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A NOTE ON TOPOLOGICAL REPRESENTATIONS OF DISTRIBUTIVE LATTICES.

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A distributive lattice*) L is said to be topologically*) represented in a topological T_0-space $\mathfrak{S}_L$ if there exists an isomorphism between L and a set-ring R of certain open subsets of $\mathfrak{S}_L$, R constituting an open basis of $\mathfrak{S}_L$. (One could also speak of representations by a set-ring of closed sets of $\mathfrak{S}_L$ but this latter representation being dual to the former it is not too interesting here.)

Between all the topological T_0-spaces $\mathfrak{S}_L$ representing a given distributive lattice L there is one „universal" T_0-space $\overline{\mathfrak{S}}_L$ containing every representation T_0-space $\mathfrak{S}_L$ as a dense subspace. $\overline{\mathfrak{S}}_L$ is essentially the space of all prime α-ideals of L. This universal representation—space $\overline{\mathfrak{S}}_L$ has been described by STONE (3).

We give another characterisation of $\overline{\mathfrak{S}}_L$ (theorem 5) under the not too specialising hypothesis that L has a lattice unit. As an easy consequence we get the assertion that any distributive lattice with unit and zero and with only maximal prime α-ideals is a BOOLEAN algebra. When omitting the hypothesis on the lattice unit we have a generalized BOOLEAN algebra in the sense of STONE (3).

We collect several known notions and theorems used in the sequel. (The term *lattice* always means a *distributive lattice* in what follows.)

A nonvoid subset I of a lattice L is called an *α-ideal* if the following holds:

(1) If $a \in I$, $b \in I$ then $a \cap b \in I$.
(2) If $a \in I$, $c \in L$ then $a \cup c \in I$.

An α-ideal P is said to be *prime* under the condition

(3) If $a \cup b \in P$ then either $a \in P$ or $b \in P$ whenever $P \neq L$.

*) We assume the reader to be familiar with the basic notions of *distributive lattice theory* (Cf. BIRKHOFF (1), Chap. V, KÖTHE, HERMES (2), C) and of *general topology* (cf. e. g. ALEXANDROFF, HOPF: Topologie I, Erster Teil, Erstes Kap. §§ 1 to 6, Zweites Kap. §§ 1, 2) as well.

The concept of a (prime) μ-ideal is dual.

Note that the lattice L itself is *not taken for a prime ideal.*

An α-ideal M is said to be *maximal* or *divisorless* (STONE) if there is no different prime α-ideal over M. Any maximal α-ideal is prime. The concept and the properties of a maximal μ-ideal are dual. A concept not used by STONE: A prime α-ideal U is called *minimal* if no different prime α-ideal is contained in U. The dualisation is obvious.

Following theorems are due to STONE (3).

Theorem 1. If the lattice L is partitioned into disjoint subclasses P and Q then (1) P is an α-ideal and Q is a μ-ideal if and only if both are prime, (2) P is a prime α-ideal if and only if Q is a prime μ-ideal.

Theorem 2. If the α-(μ-)ideal I is distinct from the whole lattice L then I is the set theoretical product of the prime α-(μ-)ideals which contain I.

Definition. A set $\mathfrak{S}$ of prime α-ideals P of a distributive lattice L is called *representative* if: (1) any $x \in L$ is contained in a suitable $P \in \mathfrak{S}$, (2) for no two different $x_1 \in L$, $x_2 \in L$ the set of all $P \in \mathfrak{S}$ with $x_1 \in P$ coincides with that of all $P \in \mathfrak{S}$ with $x_2 \in P$.

Theorem 3 and **definition.** *Taking the sets $\mathfrak{S}_x$ of all prime α-ideals $P \in \mathfrak{S}$ containing a fixed $x \in L$ for open sets, a representative set $\mathfrak{S}$ of prime α-ideals becomes a topological T_0-space with an open basis formed by the set-ring R of all the $\mathfrak{S}_x$, R being isomorphic with L by the correspondence $\mathfrak{S}_x \longleftrightarrow x$. The relations $x \in P$ in L and $P \in \mathfrak{S}_x$ in $\mathfrak{S}$ are logically equivalent. $\emptyset \in R$ if and only if L has a lattice zero n, $\mathfrak{S}_n = \emptyset$. $\mathfrak{S} \in R$, $\mathfrak{S} = \mathfrak{S}_u$ if and only if L has a lattice unit u.*

$\mathfrak{S}$ is called a representative space of L.

Proof is essentially known and very easy (Cf. STONE (3), BIRKHOFF (1)).

Hence there is no need for us to go into detail.

Note. Conversely, any topological T_0-space $\mathfrak{S}'$ possessing an open basis forming a set-ring R isomorphic to the lattice L, is homeomorphic with a space $\mathfrak{S}$ of the theorem 3, which it is easy to see if we consider the complete systems of „fundamental" neighbourhoods of a point of $\mathfrak{S}'$ as a prime α-ideal in the ring R of basic open sets of $\mathfrak{S}'$.

Theorem 4. *Suppose $\mathfrak{S}'$ is a T_0-space, R is its open basis. Let R form a set ring (i. e. a distributive lattice) with the zero $\emptyset \in R$ and the unit $\mathfrak{S}' \in R$.* (Any open basis of $\mathfrak{S}'$ obviously can be extended to such an R without changing its original cardinal number except in the case this cardinal number is finite. But in this latter case R remains finite.)

Let each minimal prime α-ideal U of R form a complete system of neighbourhoods of a certain point $\xi \in \mathfrak{S}'$. — Then $\mathfrak{S}'$ is a bicompact space.

Proof. Assume $\mathfrak{S}'$ is not bicompact under the given hypotheses. Hence there exists a family A of open sets $\mathfrak{A} \in R$ covering $\mathfrak{S}'$ such that no finite subfamily of A covers $\mathfrak{S}'$. Let us form a μ-ideal I_A in R generated by A. It is obvious that the family I_A itself is a covering of $\mathfrak{S}'$ containing no finite covering. Therefore $\mathfrak{S}'$ non $\epsilon\, I_A$ i. e. $I_A \neq R$.

Applying a usual transfinite construction let us form a maximal μ-ideal M over I_A. Then the complementary prime α-ideal $U = R - M$ is minimal by theorem 1. As a consequence of the hypothesis U is a complete system of neighbourhoods of a certain point $\xi \in \mathfrak{S}'$. Hence $\xi \in \prod_{\mathfrak{O} \in U} \mathfrak{O} \neq \emptyset$. This means ξ non $\epsilon\, \mathfrak{A}$ for any $\mathfrak{A} \in M$, and consequently for any $\mathfrak{A} \in A$. This is a contradiction since A is a covering of $\mathfrak{S}'$. — Hence $\mathfrak{S}'$ must be bicompact.

Definition. A system Q of open sets $\mathfrak{A}$ of the open basis R of any T_0-space $\mathfrak{S}'$ is called a *pseudocomplete system of neighbourhoods**) of the point $\xi \in \mathfrak{S}'$ if $\prod_{\xi \in \mathfrak{A} \in R} \mathfrak{A} = \prod_{\mathfrak{A} \in Q} \mathfrak{A}$.

A system of open sets is said to be *centred* if any of its finite subsystems has a nonvoid product.

Theorem 5. *The space $\overline{\mathfrak{S}}_L$ of all prime x-ideals $P \in \overline{\mathfrak{S}}$ of a distributive lattice L with unit u and zero n is a bicompact T_0-space possessing an open basis R' with the following properties:*

(i) *Any centred system of open sets of the basis R' has a nonvoid product.*

(ii) *Any pseudocomplete system Q' of neighbourhoods of a point $P \in \overline{\mathfrak{S}}$ containing a suitable $\mathfrak{A}_3 \in Q'$ with $\mathfrak{A}_3 \subset \mathfrak{A}_1\mathfrak{A}_2$ whenever $\mathfrak{A}_1 \in Q'$, $\mathfrak{A}_2 \in Q'$, is a complete system of neighbourhoods of P.*

Conversely, if a bicompact T_0-space $\overline{\mathfrak{S}}$ has an open basis R' fulfilling (i) *and* (ii) *then $\overline{\mathfrak{S}}$ can be taken for a space of all prime α-ideals of the distributive lattice (set ring) R generated by forming finite set sums and products upon members of the open basis R' and by adjoining $\emptyset$ and $\mathfrak{S}$.*

Proof. Let L be the given distributive lattice with unit u and zero n. Then $\overline{\mathfrak{S}}_L$ is bicompact by theorem 4.

Let A be a centred system of open sets of the basis R of $\overline{\mathfrak{S}}_L$ where R forms a set-ring isomorphic to L. Then the α-ideal I_A generated by A is distinct from the whole R. Hence there exists a prime α-ideal P over I_A in R, by theorem 2.

Let P' correspond in L to the prime α-ideal P of R by the representation isomorphism $L \cong R$ (see theorem 3). Therefore $P' \in \mathfrak{S}_x$ for each $x \in L$ with $\mathfrak{S}_x \in I_A$, i. e. $P' \in \prod_{\mathfrak{S}_x \in A} \mathfrak{S}_x$.

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*) The concept and the term are due to prof. ČECH, Čas. mat. fys. **66** (1937), p. D 232. — A system of neighbourhoods of a point is known to be complete if every open set containing this point contains a neighbourhood belonging to this system.

This is the property (i).

Let us prove the property (ii).

Let $T_{P'}$ be a pseudocomplete system of neighbourhoods of the point $P' \in \overline{\mathfrak{S}}_L$, P' being a prime α-ideal in L as well. Let P be the complete system of neighbourhoods of the point P' so that the prime α-ideal P' of L corresponds to the prime α-ideal P of R in the representation isomorphism $L \cong R$. Let finally $T_{P'}$ fulfill the hypothesis of the condition (ii). We form the α-ideal $I_{P'}$ generated by the sets of $T_{P'}$ in R. We have to prove that $T_{P'}$ is a complete system of neighbourhoods of P'. Since for any $\mathfrak{O} \in I_{P'}$ there is an $\mathfrak{O}' \in$ with $\mathfrak{O}' \subset \mathfrak{O}$, it suffices to prove $I_{P'} = P$.

Actually, $T_{P'} \subset I_{P'} \subset P$ implies $\prod_{\mathfrak{O} \in T_{P'}} \mathfrak{O} \supset \prod_{\mathfrak{O} \in I_{P'}} \mathfrak{O} \supset \prod_{\mathfrak{O} \in P} \mathfrak{O}$ and since $T_{P'}$ is a pseudocomplete system of neighbourhoods of $P' \in \overline{\mathfrak{S}}_L$ we have

$$\prod_{\mathfrak{O} \in I_{P'}} \mathfrak{O} = \prod_{\mathfrak{O} \in P} \mathfrak{O}. \qquad (*)$$

Let I' be the original in L of the α-ideal $I_{P'}$ of R. According to theorem 2, in order to get $I' = P'$ — and theretore $I_{P'} = P$ — we have to prove the identity of both the sets of all prime α-ideals over I' and over P'. But the former set is nothing else than the set of all points of the set product $\prod_{\mathfrak{O} \in I_{P'}} \mathfrak{O}$, the latter one is analogously the product $\prod_{\mathfrak{O} \in P} \mathfrak{O}$. These products being identical by (*) the condition (ii) is proved.

Now, let us return to the proof of the converse theorem.

Suppose $\overline{\mathfrak{S}}$ is a bicompact T_0-space, R' its open basis satisfying the conditions (i) and (ii). Let us generate the minimal set-ring (distributive lattice) R containing $\emptyset$ and $\mathfrak{S}$ and R'. The conditions (i), (ii) remain valid even in R. We form the set $\overline{\mathfrak{S}}_R$ of all prime α-ideals α of the lattice R.

By (i) we get $\prod_{\mathfrak{O} \in P} \mathfrak{O} \neq \emptyset$ for any prime α-ideal P. We shall prove that P is a complete system of neighbourhoods of a certain point $\pi \in \overline{\mathfrak{S}}$ where, of course, $\pi \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$.

We have to consider two alternatives, (A) and (B).

(A) The product $\prod_{\mathfrak{O} \in P} \mathfrak{O}$ contains only one point π. — In this case we apply (ii) to the pseudocomplete system P of neighbourhoods of π and have the wished result.

(B) The product $\prod_{\mathfrak{O} \in P} \mathfrak{O}$ contains more than a single point. — In this case we proceed as follows:

First prove that the product $\Pi(\bar{\tau})$ of all closed sets $(\bar{\tau})$ with $\tau \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$ cannot be void. — Indeed, suppose $\Pi(\bar{\tau}) = \emptyset$. Then $\overline{\mathfrak{S}} = \Sigma(\overline{\mathfrak{S}} - (\bar{\tau}))$ is a covering of the space $\overline{\mathfrak{S}}$ by open sets $\overline{\mathfrak{S}} - (\bar{\tau})$. Choose an open neighbourhood $\mathfrak{O}_\gamma \in R$ to any $\gamma \in \overline{\mathfrak{S}} - (\bar{\tau})$ (for $\tau \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$) such that $\mathfrak{O}_\gamma \subset \overline{\mathfrak{S}} - (\bar{\tau})$. Hence $\Sigma \mathfrak{m}_\gamma = \overline{\mathfrak{S}}$ is an open covering of $\overline{\mathfrak{S}}$. Since $\overline{\mathfrak{S}}$ is bicompact we can extract a finite covering $\sum_{i=1}^{n} \mathfrak{O}_{\gamma_i} = \overline{\mathfrak{S}}$ by certain $\mathfrak{O}_{\gamma_i}$. Now, $\overline{\mathfrak{S}} \in P$ and P is a prime α-ideal in R. This requires $\mathfrak{O}_{\gamma_j} \in P$ with a suitable j, $(1 \leqq j \leqq n)$. But we have $\mathfrak{O}_{\gamma_j} \subset \overline{\mathfrak{S}} - (\bar{\tau})$ for a certain $\tau \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$, i. e. τ non $\in \mathfrak{O}_{\gamma_j}$, which is a contradiction. — Therefore $\pi \in \Pi(\bar{\tau})$ with a suitable $\pi \in \overline{\mathfrak{S}}$.

We furthermore prove that no other point different from π is contained in $\Pi(\bar{\tau})$. — Actually, every neighbourhood $\mathfrak{O}_\pi \in R$ of π contains each $\tau \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$. Hence $\Pi(\bar{\tau}) \subset \prod_{\mathfrak{O} \in P} \mathfrak{O}$ and any further point $\pi' \neq \pi$ contained in $\Pi(\bar{\tau})$ would have the same neighbourhoods as π itself, which is impossible in a T_0-space. Hence $\Pi(\bar{\tau}) = \pi$.

Finally, we prove that the complete system P_π of all neighbourhoods of π is P itself. — Evidently $\prod_{\mathfrak{O} \in P\pi} \mathfrak{O} \subset \prod_{\mathfrak{O} \in P} \mathfrak{O}$. If these products would differ we would have a point $\tau \in \prod_{\mathfrak{O} \in P} \mathfrak{O}$ which would not be contained in a certain neighbourhood $\mathfrak{O} \in R$ of π. This would mean π non $\in (\bar{\tau})$ in contradiction with $\pi \in \Pi(\bar{\tau})$ above. Hence $\prod_{\mathfrak{O} \in P\pi} \mathfrak{O} = \prod_{\mathfrak{O} \in P} \mathfrak{O}$ and applying (ii) we get $P = P_\pi$.

It is now obvious that the correspondence $\pi \longleftrightarrow P_\pi$ is a homeomorphism of the given space $\overline{\mathfrak{S}}$ with the universal representative space $\overline{\mathfrak{S}}_R$ of all prime α-ideals of the lattice R, which completes the proof.

Corollary. *Suppose* $\overline{\mathfrak{S}}$ *is a bicompact* T_1*-space satisfying the conditions* (i), (ii) *of the preceding theorem. Then* $\overline{\mathfrak{S}}$ *is a totally disconnected bicompact* HAUSDORFF *space, i. e. a* BOOLEAN *space.*

Proof. It is easy to see that assuming $\overline{\mathfrak{S}}$ to be a T_1-space we can consider $\overline{\mathfrak{S}}$ as a universal representative space of such a distributive lattice L that no prime α-ideal in L can contain another different prime α-ideal. Hence any prime α-ideal in L is maximal. Denoting by M_1, M_2 two different maximal α-ideals in L, i. e. points of $\overline{\mathfrak{S}}_L$, we get $x_1 \cap x_2 = n$ for suitable $x_1 \in M_1$, $x_2 \in M_2$, i. e. $\mathfrak{O}_{x_1} \cap \mathfrak{O}_{x_2} = \emptyset$ for suitable open neighbourhoods, $\mathfrak{O}_{x_1}$ of the point M_1 and $\mathfrak{O}_{x_2}$ of the point M_2, if $\mathfrak{O}_{x_{1,2}}$ represents $x_{1,2} \in L$. Hence $\overline{\mathfrak{S}}$ is a HAUSDORFF space.

Now, let R be the set-ring of the open basis of the representative space so that the conditions (i), (ii) apply to R.

We have to prove $\mathfrak{O} = \overline{\mathfrak{O}}$ for any open set $\mathfrak{O} \in R$, if $R \cong L$.

Consider a complete system of neighbourhoods $\mathfrak{O}_\tau$ of a certain point $\tau \in \overline{\mathfrak{O}}$. Then the system of all products $\mathfrak{O}\mathfrak{O}_\tau \in R$ is centred. Hence

$\Pi\mathfrak{O}\mathfrak{O}_\tau \neq \emptyset$ by the preceding theorem. Therefore $\tau \in \mathfrak{O}$, q. e. d. Thus we almost immediately get the

Theorem 6. *Any distributive lattice with unit and zero each prime α-ideal* (or μ-ideal as well) *of which is maximal is a* BOOLEAN *algebra.*

Theorem 7. *Any distributive lattice with zero each prime α-ideal of which is maximal is a generalized* BOOLEAN *algebra in the sense of* STONE (3).

Proof. The proof can be given by an appropriate generalization of theorem 5. We prefer to prove it by an easy sharpening of a theorem of STONE (3).

STONE (3) defines the topology in the set $\mathfrak{C}$ of all prime μ-ideals of a distributive lattice L (with zero) as follows: An open set $\mathfrak{C}_x$ in $\mathfrak{C}$ is the set of all prime μ-ideals not containing the given element $x \in L$.

Comparing STONE'S representative space $\mathfrak{C}$ of a distributive lattice with our universal representative space $\overline{\mathfrak{S}}_L$ of all prime α-ideals (in the sense of theorem 3) we easily see that the correspondence $P \longleftrightarrow L - P = Q$, between prime α- and the complementary prime μ-ideals (see theorem 1), is in fact a homeomorphism between both the representative spaces.

Now, STONE'S theorem 17 ((3), p. 17) says:

The space $\mathfrak{C}$ is a T_1-space if and only if every prime μ-ideal in the distributive lattice L is maximal.

But applying the preceding remark on the homeomorphism of $\mathfrak{C}$ and $\overline{\mathfrak{S}}$, we easily conclude by the reasoning used in the proof of the preceding corollary of theorem 5 that STONE'S theorem 17 (3) remains valid if „T_1-space" changes into „HAUSDORFF space".

The next theorem 18 of STONE (3) asserts:

The space $\mathfrak{C}$ is a HAUSDORFF space if and only if L is a generalized BOOLEAN algebra. Hence theorem 7 is proved.

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Poznámka o topologických representacích distributivních svazů.

(Obsah předešlého článku.)

Topologickou representací distributivního svazu L nazýváme jeho isomorfní zobrazení na množinový okruh R otevřených množin jistého topologického T_0-prostoru $\mathfrak{S}_L$, v němž R tvoří otevřenou basi. STONE v (3) charakterisoval jistý topologický T_0-prostor, který dává topologickou representaci daného distributivního svazu a který lze považovati v podstatě za prostor všech jeho (průnikových) primideálů. Zde podáváme jinou charakterisaci tohoto prostoru pro případ, že daný svaz má jednotku, která zní takto: Každý topologický T_0-prostor, který je bikompaktní a jehož jistá otevřená base má následující dvě vlastnosti:

(i) každý centrovaný systém množin z base má neprázdný průnik,

(ii) každý pseudoúplný systém okolí bodu, který se dvěma okolími obsahuje vždy jisté další okolí obsažené v průniku těchto, je již úplným systémem okolí tohoto bodu —

je (až na homeomorfismus) STONEOVÝM prostorem všech primideálů jistého svazu s jednotkou. Obráceně, STONEŮV prostor všech primideálů má řečené vlastnosti, jakmile jen daný distributivní svaz má jednotku. — Z toho plyne tento důsledek: Každý distributivní svaz s jednotkou a nulou, jehož všechny primideály jsou maximální, je BOOLEOVA algebra. Po vynechání předpokladu o existenci jednotky dostáváme zobecněnou BOOLEOVU algebru.