Časopis pro pěstování matematiky

Miroslav Bartušek

On L^p-solutions of the differential equation $y'' = q(t)y$

Časopis pro pěstování matematiky, Vol. 100 (1975), No. 2, 109--115

Persistent URL: http://dml.cz/dmlcz/108768

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 100 * PRAHA 16. 5. 1975 * ČÍSLO 2

ON L^p-SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$

MIROSLAV BARTUŠEK, Brno

(Received September 3, 1973)

1.1. Consider a differential equation

$$\text{(q)} \qquad y'' = q(t)\,y\,, \quad q \in C^0[a, b)\,, \quad b \leqq \infty\,, \quad q(t) < 0\,, \quad t \in [a, b)$$

where $C^n[a, b)$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$.

Let y_1 be a non-trivial solution of (q) vanishing at $t \in [a, b)$ and y_2 a non-trivial one the derivative of which vanishes at t. If $\varphi(t)$, $\psi(t)$, $\chi(t)$, $\omega(t)$ is the first zero respectively of y_1, y_2', y_1', y_2 lying to the right from t, then φ, ψ, χ, ω is called the basic central dispersion of the 1-st, 2-nd, 3-rd, 4-th kind, respectively (briefly, dispersion of the 1-st, 2-nd, 3-rd, 4-th kind).

Throughout the paper we shall deal with oscillatory ($t \to b_-$) differential equations (i.e., every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$).

Let δ be the dispersion of the k-th kind, $k = 1, 2, 3, 4$. Then δ has the following properties (see [4] § 13)

(1)
1) $\delta \in C^3[a, b)$ if $k = 1$
   $\delta \in C^1[a, b)$ if $k = 2, 3$ or 4
2) $\delta(t) > t$ on $[a, b)$
3) $\delta'(t) > 0$ on $[a, b)$
4) $\lim_{t \to b_-} \delta(t) = b$.

Let n be a positive integer. If δ_n is the n-th iteration of the dispersion δ of the k-th kind, then δ_n has the same properties (1), see [4] § 13.

We shall need also some other properties of dispersions. Let y be a non-trivial solution of (q) and let φ_n, ψ_n be the n-th iteration of the dispersion φ, ψ of the 1-st or 2-nd kind, respectively. Then we have (see [4] § 13):

$$\begin{aligned}(2)\qquad \varphi_n'(t) &= y^2(\varphi_n(t))/y^2(t) &&\text{for}\quad y(t) \neq 0\\ &= y'^2(t)/y'^2(\varphi_n(t)) &&\text{for}\quad y(t) = 0\\ \psi_n'(t) &= \frac{q(t)}{q(\psi_n(t))}\cdot\frac{y'^2(\psi_n(t))}{y'^2(t)} &&\text{for}\quad y'(t) \neq 0\\ &= \frac{q(t)}{q(\psi_n(t))}\cdot\frac{y^2(t)}{y^2(\psi_n(t))} &&\text{for}\quad y'(t) = 0\,.\end{aligned}$$

1.2. First we summarize the results that we shall need in the sequel. See [1], [6], [2] (Theorems 5, 9, 10).

Theorem 1. *Let* (q), $q \in C^0[a, b)$, $q(t) < 0$, $t \in [a, b)$ *be an oscillatory* $(t \to b_-)$ *differential equation and* $\varphi_n(\psi_n)$ *the n-th iteration of its dispersion* φ (ψ) *of the first (second) kind. Let* $t_0 \in [a, b)$.

a) *Every solution of* (q) *is bounded on* $[t_0, b)$ *if and only if a constant N exists such that*

$$\varphi_n'(x) \leqq N\,, \quad x \in [t_0, \varphi(t_0))\,, \quad n = 1, 2, 3, \ldots$$

b) *Every solution of* (q) *belongs to* $L^p[t_0, b)$, $p > 0$ *if and only if*

$$\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+p/2}\, dt < \infty$$

holds.

c) *If q is non-increasing (non-decreasing), then*

$$\frac{q(t)}{q(\delta(t))} \leqq \delta'(t) \leqq 1 \quad (\delta'(t) \geqq 1)\,, \quad t \in [a, b)$$

holds where δ *is the dispersion of the k-th kind of* (q), $k = 1, 2$.

d) *Let* $0 > q(t) \geqq \text{const} > -\infty$. *If there exists a solution y of* (q) *tending to zero for* $t \to b_-$, *then every solution of* (q) *linearly independent of y is unbounded on* $[a, b)$.

e) *Let* $0 > \text{const.} \geqq q(t) > -\infty$. *If there exists a solution y of* (q) *the derivative of which tends to zero* $t \to b_-$, *then the derivative of every solution of* (q) *linearly independent of y is unbounded on* $[a, b)$.

f) *Consider the following assertions on* $[a, b)$:

A) *The sequence of absolute values of local extremes of (the derivative of) an arbitrary solution of* (q) *is non-increasing.*

B) *The sequence of absolute values of local extremes of the derivative of an arbitrary solution (of an arbitrary solution) is non-decreasing.*

C) $\frac{q(\psi(t))}{q(t)}\,\psi'(t) \geqq 1$ *($\varphi(t) - t$ is non-decreasing).*

D) *$\varphi(t) - t$ is non-increasing* $\left(\frac{q(\psi(t))}{q(t)}\,\psi'(t) \leqq 1\right)$.

Then $A \Leftrightarrow C \Rightarrow D \Leftrightarrow B$ *holds.*

2. This paragraph deals with the relation of the dispersions of the 1-st and 2-nd kind of (q) and the property of every solution (of the derivative of every solution) of (q) to belong to $L^p[a, b)$, $p > 0$. Theorem 1b) gives the necessary and sufficient condition for every solution to belong to $L^p[a, b)$, $p > 0$. The situation for the derivative of an arbitrary solution is described by the following

Theorem 2. *Let* (q), $q \in C^0[a, b)$, $q(t) < 0$, $t \in [a, b)$ *be oscillatory on* $[a, b)$ *and let* ψ_n *be the n-th iteration of the dispersion* ψ *of the 2-nd kind. Then the derivative of every solution of* (q) *belongs to* $L^p[a, b)$, $p > 0$ *if and only if*

$$\sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n(t))|^{p/2}\, \psi_n'(t)^{1+p/2}\, dt < \infty \tag{4}$$

holds.

Proof. Let the condition (4) be satisfied. According to (3) we have for an arbitrary solution y

$$\int_a^b |y'(t)|^p\, dt = \sum_{n=0}^{\infty} \int_{\psi_n(a)}^{\psi_{n+1}(a)} |y'(t)|^p\, dt = \sum_{n=0}^{\infty} \int_a^{\psi(a)} |y'(\psi_n)|^p\, \psi_n'\, dt =$$

$$= \sum_{n=0}^{\infty} \int_a^{\psi(a)} \psi_n'^{1+p/2} \left(|y'(t)|^2 \frac{|q(\psi_n)|}{|q(t)|}\right)^{p/2} dt \leqq M \sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n(t))|^{p/2}\, \psi_n'^{1+p/2}\, dt < \infty$$

where

$$M = \max_{t \in [a, \psi(a)]} \left|\frac{y'^2(t)}{q(t)}\right|^{p/2}.$$

We can see that y' belongs to $L^p[a, b)$, $p > 0$. Let y' belong to $L^p[a, (b)$, $p > 0$ for an arbitrary solution y. Let y_1, y_2 be two linearly independent solutions of (q) such that $y_1' \neq 0$ on $[a, t_1]$, $y_2' \neq 0$ on $[t_1, \psi(a)]$, $t_1 = (a + \psi(a))/2$. Then

$$\sum_{n=0}^{\infty} \int_a^{\psi(a)} |q(\psi_n)|^{p/2}\, \psi_n'^{1+p/2}\, dt = \sum_{n=0}^{\infty} \left[\int_a^{t_1} \left|\frac{y_1'(\psi_n)}{y_1'(t)}\right|^p |q(t)|^{p/2}\, \psi_n'\, dt + \right.$$

$$\left. + \int_{t_1}^{\psi(a)} \left|\frac{y_2'(\psi_n)}{y_2'(t)}\right|^p \cdot |q(t)|^{p/2}\, \psi_n'\, dt\right] \leqq M_1 \cdot \left(\int_a^b |y_1'|^p\, dt + \int_a^b |y_2'|^p\, dt\right) < \infty$$

where

$$M_1 = \max\left(\max_{t\in[a,t_1]}\left|\frac{q(t)}{y_1'^2(t)}\right|^{p/2}, \quad \max_{t\in[t_1,\psi(a)]}\left|\frac{q(t)}{y_2'^2(t)}\right|^{p/2}\right)$$

and we can see that the condition (4) is satisfied.

Lemma 1. *Let* (q), $q \in C^0[a, b)$, $q(t) < 0$, $t \in [a, b)$ *be oscillatory on* $[a, b)$ *and let* y *be an arbitrary solution. Let* $\varphi_n, \psi_n, \chi_n, \omega_n$ *be the n-th iteration of the dispersion* $\varphi, \psi, \chi, \omega$ *of the 1-st, 2-nd, 3-rd, 4-th kind, respectively. Let* $t_0, t_1 \in [a, b)$, $y(t_0) = 0$, $y'(t_1) = 0$.

a) *The solution* y *belongs to* $L^p[a, b)$, $p > 0$ *ij and only if*

$$\sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) < \infty \tag{5}$$

holds.

b) *The derivative of* y *belongs to* $L^p[a, b)$, $p > 0$ *if and only if*

$$\sum_{n=0}^{\infty} |y'(\omega_{n+1}(t_1))|^p (\psi_{n+1}(t_1) - \psi_n(t_1)) < \infty \tag{6}$$

holds.

Proof. a) Let y belong to $L^p[a, b)$, $p > 0$. Then

$$\tfrac{1}{2}\sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) =$$

$$= \sum_{n=0}^{\infty}\left[\int_{\varphi_n(t_0)}^{\chi_{n+1}(t_0)} (t - \varphi_n(t_0)) \frac{|y(\chi_{n+1}(t_0))|^p}{\chi_{n+1}(t_0) - \varphi_n(t_0)}\,dt + \right.$$

$$\left. + \int_{\chi_{n+1}(t_0)}^{\varphi_{n+1}(t_0)} (t - \varphi_{n+1}(t_0)) \frac{|y(\chi_{n+1}(t_0))|^p}{\chi_{n+1}(t_0) - \varphi_{n+1}(t_0)}\,dt\right] \leqq \sum_{n=0}^{\infty}\int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(t)|^p\,dt < \infty$$

(because $|y|^p$ has not smaller values on the interval $[\varphi_n(t_0), \chi_{n+1}(t_0)]$ or on $[\chi_{n+1}(t_0), \varphi_{n+1}(t_0)]$ than the function the graph of which is the line segment connecting the points $(\varphi_n(t_0), |y(\varphi_n(t_0))|^p)$ and $(\chi_{n+1}(t_0), |y(\chi_{n+1}(t_0))|^p)$ or $(\chi_{n+1}(t_0), |y(\chi_{n+1}(t_0))|^p)$ and $(\varphi_{n+1}(t_0), |y(\varphi_{n+1}(t_0))|^p)$, respectively. Thus we can see that (5) is valid.

Let (5) be valid. Then

$$\int_a^b |y(t)|^p\,dt = M + \sum_{n=0}^{\infty}\int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(t)|^p\,dt \leqq M + \sum_{n=0}^{\infty}\int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} |y(\chi_{n+1}(t_0))|^p\,dt =$$

$$= M + \sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) < \infty\,, \quad M = \int_a^{t_0} |y(t)|^p\,dt$$

and the theorem is poved in this case.

b) The statement for y' can be proved in the same way. We only use ψ, ω instead of φ, χ.

Theorem 3. *Let* (q), $q \in C^0[a, \infty)$, $q(t) < 0$, $t \in [a, \infty)$ *be oscillatory on* $[a, \infty)$, *q monotone.*

a) *If there exists a solution y belonging to* $L^p[a, \infty)$, $p > 0$, *then* $\lim_{t \to \infty} y(t) = 0$.

b) *If every solution belongs to* $L^p[a, \infty)$, $p > 0$, *then every solution converges to zero for* $t \to \infty$ *and* $\lim_{t \to \infty} q(t) = -\infty$.

c) *If there exists a solution y such that y' belongs to* $L^p[a, \infty)$, $p > 0$, *then y' converges to zero for* $t \to \infty$.

d) *If the derivative of every solution belongs to* $L^p[a, \infty)$, $p > 0$, *then* $\lim_{t \to \infty} q(t) = 0$ *and the derivative of every solution tends to zero for* $t \to \infty$.

Proof. a) Let y be a non-trivial solution of (q) such that $y \in L^p[a, \infty)$, $p > 0$. Let $t_0 \in [a, \infty)$, $y(t_0) = 0$. According to Theorem 1c) f) the sequence of absolute values of local extremes of y is monotone. Hence $\lim_{n \to \infty} |y(\chi_n(t_0))| = M \geqq 0$ where χ_n is the n-th iteration of the dispersion χ of the 3-rd kind of (q). If $M = 0$, then $\lim_{t \to \infty} y(t) = 0$. If $M \neq 0$, then there exists a constant M_1 such that $|y(\chi_n(t_0))| \geqq M_1 > 0$, $n = 1, 2, \ldots$ and according to Lemma 1 we have

$$\sum_{n=0}^{\infty} |y(\chi_{n+1}(t_0))|^p (\varphi_{n+1}(t_0) - \varphi_n(t_0)) \geqq M_1 \sum_{n=0}^{\infty} (\varphi_{n+1}(t_0) - \varphi_n(t_0)) = \infty .$$

However, this contradicts our assumption.

c) This case can be proved in the same way as a).

b) d) The statement follows from a) c) and Theorem 1d) e).

Remark 1. A result of Bellman [3] § 6.8 concerns problems of this paragraph.

Let $a \in C^0[t_0, \infty)$, $b \in C^0[t_0, \infty)$, $|b(t)| \leqq$ const. $< \infty$ for $t \in [t_0, \infty)$. Let $p > 1$ be a number and $p' = p/(p - 1)$. If every solution of $y'' = a(t)\, y$ belongs to $L^p[t_0, \infty)$ and $L^{p'}[t_0, \infty)$, then every solution of $y'' = (a(t) + b(t))\, y$ has the same property.

For $p = 2$ the statement $\lim_{t \to \infty} q(t) = -\infty$ from Theorem 3b) follows from this result by indirect proof: Let $\lim_{t \to \infty} q(t) = -C > -\infty$. Put $a(t) = q(t)$, $b(t) = -1 - q(t)$. Then every solution of $y'' = -y$ belongs to $L^2[t_0, \infty)$ but this is not true.

3. In the last paragraph we shall prove some new results concerning the existence of integral $\int_a^b y(t)\, dt$ where y is a non-trivial solution of (q).

Lemma 2. *Let* (q), $q \in C^0[a, b)$ *be oscillatory on* $[a, b)$ *and let* y *be its solution. Let* φ_n *be the n-th iteration of its dispersion* φ *of the* 1-*st kind.*

Let $t_0 \in [a, b)$. Then

$$\int_{t_0}^{b} y(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}\, y(t)\,dt\,. \tag{7}$$

Proof. According to (2) we have

$$\int_{t_0}^{b} y(t)\,dt = \sum_{n=0}^{\infty} \int_{\varphi_n(t_0)}^{\varphi_{n+1}(t_0)} y(t)\,dt = \sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} y(\varphi_n(t))\,\varphi_n'(t)\,dt =$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}(t)\, y(t)\,dt$$

and thus the stament is valid.

Theorem 4. *Let* (q), $q \in C^0[a, b)$, $b < \infty$ *be a differential equation, q non-increasing,* $\lim_{t \to b-} q(t) = -\infty$. *Let* y *be an arbitrary solution of* (q). *Then*

$$\left| \int_{a}^{b} y(t)\,dt \right| = M_y = \text{const.} < \infty \tag{8}$$

holds.

Proof. As $\lim_{t \to b-} q(t) = -\infty$, the equation (q) is oscillatory on $[a, b)$. Let y be a non trivial solution of (q). Let $t_0 \in [a, b)$, $y(t_0) = 0$, $q(t) < 0$, $t \in [t_0, b)$. As $\varphi' \leqq 1$ on $[t_0, b)$ (see Theorem 1c)) we have

$$\left| \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}\, y(t)\,dt \right| \leqq \left| \int_{t_0}^{\varphi(t_0)} \varphi_{n-1}'(t)\, y(t)\,dt \right|, \quad n = 2, 3, \ldots$$

and according to the alternating series test the infinite series in (7) converges if and only if

$$\lim_{n \to \infty} \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}(t)\, y(t)\,dt = 0 \tag{9}$$

holds. Hence $\int_a^b y(t)\,dt$ converges iff the condition (9) is valid.

Let $c < 0$ be a number. As $\lim_{t \to b-} q(t) = -\infty$, there exists a number t_1, $t_1 \in [a, b)$ such that $q(t) < c$, $t \in [t_1, b)$. Then the Sturm Comparison Theorem for the equations (q) and $y'' = c\,.\,y$ implies $0 < \varphi(t) - t \leqq \pi/\sqrt{-c}$. Thus $\lim_{t \to b-} (\varphi(t) - t) = 0$. According to Theorem 1a) c) an arbitrary solution of (q) is bounded on $[a, b)$ and

$\varphi_n'(x) \leqq 1$, $x \in [a, b)$. There exists a constant $M > 0$ such that $|y(t)| \leqq M$, $t \in [a, b)$ holds and we have

$$0 \leqq \lim_{n\to\infty} \left| \int_{t_0}^{\varphi(t_0)} \varphi_n'^{3/2}(t)\, y(t)\, dt \right| \leqq \lim_{n\to\infty} M \int_{t_0}^{\varphi(t_0)} \varphi_n'(t)\, dt =$$

$$= \lim_{n\to\infty} M(\varphi_{n+1}(t_0) - \varphi_n(t_0)) = 0$$

(because $\lim_{t\to b_-} (\varphi(t) - t) = 0$, $\lim_{n\to\infty} \varphi_n = b$).

Thus (9) is valid and the theorem is proved.

Remark 2. Theorem 4 is a generalization of a result in [5] XIV, § 3, where the assumptions are: $q \in C^0[a, b)$, q non-increasing, (q) oscillatory on $[a, b)$. However, (8) was proved only for solutions tending to zero for $t \to b_-$. Theorem 4 is a generalization of this result because if $\lim_{t\to b} q(t) = c$, $0 > c > -\infty$, then no non-trivial solution of (q) tends to zero for $t \to b_-$. This follows from the following argument: Suppose that $\lim_{t\to b} y_1(t) = 0$. According to Theorem 1d) we have that the function y_2 is unbounded on $[a, b)$ where y_1, y_2 are linearly independent solutions of (q). Theorem 1c) gives $q(\psi(t))\, \psi'(t)/q(t) \geqq 1$, $t \in [a, b)$ where ψ is the dispersion of the 2-nd kind of (q). On the other hand it follows from Theorem 1f) that the sequence of absolute values of local extremes of y_2 is non-increasing and thus y_2 is bounded on $[a, b)$ which is a contradiction.

References

[1] *Bartušek M.:* Connection between Asymptotic Properties and Zeros of Solutions of $y'' = q(t)\,y$. Arch. Math. *3*, VIII, 1972, 113—124.

[2] *Bartušek M.:* On Asymptotic Properties and Distribution of Zeros of Solutions of $y'' = q(t)\,y$. Acta F.R.N. Univ. Comenian. To appear.

[3] *Belman R.:* Теория устойчивости решений дифференциальных уравнений, Москва 1954.

[4] *Borůvka O.:* Lineare Differentialtransformationen 2. Ordnung. VEB Berlin 1967.

[5] *Hartman Ph.:* Ordinary Differential Equations. New York—London—Sydney, 1964.

[6] *Neuman F.:* Distribution of Zeros of Solutions of $y'' = q(t)\,y$ in Relation to Their Behaviour in Large. Acta Math. Acad. Scien. Hungaricae. *8* (1973) 177—185.

Author's address: 662 95 Brno, Janáčkovo nám. 2a (Přírodovědecká fakulta UJEP).