If $f: C \longrightarrow \overline{C}$, $f': \overline{C} \longrightarrow \tilde{C}$ be $\mathcal{DL}$-morphisms with corresponding quadruples (f_1, f_2, f_3, σ) and $(f_1', f_2', f_3', \sigma')$ then the quadruple $(f_1' \circ f_1, f_2' \circ f_2, f_3' \circ f_3, \sigma'(f_1, f_2) + f_3' \circ \sigma)$ corresponds to the product $f' \circ f: C \longrightarrow \tilde{C}$. The proof is straightforward.

The mapping $\mathcal{H}$ enables us to identify the sets $\mathrm{Hom}\,(C, \overline{C})$ and $\mathrm{Hom}\,(A, \overline{A}) \times \mathrm{Hom}\,(B, \overline{B}) \times \mathrm{Hom}\,(V, \overline{V}) \times \mathrm{Hom}\,(A \quad B, \overline{V})$. Note that $(f_1, f_2, f_3, \sigma) \in \mathrm{Hom}\,(C, \overline{C})$ is an isomorphism if and only if $f_1 \in \mathrm{Hom}\,(A, \overline{A})$, $f_2 \in \mathrm{Hom}\,(B, \overline{B})$, $f_3 \in \mathrm{Hom}\,(V, \overline{V})$ are isomorphisms.

Let $\mathrm{Aut}(C)$ be the group of all automorphisms of the $\mathcal{DL}$-space C, let $\mathrm{Aut}(A)$ denote the group of all automorphisms of the vector space A etc. The mapping $\mathcal{H}$ gives an identification

$$\mathrm{Aut}(C) \longrightarrow \widetilde{\mathrm{Aut}}(A,B,V) \times \mathrm{Hom}\,(A \times B, \overline{V})$$

where $\widetilde{\mathrm{Aut}}(A,B,V) = \mathrm{Aut}(A) \times \mathrm{Aut}(B) \times \mathrm{Aut}(V)$ is a direct product of groups. Define

$$j: \mathrm{Aut}(C) \longrightarrow \widetilde{\mathrm{Aut}}(A,B,V)$$

by $j(f_1, f_2, f_3, \sigma) = (f_1, f_2, f_3)$. It is easily seen that this mapping is a group epimorphism. Its kernel is a commutative group $\mathrm{Hom}(A \times B, \overline{V})$ with its usual additive group structure. Hence we have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(A \times B, \overline{V}) \longrightarrow \mathrm{Aut}(C) \longrightarrow \widetilde{\mathrm{Aut}}(A,B,V) \longrightarrow 0$$

where i is an embedding. If we define $q: \widetilde{\mathrm{Aut}}(A,B,V) \longrightarrow \mathrm{Aut}(C)$ by $q(f_1, f_2, f_3) = (f_1, f_2, f_3, 0)$ then q is a group homomorphism and it is a splitting of the above sequence. It follows

Theorem 3.5. *The group $\mathrm{Aut}(C)$ is a semi-direct product of the groups $\widetilde{\mathrm{Aut}}(A,B,V)$ and $\mathrm{Hom}\,(A \times B, \overline{V})$. Moreover, the group operation of $\widetilde{\mathrm{Aut}}(A,B,V)$ on $\mathrm{Hom}\,(A \times B, \overline{V})$ establishing this semi-direct product is of the form $(f_1, f_2, f_3)\,\sigma = f_3^{-1}\sigma(f_1, f_2)$.*

Example 3.1. If $p_E : E \longrightarrow M$ is a vector bundle then its tangent bundle TE admits two vector bundle projections

$p_{TE} : TE \longrightarrow E$ and $T_{p_E} : TE \longrightarrow TM$. Each fibre $(TE)_x =$
$= (p_{TE}\ p_E)^{-1}(x)$ for $x \in M$ is a $\mathcal{DL}$ -space with projection
$\mathcal{T}: (TE)_x \longrightarrow E_x \times T_xM$, $\mathcal{T}(\xi) = (p_{TE}(\xi), T_{p_E}(\xi))$ for $\xi \in (TE)_x$.

Example 3.2. Let T^*E be a cotangent bundle of the vector bundle $p_E : E \longrightarrow M$. Besides a natural projection $p^* : T^*E \longrightarrow E$, there exists a projection $q : T^*E \longrightarrow E^*$ of the vector bundle given as follows. For $y \in E$ and $\omega : T_yE \longrightarrow R$, assume the restriction $\overline{\omega}$ of ω to the vertical subspace V_yE : $\overline{\omega} = \omega | V_yE \longrightarrow R$.

The vector space V_yE may be identified with E_x (where $x = p_E(y)$) via translation. Let $\mathcal{T} : V_yE \longrightarrow E_x$ denotes the corresponding isomorphism. Then we define $q\,\omega = \mathcal{T}^* \overline{\omega}\, E_x \longrightarrow R$. A fibre $(T^*E)_x$ is a $\mathcal{DL}$-space with projection
$\mathcal{T}^* : (T^*E)_x \longrightarrow E_x \times E^*_x$, $\mathcal{T}^*(\omega) = (p^*(\omega), q(\omega))$.

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SOUHRN

Dvojné vektorové prostory

A l e n a   V a n ž u r o v á

V článku je podána geometrická axiomatizace kategorie dvojně lineárních prostorů a jejich morfismů, kterou zavedl J.Pradines v [1]. Ukazuje se, že každý $\mathcal{DL}$ -prostor je isomorfní s některým triviálním $\mathcal{DL}$-prostorem. V závěru je vyšetřována grupa všech $\mathcal{DL}$-automorfismů triviálního $\mathcal{DL}$-prostoru.

Р Е З Ю М Е

Двойно векторные пространства

А л е н а В а н ж у р о в а

В статье дается аксиоматическое описание двойно векторных пространств и двойно линейных морфизмов, которое более геометрично чем оригинальное определение введенное Прадином. С двойно векторными пространствами встречаемся в дифференциальной геометрии второго порядка, где служат слоями некоторых расслоений.

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