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ON SUMMABILITY IN CONVERGENCE l-GROUPS

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Summary. In connection with two questions on convergence groups proposed by J. Novák there are constructed convergence l-groups which have some rather pathological properties concerning the summability of sequences.

Keywords: Convergence group, convergence l-group, summability.

AMS Subject Classification: 06F15, 06F20.

Convergence groups were studied by J. Novák [13], [14], [15]; cf. also R. Frič [2], [33], R. Frič and V. Koutník [4], C. Kliś [10], V. Koutník [12], C. Schwartz [17] and F. Zanolin [18].

Let $\mathscr{A}$ be the class of all convergence groups G containing a sequence (x_n) which converges to 0 but each subsequence (y_n) of which is not summable. (A sequence (z_n) is summable if the series $\sum_{n=1}^{\infty} z_n$ converges.)

Next, let $\mathscr{B}$ be the class of all convergence groups G containing a sequence (x_n) such that each subsequence (y_n) of (x_n) contains a subsequence which is summable and another subsequence which is not summable.

Problems 14 and 16 proposed by J. Novák [15] consist in asking whether the class $\mathscr{A}$ (or the class $\mathscr{B}$, respectively) is nonempty.

Problem 14 was solved affirmatively by F. Zanolin [18] and by R. Frič and V. Koutník [4]. C. Schwartz [17] found a normed linear space belonging to the class $\mathscr{A}$.

C. Kliś [10] solved Problem 15 affirmatively by appliyng orthonormal vector measures with values in the Hilbert space l_2.

The notion of the convergence l-group was introduced by M. Harminc [6]; cf. also Harminc [7], [8], and the author [9]. While in [6] a convergence α on an l-group G is a subset of $G^N \times G$ consisting of pairs $((x_n), x)$ where x_n converges to x, here we understand by a convergence α a subset of G^N consisting of sequences (x_n) converging to 0.

Each convergence l-group is a convergence group. A natural question arises whether there exists a convergence l-group belonging to the class $\mathscr{A}$; a similar question can be asked for the class $\mathscr{B}$.

For an l-group G we denote by Conv G the set of all convergences α on G such that (G, α) turns out to be a convergence l-group. If H is an l-subgroup of G and

$\alpha \in \operatorname{Conv} G$, then $\alpha(H) = \alpha \cap H^N$ is a convergence on H induced by α; in such a case $(H; \alpha(H))$ is a convergence l-group as well. The l-group G is said to be of infinite breadth if there exists an infinite disjoint subset of G (a subset M of G is called disjoint if $x_1 \wedge x_2 = 0$ whenever x_1 and x_2 are distinct elements of M, and $x > 0$ for each $x \in M$). For example, each direct product of an infinite number of nonzero l-groups is of infinite breadth.

In the present note it will be shown that convergence l-groups belonging to the class $\mathscr{A}$ occur rather frequently. Also, there exists a convergence l-group which belongs to the class $\mathscr{B}$. Namely, the following results will be established:

(A) *Let G be an abelian lattice ordered group of infinite breadth. There exist $\alpha_m \in \operatorname{Conv} G$ $(m = 1, 2, \ldots)$ and convex l-subgroups G_m $(m = 1, 2, \ldots)$ of G such that*

(i) *$\alpha_{m(1)} \neq \alpha_{m(2)}$ and $G_{m(1)} \cap G_{m(2)} = \{0\}$ whenever $m(1)$ and $m(2)$ are distinct positive integers;*

(ii) *for each positive integer m, $(G_m, \alpha_m(G_m))$ belongs to the class $\mathscr{A}$.*

(B) *There exists a linearly ordered group G such that*

(i) *$(G, \alpha_0) \in \mathscr{B}$, where α_0 is the set of all sequences (x_n) in G which o-converge to 0 in G;*

(ii) *G is a subgroup of the lexicographic product of linearly ordered groups G_n $(n \in N)$, where each G_n is isomorphic to Z.*

(Here, Z denotes the additive group of all integers with the natural linear order.)

1. PRELIMINARIES

For the terminology and notation concerning linearly ordered groups and lattice ordered groups (= l-groups) cf. L. Fuchs [5] and V. M. Kopytov [11]. The group operation will be denoted additively. Throughout the paper we assume that all l-groups under consideration are abelian.

We recall some relevant notions on convergence l-groups.

Let N be the set of all positive integers and let G be an l-group. The direct product $\prod_{n \in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by G^N. The elements of G^N are denoted by $(g_n)_{n \in N}$, or simply (g_n). If there exists $g \in G$ such that $g_n = g$ for each $n \in N$, then we put $(g_n) = \operatorname{const} g$.

(g_n) is said to be a sequence in G. The notion of a subsequence has the usual meaning.

For each l-group G we set $G^+ = \{g \in G: g \geqq 0\}$. Let α be a convex subsemigroup of $(G^N)^+$ such that the following conditions are satisfied:

(I) *If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.*

(II) *Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.*

(III) *Let $g \in G$. Then* const g *belongs to α if and only if $g = 0$.*

Under these assumptions α is said to be a convergence in G. The system of all convergences in G will be denoted by Conv G.

For $(g_n) \in G^N$, $\alpha \in \text{Conv } G$ and $g \in G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$. If $(x_n), (y_n) \in G^N$, $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n + y_n \to_\alpha x + y$ and $-x_n \to_\alpha -x$.

If $\alpha \in \text{Conv } G$, then the pair (G, α) will be called a convergence l-group. It is clear that each convergence l-group is a convergence group.

Let H be an l-subgroup of G and let $\alpha \in \text{Conv } G$. Put $\alpha(H) = \alpha \cap H^N$. Then $\alpha(H)$ belongs to Conv H; it is said to be induced by α. For a sequence (h_n) in H and for $h \in H$ we often write $x_n \to_\alpha x$ instead of $x_n \to_{\alpha(H)} x$.

Let A be a nonempty subset of $(G^N)^+$. We denote by δA the system of all subsequences of sequences belonging to A. The symbol $[A]$ will denote the convex closure of the set $A \cup \{\text{const } 0\}$ in G^N. Let $\langle A \rangle$ be the subsemigroup of G^N generated by the set A. Next, A^* will denote the set of all sequences (x_n) in G such that each subsequence (y_n) of (x_n) has a subsequence belonging to A.

1.1. Proposition. (Cf. [8], Theorem 1.18 or [6], Theorem 2.) *Let $\emptyset \neq A \subseteq (G^N)^+$. Then the following conditions are equivalent:*

(a) *If $g \in G$, const $g \in [\langle \delta A \rangle]$, then $g = 0$.*

(b) $[\langle \delta A \rangle]^* \in \text{Conv } G$.

For $X \subseteq G$ we put

$$X^\perp = \{g \in G: |g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

If a nonempty subset A of $(G^N)^+$ satisfies the condition (a) from 1.1, then A will be said to be regular.

The following two assertions are easy consequences of 1.1 (cf. also [8] for related results):

1.2. Lemma. *Let $(x_n) \in (G^N)^+$. Assume that $x_n \wedge x_m = 0$ whenever n and m are distinct elements of N. Then the one-element set (x_n) is regular.*

1.3. Lemma. *Let A be regular. Let (x_n) be a sequence in G such that all x_n belong to $A^\perp$ and $(x_n) \in [\langle \delta A \rangle]^*$. Then there is $m \in N$ such that $x_n = 0$ for each $n > m$.*

2. THE CLASS $\mathscr{A}$

Proof of Theorem (A). Let G be an l-group of infinite breadth. Hence there exists an infinite disjoint subset X in G. Thus there is a system $S = \{X_n\}_{n \in N}$ such that each X_n is a countably infinite subset of X and $X_n \cap X_m = \emptyset$ whenever n and m are distinct elements of N.

Let $m \in N$. Arrange the elements of X_m into a one-to-one sequence $(x_n^m)_{n \in N}$ in G. In view of 1.2, the set $(x_n^m)_{n \in N}$ is regular. Denote $\alpha_m = [\langle \delta\{(x_n^m)_{n \in N}\} \rangle]^*$. According to 1.1, α_m belongs to Conv G.

Let $m(1)$ and $m(2)$ be distinct elements of N. Then we have

$$(x_n^{m(1)})_{n\in N} \in \alpha_{m(1)},$$

but in view of 1.3, $(x_n^{m(2)})_{n\in N}$ does not belong to $\alpha_{m(1)}$. Hence $\alpha_{m(1)} \neq \alpha_{m(2)}$.

We denote by G_m the convex l-subgroup of G generated by the set $\{x_n^m\}_{n\in N}$. Since $(x_n^m)_{n\in N} \in \alpha_m$, we have $x_n^m \to_{\alpha_m} 0$. Let $\{z_n^m\}_{n\in N}$ be a subsequence of $(x_n^m)_{n\in N}$. Put $y_n^m = z_1^m + z_2^m + \ldots + z_n^m$ for each $n \in N$. Assume that there is $y^m \in G_m$ such that $y_n^m \to_{\alpha_m} y^m$.

We have $y_n^m > 0$ for each $n \in N$. Hence

$$y_n^m = y_n^m \vee 0 \to_{\alpha_m} y^m \vee 0,$$

thus $y^m \geqq 0$. Let $k \in N$. Consider the sequence $(z_n^m)_{k \leqq n\in N}$. For each $n \in N$ with $n \geqq k$ we have $y_n^m = y_n^m \vee z_k^m$, thus

$$y_n^m \to_{\alpha_m} y^m \vee z_k^m;$$

therefore

$$z_k^m \leqq y^m \quad \text{for each} \quad k \in N. \tag{1}$$

Since $y^m \in G_m$, there is $t \in N$ such that

$$0 \leqq y^m \leqq c_1 x_1^m + c_2 x_2^m + \ldots + c_t x_t^m, \tag{2}$$

where $c_1, c_2, \ldots, c_t$ are positive integers. Choose $k \in N$, $k > t$. Then $z_k^m \wedge x_1^m = 0, \ldots, z_k^m \wedge x_t^m = 0$, which in view of (2) implies $z_k^m \wedge y^m = 0$. Taking (1) into account, we arrive at a contradiction. We have proved that $\sum_{n=1}^{\infty} z_n^m$ does not exist in the convergence l-group $(G_m, \alpha_m(G_m))$. According to the construction of G_m we have $G_{m(1)} \cap G_{m(2)} = \{0\}$ whenever $m(1)$ and $m(2)$ are distinct elements of N. Hence we have proved Theorem (A).

3. THE CLASS $\mathscr{B}$

In this section, Theorem (B) will be established.

Let Q be the additive group of all rationals (with the natural linear order). For each $m \in N$ let $G_m = Q$. Consider the lexicographic product.

$$H = \Gamma_{m\in N} G_m$$

(cf., e.g., Fuchs [5]). Then H is a linearly ordered group. The elements of H will be denoted as $h = (h^m)_{m\in N}$.

For $r \in Q$ and $h \in H$ we put $rh = (rh^m)_{m\in N}$. Then H turns out to be a linear space over Q.

For each $n \in N$ let $e_n = (e_n^m)_{m\in N}$ be the element of H such that $e_n^m = 1$ for $m = n$ and $e_n^m = 0$ otherwise.

Let H_1 be a subgroup of H (with the induced linear order). Assume that $e_n \in H_1$ for each $n \in N$. Let $r_n \neq 0$ be a rational number for each $n \in N$. Denote $y_n = r_1 e_1 + r_2 e_2 + \ldots + r_n e_n$. There exists $y \in H$ with $y^m = r_m$ for each $m \in N$. Further, let α_0 be the set of all sequences (x_n) in H such that (x_n) o-converges to 0 in H_1.

From the fact that all elements e_n $(n \in N)$ belong to H_1 we obtain

3.1. Lemma. *Assume that $r_n e_n \in H_1$ for each $n \in N$. If $y \in H_1$, then $y = \bigvee_{n=1}^{\infty} y_n$. If y does not belong to H_1, then $\bigvee_{n=1}^{\infty} y_n$ does not exist in H_1.*

Since $y_1 \leqq y_2 \leqq y_3 \leqq \ldots$, Lemma 3.1 yields

3.2. Lemma. *Assume that $r_n e_n \in G$ for each $n \in N$. If $y \in H_1$, then $y_n \to_0 y$ in G (hence $\sum_{n=1}^{\infty} r_n e_n$ is summable in H_1 with respect to the o-convergence). If y does not belong to H_1, then (y_n) is not o-convergent in H_1 (hence $\sum_{n=1}^{\infty} r_n e_n$ fails to be summable in H_1 with respect to the o-convergence).*

We define a mapping $m: 2^N \to H$ as follows: for each $\emptyset \neq A \subseteq 2^N$ we put $m(A) = h$, where $h^m = 1$ if $m \in A$, and $h^m = 0$ otherwise; next we set $m(\emptyset) = 0$.

By applying the results established in [10], Part II we obtain the following assertion as a particular case:

3.3. Lemma. *There exists a linear subspace E of the linear space H with the property that for each infinite subset A of N there are elements $u \in E$, $v \in H \setminus E$ with*

$$m^{-1}(u), m^{-1}(v) \subseteq A .$$

3.4. Lemma. *Let E be as in 3.3 and let (e_n) be as above. Let E be viewed as a convergence group with respect to the o-convergence. Then*

(i) *$e_n \in E$ for each $n \in N$;*

(ii) *each subsequence of (e_n) contains a subsequence which is summable in E, and another subsequence which is not summable in E.*

Proof. (i) follows from the proof of Theorem in [10] since, in the notation of [10], $e_n \in E_0^1 \subset E$ for each $n \in N$. The assertion (ii) is a consequence of 3.2 and 3.3; in 3.2 we put $r_n = 1$, $n \in N$, and hence $y_n = \sum_{i=1}^{n} e_i$.

Let $G = \{h \in E: h^m$ is an integer for each $m \in N\}$. Then G is a subgroup of E; it is linearly ordered by the induced linear order. It is obvious that the assertion of 3.4 remains valid if E is replaced by G. Moreover, G is a subgroup of $\Gamma_{m \in M} G'_m$, where $G'_m = Z$ for each $m \in N$. Thus Theorem (B) is proved.

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Súhrn

O SUMOVATEĽNOSTI V KONVERGENČNÝCH *l*-GRUPÁCH

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V súvislosti s dvoma otázkami o konvergenčných grupách položenými J. Novákom zostrojujú sa v tomto článku konvegenčné zväzovo usporiadené grupy s určitými „patologickými" vlastnosťami týkajúcimi sa sumovateľnosti radov.

Резюме

О СУММИРУЕМОСТИ В РЕШЕТОЧНО УПОРЯДОЧЕННЫХ ГРУППАХ СХОДИМОСТИ

JÁN JAKUBÍK

В связи с двумя вопросами о сходимости, поставленными Й. Новаком, конструируются решеточно упорядоченные группы сходимости с „патологическими" свойствами, касающимися суммируемости рядов.

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