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# ON HADWIGER NUMBER OF A GRAPH

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In this paper, by the word graph we always mean a finite undirected graph without loops and multiple edges.

We say that a graph $G$ can be contracted onto a graph $G'$ if and only if $G'$ can be obtained from $G$ by a finite number of the following operations:

(1) deleting an edge;

(2) deleting an isolated vertex;

(3) identifying two adjacent vertices, i.e., substituting two adjacent vertices $x$ and $y$ by a new vertex adjacent exactly with those vertices which were adjacent at least with one of the vertices $x$ and $y$.

If we speak about contracting an edge, we mean the operation (3), where $x$ and $y$ are the end vertices of the edge in question.

The Hadwiger number $\eta(G)$ of a graph $G$ is the maximal number of vertices of a complete graph onto which the graph $G$ can be contracted.

V. G. VIZING in [2] suggests the following problem:

*Let the degree of each vertex of a graph $G$ be at least $k$. Is then true that $\eta(G) \to \infty$ when $k \to \infty$?*

We shall prove a theorem whose corollary will solve this problem affirmatively. For this purpose we define the relative edge number of a graph $G$ to be $m/n$, where $m$ is the number of edges and $n$ is the number of vertices of $G$. The symbol $\Gamma x$ will denote the set of vertices of the graph $G$ which are adjacent with a vertex $x$ of this graph (following [1]).

First we prove

**Lemma.** *Let $G$ be a graph, $r$ its relative edge number, $k$ the minimal degree of a vertex of $G$. Then*

$$2r \geqq k .$$

Proof. Let $n$ be the number of vertices of $G$ and $m$ the number of its edges. If $d_1, \ldots, d_n$ are degrees of the vertices of $G$, then

$$m = \tfrac{1}{2}\sum_{i=1}^{n} d_i .$$

For each $i = 1, \ldots, n$ we have $d_i \geqq k$, therefore

$$m \geqq \tfrac{1}{2}nk .$$

However, on the other hand,

$$m = nr,$$

which implies

$$2r \geqq k .$$

Now we prove

**Theorem 1.** *Let $r$ be a non-negative rational number, let $\eta^*(r)$ be the least possible Hadwiger number of a graph whose relative edge number is greater than or equal to $r$. Then*

$$\lim_{r \to \infty} \eta^*(r) = \infty .$$

Proof. We shall prove the theorem by contradiction. Suppose that the assertion is not true. The function $\eta^*(r)$ is evidently a non-decreasing function defined on the set of all non-negative rational numbers and its values are positive integers. As it does not tend to infinity, there must exist a non-negative rational number $r_0$ such that

$$\eta^*(r) = \eta^*(r_0)$$

for each rational $r \geqq r_0$. Take this number $r_0$ and put $r_1 = 3r_0$. We have obviously $\eta^*(r_1) = \eta^*(r_0)$; denote this number by $N_0$. There exist graphs whose relative edge number is greater than or equal to $r_1$ and whose Hadwiger number is equal to $N_0$. Let $G$ be a graph with these properties and with the minimal possible number of vertices; let this number be $n$. Let $e$ be an edge of $G$, let its end vertices be $v_1$ and $v_2$. Let $G'$ be the graph obtained from $G$ by contracting the edge $e$, let $v$ be the vertex of $G'$ obtained from $v_1$ and $v_2$ by this contraction. The graph $G'$ has $n - 1$ vertices. The minimality of $n$ implies that either $G'$ has the relative edge number less than $r_1$, or the Hadwiger number of $G'$ is different from $N_0$. However, if $G'$ had the relative edge number greater than or equal to $r_1$, its Hadwiger number would be at least $\eta^*(r_1) = N_0$. But the Hadwiger number of $G'$ cannot exceed the Hadwiger number of $G$ (if $G'$ can be contracted onto a complete graph, then evidently $G$ can be contracted onto the same graph, starting by contracting $G$ onto $G'$), therefore it would be $N_0$, which would be a contradiction with the minimality of $n$. Thus the relative edge number of $G'$ must be less than $r_1$. Let $e_0$ be the number of edges of the induced subgraph of $G$ obtained by deleting $v_1$ and $v_2$ and all edges incident with these vertices. Let $e_1 = |\Gamma v_1 \dot{-} (\Gamma v_2 \cup \{v_2\})|$, $e_2 = |\Gamma v_2 \dot{-} (\Gamma v_1 \cup \{v_1\})|$, $e_3 = |\Gamma v_1 \cap \Gamma v_2|$. The degree of $v_1$ is $e_1 + e_3 + 1$, the degree of $v_2$ is $e_2 + e_3 + 1$ in $G$. The degree of $v$ in $G'$ is $e_1 + e_2 + e_3$. The number of edges of $G$ is $e_0 + e_1 + e_2 + 2e_3 + 1$, the number of edges of $G'$ is $e_0 + e_1 + e_2 + e_3$. From our considerations on the relative edge number we have

$$e_0 + e_1 + e_2 + 2e_3 + 1 \geqq nr_1 ,$$

$$e_0 + e_1 + e_2 + e_3 < (n - 1)\, r_1 .$$

This implies

$$e_3 > r_1 - 1 .$$

As $e_3$ is an integer, we may write

$$e_3 \geqq ]r_1[ - 1 ,$$

where the symbol $]a[$ for each real number $a$ denotes the least integer which is greater than or equal to $a$ (the so-called "post-office function"). As $e_3 = |\Gamma v_1 \cap \Gamma v_2|$, this means that the edge $e$ is contained at least in $]r_1[ - 1$ triangles of $G$. As the edge $e$ was chosen arbitrarily, this holds for each edge of $G$. Now let $u$ be a vertex of $G$. If $T$ is a triangle of $G$ containing $u$, then the edge of $T$ opposite to $u$ is an edge of the subgraph $H_0(u)$ of $G$ induced by the set $\Gamma u$. Let $v \in \Gamma u$. The edge $uv$ is contained at least in $]r_1[ - 1$ triangles of $G$; these triangles obviously contain $u$. This means that the degree of $v$ in $H_0(u)$ is at least $]r_1[ - 1$. As $v$ was chosen arbitrarily from $H_0(u)$, this holds for each vertex of $H_0(u)$. It follows from Lemma that the relative edge number of $H_0(u)$ is at least $\frac{1}{2}(]r_1[ - 1) = \frac{1}{2}(]3r_0[ - 1)$. As evidently $r_0 > 2$, this is greater than $r_0$ and therefore the Hadwiger number of $H_0(u)$ is at least $N_0$. This means that $H_0(u)$ can be contracted onto a complete graph with $N_0$ vertices. Now let $H(u)$ be the subgraph of $G$ induced by the set $\{u\} \cup \Gamma u$. The graph $H_0(u)$ is an induced subgraph of $H(u)$ and the vertex $u$ is joined in $H(u)$ with all vertices of $H_0(u)$. Thus if we contract $H_0(u)$ onto a complete graph with $N_0$ vertices, the graph $H(u)$ will be contracted by this contraction onto a complete graph with $N_0 + 1$ vertices. Therefore the Hadwiger number of $H(u)$ is at least $N_0 + 1$. However, as $H(u)$ is a subgraph of $G$, the Hadwiger number of $G$ is also at least $N_0 + 1$, which is a contradiction.

**Corollary.** *Let $\eta^{**}(k)$ be the least possible Hadwiger number of a graph in which the minimal degree of a vertex is $k$. Then*

$$\lim_{k \to \infty} \eta^{**}(k) = \infty .$$

According to Lemma $r \geqq \frac{1}{2}k$, therefore if $k$ tends to infinity, then so does $r$.

Now we shall study the function $\lambda_k(n)$ which denotes the maximal possible number of edges of a graph with $n$ vertices and with the Hadwiger number $k$. This study was also suggested by V. G. Vizing [2]. In [3] it is conjectured that

$$\lambda_k(n) = (k - 1)\, n - \binom{k}{2}$$

for any two positive integers $n$ and $k$, where $n \geqq k$. This conjecture was proved in [3] for $k$ equal to 1, 2 and 3 and for an arbitrary $n \geqq k$. Here we shall not prove this conjecture, but we shall prove that the right-hand side of this equality is a lower estimate for $\lambda_k(n)$.

**Theorem 2.** *Let $\lambda_k(n)$ be the maximal possible number of edges of a graph with $n$ vertices and with the Hadwiger number $k$. Then*

$$\lambda_k(n) \geqq (k - 1)\, n - \binom{k}{2}$$

*for any two positive integers $n$ and $k$ such that $n \geqq k$.*

Remark. For $n < k$, evidently any graph with $n$ vertices has the Hadwiger number less than $k$.

Proof. Let $k$ and $n$ be two positive integers, $n \geqq k$. Let $P_{n-k+1}$ be a simple path of the length $n - k + 1$, let $K_{k-2}$ be a complete graph with $k - 2$ vertices (if $k \geqq 2$; for $k < 2$ the proof is trivial). Let $P_{n-k+1}$ and $K_{k-2}$ be vertex-disjoint. Let $G_k(n)$ be the graph obtained from $P_{n-k+1}$ and $K_{k-2}$ by joining each vertex of $P_{n-k+1}$ with each vertex of $K_{k-2}$ by an edge. We shall prove that $G_k(n)$ has the Hadwiger number $k$. If we contract $P_{n-k+1}$ onto a graph consisting of one edge and its end vertices (this is possible because $P_{n-k+1}$ is a path), the graph $G_k(n)$ will be contracted onto a complete graph with $k$ vertices. Therefore

$$\eta(G_k(n)) \geqq k\,.$$

The Hadwiger number of $P_{n-k+1}$ is evidently 2, the Hadwiger number of $K_{k-2}$ is $k - 2$. As the vertex set of $G_k(n)$ is the union of the vertex sets of $P_{n-k+1}$ and $K_{k-2}$, the Hadwiger number of $G_k(n)$ evidently cannot exceed the sum of Hadwiger numbers of these graphs, which is $k$. Therefore

$$\eta(G_k(n)) = k\,.$$

The graph $G_k(n)$ has $n$ vertices and $(k - 1)\, n - \binom{k}{2}$ edges. We have proved that a graph with $n$ vertices and $(k - 1)\, n - \binom{k}{2}$ edges can have the Hadwiger number equal to $k$, therefore

$$\lambda_k(n) \geqq (k - 1)\, n - \binom{k}{2}.$$

Now we shall return to the function $\eta^*(r)$.

**Theorem 3.** *Let $\eta^*(r)$ be the least possible Hadwiger number of a graph whose relative edge number is greater than or equal to r. Then*

$$\eta^*(r) \leqq [r] + 2$$

*for each non-negative rational number r.*

Proof. Let $r$ be a non-negative rational number. Put $k = [r] + 2$. Now let $n$ be an integer such that

$$n \geqq \frac{\binom{[r + 2]}{2}}{[r] - r + 1}. \tag{4}$$

Construct the graph $G_k(n)$ from the proof of Theorem 2. This graph has $n$ vertices and $([r] + 1)\, n - \binom{[r] + 2}{2}$ edges; its Hadwiger number is $[r] + 2$. The relative edge number of $G_k(n)$ is $[r] + 1 - \binom{[r] + 2}{2}/n$; according to (4) this is greater than or equal to $r$. We have a graph with the relative edge number greater than or equal to $r$ and with the Hadwiger number $[r] + 2$. This means that

$$\eta^*(r) \leqq [r] + 2\,.$$

Thus we have obtained an upper estimate for $\eta^*(r)$. From the proof of Theorem 1 we see that

$$\eta^*(3r) \geqq \eta^*(r) + 1\,,$$

therefore $\eta^*(r)$ has the lower estimate $c \log r$, where $c$ is a positive constant. Thus

$$c \log r \leqq \eta^*(r) \leqq [r] + 2\,.$$

**Conjecture.** $\eta^*(r) = [r] + 2$.

From the results of [3] it follows that this conjecture is true for $0 \leqq r < 3$.

### *References*


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