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EXISTENCE OF n-FACTORS IN POWERS OF CONNECTED GRAPHS

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Summary. In this paper the following theorem is proved: Let G be a connected graph of an order $p \geqq n + 2$, where $n \geqq 1$. Assume that if n is odd, then p is also odd. Then for an arbitrary vertex $v \in V(G)$, the graph $G^{n+1} - v$ has an n-factor.

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By a graph we mean a finite undirected graph with no loops or multiple edges (a graph in the sense of [1] and [3]). If G is a graph, then the vertex set of G and the edge set of G will be denoted by $V(G)$ and $E(G)$, respectively. The number $|V(G)|$ is called the order of G. If $W \subseteq V(G)$, then we denote by $\langle W \rangle_G$ the subgraph of G induced by W. For a finite nonempty set M we denote by $K(M)$ the complete graph whose vertex set is M.

Suppose that T is a tree and $u \in V(T)$. We shall say that $W \subseteq V(T)$ is a u-set in T, if either $W = \{u\}$ or there exist distinct components $T_1, \ldots, T_i$ $(i \geqq 1)$ of $T - u$ such that either $W = V(T_1) \cup \ldots \cup V(T_i)$ or $W = \{u\} \cup V(T_1) \cup \ldots \cup V(T_i)$.

For every integer $n \geqq 1$, by the n-th power G^n of G we mean the graph with $V(G^n) = V(G)$ and $E(G^n) = \{uv;\ u, v \in V(G)$ and $1 \leqq d_G(u, v) \leqq n\}$, where d_G denotes the distance between vertices in G.

If a spanning subgraph F of G is a regular graph of a degree $m \geqq 0$, then we say that F is an m-factor of G. Recall that if $m \geqq 1$ is an odd integer and G has an m-factor, then the order of G is even.

The following theorem was proved in [4]:

Theorem 0. *Let G be a connected graph of an order $p \geqq n + 1$, where $n \geqq 1$. Assume that if n is odd, then p is even. Then G^{n+1} has an n-factor.*

(Moreover, it was shown in [4] that for any integers $n \geqq 1$ and $p > n(n + 1)$, there exists a tree T of order p such that T^n has no n-factor).

The main result of the present paper is the following:

Theorem 1. *Let G be a connected graph of an order $p \geqq n + 2$, where $n \geqq 1$. Assume that if n is odd, then p is also odd. Then for any arbitrary vertex $v \in V(G)$, the graph $G^{n+1} - v$ has an n-factor.*

To prove Theorem 1 we use Theorem 0, four lemmas (one of them was proved in [4]) and four remarks.

Lemma 1. [4] *Let T be a tree of an order $p > n + 1$, where $n \geqq 1$. Then there exist $u \in V(T)$ and disjoint u-sets W' and W'' in T such that*

(1) $W' \cup W'' \neq V(T)$,
(2) $T - (W' \cup W'')$ *is connected,*
(3) $|W'| \leqq |W''| \leqq n < |W' \cup W''|$, *and*
(4) *if* $|W' \cup W''| \neq n + 1$, *then* $|W' \cup W''|$ *is even.*

Remark 1. Let T be a tree, $u \in V(T)$, $n \geqq 1$, and let $W_1, \ldots, W_k$ $(k \geqq 2)$ be disjoint u-sets such that $|W_1| \leqq n, \ldots, |W_k| \leqq n$. Then every set W_h, $1 \leqq h \leqq k$, can be arranged into a sequence $w_{h,1}, w_{h,2}, \ldots, w_{h,|W_h|}$ such that, for every g, $1 \leqq g \leqq |W_h|$, we have

$$d_T(w_{h,g}, u) < g \quad \text{if} \quad u \in W_h\,,$$

$$d_T(w_{h,g}, u) \leqq g \quad \text{if} \quad u \notin W_h\,.$$

This means that if $u \in W_h$, then $w_{h,1} = u$.

Let h' and h'' be arbitrary integers such that $1 \leqq h' < h'' \leqq k$. It follows from Remark 1 in [4] that the set $W_{h'} \cup W_{h''}$ can be arranged into a sequence

$$w_1, w_2, \ldots, w_m\,,$$

where $m = |W_{h'}| + |W_{h''}|$, with the following property: Assume that $1 \leqq i \leqq j \leqq m$. Let $j - i \leqq n$ for $u \notin W_{h'} \cup W_{h''}$, and $j - i \leqq n + 1$ for $u \in W_{h'} \cup W_{h''}$. Then $d_T(w_i, w_j) \leqq n + 1$.

Remark 2. Let T be a tree, $n \geqq 1$, and let $w_1, \ldots, w_m$ be a sequence of distinct vertices in T which has the properties described in Remark 1. Let m be even and $n + 1 \leqq m \leqq 2n$. Denote

$$\begin{aligned} E_0 = \{&w_1 w_{(m/2)+1}, w_1 w_{(m/2)+2}, \ldots, w_1 w_{n+1}\,, \\ &w_2 w_{(m/2)+2}, w_2 w_{(m/2)+3}, \ldots, w_2 w_{n+2}\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &w_{m/2} w_m, w_{m/2} w_{m+1}, \ldots, w_{m/2} w_{n+(m/2)}\}\,, \end{aligned}$$

where every index $i > m$ is to be replaced by the index $i - (m/2)$. We denote by F the graph with $V(F) = \{w_1, \ldots, w_m\}$ and $E(F) = E(K(\{w_1, \ldots, w_{m/2}\})) \cup \cup E(K(\{w_{(m/2)+1}, \ldots, w_m\})) \cup E_0$. Then F is an n-factor of the graph $\langle\{w_1, \ldots, w_m\}\rangle_{T^{n+1}}$.

Remark 3. Let m and n be integers such that $0 < m < n$. It follows from Theorems 9.1 and 9.6 in [3] that K_n has an m-factor if and only if at least one of the integers m and n is even.

Remark 4. Let T be a tree of an order $p > n + 2$, where $n \geqq 1$. Assume that $W_1, \ldots, W_k$ $(k \geqq 2)$ are disjoint u-sets in T such that $|W_1| \leqq n, \ldots, |W_k| \leqq n$ and $W_1 \cup \ldots \cup W_k = V(T) - \{u\}$. In accordance with Remark 1 every set W_h, $1 \leqq h \leqq$ $\leqq k$, can be arranged into a sequence $w_{h,1}, \ldots, w_{h,|W_h|}$ such that $d_T(w_{h,g}, u) \leqq g$ for every g, $1 \leqq g \leqq |W_h|$.

For every vertex $x \in V(T)$ and for every h, $1 \leqq h \leqq k$, we have

$$d_T(x, u) \leqq n\,, \quad d_T(x, w_{h,1}) \leqq n + 1\,.$$

Assume that $y \in \{u, w_{1,1}, w_{2,1}, \ldots, w_{k,1}\}$ and $T^{n+1} - y$ has an n-factor, say F^*. Let $v \in V(T)$ be an arbitrary vertex of T, $v \neq y$, and let $vx_1, vx_2, \ldots, vx_n \in E(F^*)$.

Then the graph

$$F^* - v + y + \{yx_1, yx_2, \ldots, yx_n\}$$

is an n-factor of $T^{n+1} - v$.

Lemma 2. *Let T be a tree of an order $p \geqq n + 2$, where $n \geqq 1$. Assume that*
(1) there exists $u \in V(T)$ and disjoint u-sets W_1 and W_2 in T such that $|W_1| \leqq$ $\leqq |W_2| \leqq n$ and $W_1 \cup W_2 = V(T) - \{u\}$,
(2) if n is odd, then p is also odd.
Then for an arbitrary vertex $v \in V(T)$, the graph $T^{n+1} - v$ has an n-factor.

Proof. If $p = n + 2$, then $T^{n+1} - v = K(V(T) - \{v\})$ and thus $T^{n+1} - v$ is a regular graph of the degree n. Assume that $p > n + 2$. We distinguish the following cases:

1. p is odd. Then according to Remark 1, the set $W_1 \cup W_2$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = p - 1 > n + 1$ and m is even). According to Remark 2, the graph

$$\langle\{w_1, \ldots, w_m\}\rangle_{T^{n+1}} = T^{n+1} - u$$

has an n-factor.

2. p is even. Then n is even, $|W_1| < |W_2|$ and $|W_1 \cup \{u\}| \leqq n$. According to Remark 1, the set $W_1 \cup \{u\} \cup W_2$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = p > n + 2$, $u = w_1$ and $1 < l \leqq m/2$). Denote

$$\begin{aligned} E_1 = \{ & w_1 w_{(m/2)+1},\ w_1 w_{(m/2)+2},\ \ldots,\ w_1 w_{n+2}\,, \\ & w_2 w_{(m/2)+2},\ w_2 w_{(m/2)+3},\ \ldots,\ w_2 w_{n+3}\,, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & w_{l-1} w_{(m/2)+l-1},\ w_{l-1} w_{(m/2)+l},\ \ldots,\ w_{l-1} w_{n+l}\,, \\ & w_{l+1} w_{(m/2)+l},\ w_{l+1} w_{(m/2)+l+1},\ \ldots,\ w_{l+1} w_{n+l+1}\,, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & w_{m/2} w_{m-1},\ w_{m/2} w_m,\ \ldots,\ w_{m/2} w_{n+(m/2)}\}\,, \end{aligned}$$

where every index $i > m$ is to be replaced by the index $i - (m/2)$. Furthermore, we denote by F_1' the graph with $V(F_1') = \{w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_m\}$ and

$$E(F_1') = E(K(\{w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_{m/2}\})) \cup E(K(\{w_{(m/2)+1}, \ldots, w_m\})) \cup E_1 .$$

Then the graph

$$F_1 = F_1' - \{w_{n+2}w_{n+3}, w_{n+4}w_{n+5}, \ldots, w_{m-2}w_{m-1}\}$$

is an n-factor $\langle\{w_1, \ldots, w_m\}\rangle_{T^{n+1}} - w_1 = T^{n+1} - u$. It follows from Remark 4 that if $T^{n+1} - u$ has an n-factor, then for an arbitrary vertex $v \in V(T)$, the graph $T^{n+1} - v$ has an n-factor, too.

Thus the lemma is proved.

Lemma 3. *Let T be a tree of an order $p \geqq n + 2$, where $n \geqq 1$. Assume that*
(1) *there exists $u \in V(T)$ and disjoint u-sets A, B, C in T such that $n \geqq |A| \geqq |B| \geqq |C|$, $|A \cup B| > n$, $|A \cup C| > n$, $|B \cup C| > n$ and $A \cup B \cup C = V(T) - \{u\}$,*
(2) *if n is odd, then p is also odd.*
Then for an arbitrary vertex $v \in V(T)$, the graph $T^{n+1} - v$ has an n-factor.

Proof. If $p = n + 2$, then $T^{n+1} - v = K(V(T) - \{v\})$ and thus $T^{n+1} - v$ is a regular graph of the degree n.

Assume that $p > n + 2$. Let r be a vertex of T such that

$$r \in C \quad \text{and} \quad ru \in E(T) .$$

Denote $a = |A|$, $b = |B|$ and $c = |C|$.

If $a + b$ is even, we put $\bar{A} = A$, $\bar{B} = B$, $\bar{C} = C$ and $y = u$. If $a + b$ is odd, then $n \geqq a > b$, and we put $\bar{A} = A$, $\bar{B} = B \cup \{u\}$, $\bar{C} = C - \{r\}$ and $y = r$. Denote $\bar{a} = |\bar{A}|$, $\bar{b} = |\bar{B}|$ and $\bar{c} = |\bar{C}|$. Thus $n \geqq \bar{a} \geqq \bar{b} \geqq \bar{c}$, $\bar{a} + \bar{b} > n$, $\bar{b} + \bar{c} > n$, $\bar{a} + \bar{c} > n$, $\bar{a} + \bar{b}$ is even and $\bar{A} \cup \bar{B} \cup \bar{C} = V(T) - \{y\}$. In accordance with Remark 1, the set $\bar{C}$ can be arranged into a sequence $z_1, \ldots, z_{\bar{c}}$ such that $d_T(z_g, u) \leqq g + 1$ for every g, $1 \leqq g \leqq \bar{c}$ (hence, if $r \in \bar{C}$, then $z_1 = r$). Analogously, we can arrange the sets $\bar{A}$ and $\bar{B}$. Moreover, in accordance with Remark 1, the set $\bar{A} \cup \bar{B}$ can be arranged into a sequence $w_1, \ldots, w_m$ (where $m = \bar{a} + \bar{b}$) such that $w_1, \ldots, w_{\bar{a}} \in \bar{A}$ and $w_{\bar{a}+1}, \ldots, w_m \in \bar{B}$ (if $u \in \bar{B}$, then $w_{\bar{a}+1} = u$). Let F be the regular graph constructed in Remark 2. Thus, $V(F) = \{w_1, \ldots, w_m\}$.

Let $\bar{c}$ be odd. Since $p = \bar{a} + \bar{b} + \bar{c} + 1$ and $\bar{a} + \bar{b}$ is even, we have that p is even and therefore n is even. This means that at least one of the integers $\bar{c}$ and n is even. Thus at least one of the integers $\bar{c}$ and $n - \bar{c} + 1$ is even.

According to Remark 1, for $1 \leqq i \leqq \bar{c}$ and $1 \leqq j \leqq \bar{b}$, the inequality $i + j \leqq n + 1$ implies $d_T(z_i, w_{\bar{a}+j}) \leqq n + 1$. We distinguish the following cases:

1. $\bar{c} < (n + 1)/2$. Then $\bar{c} < n - \bar{c} + 1$. Since $\bar{b} + \bar{c} \geqq n + 1$, we have $m - \bar{a} = \bar{b} \geqq n - \bar{c} + 1 > \bar{c}$. It follows from Remark 3 that $K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+1+n-\bar{c}}\})$ has a $\bar{c}$-factor, say H_1. The graph obtained from the graphs $F - E(H_1)$ and $K(\bar{C})$ by adding the edges

$$z_{\bar{c}}w_{\bar{a}+1},\ z_{\bar{c}}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}},$$
$$z_{\bar{c}-1}w_{\bar{a}+1},\ z_{\bar{c}-1}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}-1}w_{\bar{a}+1+n-\bar{c}},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$z_1w_{\bar{a}+1},\ z_1w_{\bar{a}+2},\ \ldots,\ z_1w_{\bar{a}+1+n-\bar{c}}$$

is an n-factor of $T^{n+1} - y$.

2. $\bar{c} > (n + 1)/2$. Then $n - \bar{c} + 1 < \bar{c} \leqq \bar{b}$. According to Remark 3, $K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+\bar{c}}\})$ has an $(n - \bar{c} + 1)$-factor, say H_2. The graph obtained from the graphs $F - E(H_2)$ and $K(\bar{C})$ by adding the edges

$$z_{\bar{c}}w_{\bar{a}+1},\ z_{\bar{c}}w_{\bar{a}+2},\ \ldots,\ z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}},$$
$$z_{\bar{c}-1}w_{\bar{a}+2},\ z_{\bar{c}-1}w_{\bar{a}+3},\ \ldots,\ z_{\bar{c}-1}w_{\bar{a}+2+n-\bar{c}},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$z_1w_{\bar{a}+\bar{c}},\ z_1w_{\bar{a}+\bar{c}+1},\ \ldots,\ z_1w_{\bar{a}+n},$$

where every index $k > \bar{a} + \bar{c}$ is to be replaced by the index $k - \bar{c}$, is an n-factor of $T^{n+1} - y$.

3. $\bar{c} = (n + 1)/2$. Then n is odd, and thus $\bar{c}$ is even. Obviously, $\bar{c} = n - \bar{c} + 1$. We denote by d the integer $\bar{a}$ if $u \notin \bar{B}$, or the integer $\bar{a} + 1$ if $u \in \bar{B}$. Obviously, $m - d \geqq \bar{c}$. We denote by d' that of the integers $d - 1$ and d which has the same parity as $m/2$. It is not difficult to see that $d' \geqq \bar{c}$. For $1 \leqq i \leqq \bar{c}$ and $0 \leqq j \leqq d' - 1$, the inequality $i + j \leqq n - 1$ implies $d_T(z_i, w_{d'-j}) \leqq n + 1$. Since n is odd, $\bar{c}$ is even and $\bar{c} \leqq n$, we have that $\bar{c} < n$. The graph obtained from the graphs $K(\bar{C})$ and

$$F - E(K(\{w_{d+1}, \ldots, w_{d+\bar{c}}\})) - \{w_{d'}w_{d'-1}, w_{d'-2}w_{d'-3}, \ldots, w_{d'-\bar{c}+2}w_{d'-\bar{c}+1}\}$$

by adding the edges

$$z_{\bar{c}}w_{d+1},\ \ldots,\ z_{\bar{c}}w_{d+\bar{c}-1},$$
$$\cdots\cdots\cdots\cdots$$
$$z_1w_{d+\bar{c}},\ \ldots,\ z_1w_{d+2\bar{c}-2},$$

where every index $k > d + \bar{c}$ means $k - \bar{c}$, and the edges

$$z_{\bar{c}}w_{d'},\ z_{\bar{c}-1}w_{d'-1},\ \ldots,\ z_1w_{d'-\bar{c}+1},$$

is an n-factor of $T^{n+1} - y$.

It follows from Remark 4 that if $T^{n+1} - y$ has an n-factor, then for an arbitrary vertex $v \in V(T)$, the graph $T^{n+1} - v$ has an n-factor, too.

Thus the lemma is proved.

Lemma 4. *Let T be a tree of an order $p \geqq n + 2$, where $n \geqq 1$. Assume that if n is odd, then p is also odd. Then for an arbitrary vertex $v \in V(T)$, the graph $T^{n+1} - v$ has an n-factor.*

Proof. If $p = n + 2$, then $T^{n+1} - v = K(V(T) - \{v\})$ and thus $T^{n+1} - v$ is a regular graph of the degree n. Assume that $p > n + 2$, and that for every tree T^* of an order p^*, where

(i) $n + 2 \leqq p^* < p$,
(ii) if n is odd, then p^* is also odd,

and for an arbitrary vertex v of T^* it is proved that $(T^*)^{n+1} - v$ has an n-factor. We distinguish the following cases and subcases:

1. There exists a component R of $T - v$ such that

$$|V(R)| \geqq n + 1 \quad \text{and} \quad |V(T) - V(R)| = 1 \quad \text{or} \quad |V(T) - V(R)| \geqq n + 2 .$$

If $T - V(R)$ has exactly one vertex, this must be the vertex v. Then $|V(R)| = p - 1$ and $|V(R)|$ is even if n is odd. It follows from Theorem 0 that $R^{n+1} = T^{n+1} - v$ has an n-factor. Assume that $|V(T) - V(R)| \geqq n + 2$.

1.1. $|V(R)|$ is even or $|V(R)| = n + 1$. It follows from Theorem 0 that R^{n+1} has an n-factor. The induction hypothesis implies that $(T - V(R))^{n+1} - v$ has an n-factor. Hence, the statement of the lemma is correct.

1.2. $|V(R)|$ is odd and $|V(R)| > n + 1$. Let v' be a vertex of T such that

$$(*) \qquad v' \in V(R) \quad \text{and} \quad vv' \in E(T) .$$

It follows from the induction hypothesis that $R^{n+1} - v'$ has an n-factor and $\langle (V(T) - V(R)) \cup \{v'\} \rangle_{T^{n+1}} - v$ has an n-factor. Hence, the statement of the lemma is correct.

2. There exists a component R of $T - v$ such that

$$|V(R)| \geqq n + 1 \quad \text{and} \quad 1 < |V(T) - V(R)| < n + 2 .$$

Let v' be a vertex of T which fulfils $(*)$. We denote by R' the graph with $V(R') = V(R) \cup \{v\}$ and $E(R') = E(R) \cup \{vv'\}$. Since $|V(R')| > n + 1$, it follows from Lemma 1 that there exist $u \in V(R')$ and disjoint u-sets W' and W'' in R' which fulfil (1)–(4) whereby $v \in (V(R') - (W' \cup W''))$.

2.1. $|V(T) - (W' \cup W'')| \geqq n + 2$. In accordance with Remark 1, the set $W' \cup W''$ can be arranged into a sequence $w_1, \ldots, w_m$. Let F be the regular graph of the degree n constructed in Remark 2. Thus, $V(F) = \{w_1, \ldots, w_m\}$. The induction hypothesis implies that $(T - (W' \cup W''))^{n+1} - v$ has an n-factor. Hence, $T^{n+1} - v$ has an n-factor.

2.2. $|V(T) - (W' \cup W'')| = n + 1$. If $|W' \cup W''| = n + 1$, then $p = 2(n + 1)$ is even and so n is even. Let $|W' \cup W''| > n + 1$. It follows from Lemma 1 that $|W' \cup W''|$ is even. Since $p = |W' \cup W''| + n + 1$, we have that n is even.

2.2.1. $u \in (V(T) - (W' \cup W''))$. Denote $A = W'$, $B = W''$ and $C = (V(T) - (W' \cup W'')) - \{u\}$. Then $|A| \leqq n$, $|B| \leqq n$, $|C| = n$, $|A \cup B| > n$, $|A \cup C| > n$, $|B \cup C| > n$ and $A \cup B \cup C = V(T) - \{u\}$. It follows from Lemma 3 that $T^{n+1} - v$ has an n-factor.

2.2.2. $u \in W' \cup W''$ and $|W' \cup W''| > n + 1$. Since $n + 1 < |W' \cup W''| < p$ and n is even, it follows from the induction hypothesis that $\langle W' \cup W'' \rangle_{T^{n+1}} - u$ has an n-factor. Since $|(V(T) - (W' \cup W'')) \cup \{u\}| = n + 2$, we have

$$\langle (V(T) - (W' \cup W'')) \cup \{u\} \rangle_{T^{n+1}} - v = K(((V(T) - (W' \cup W'')) \cup \{u\}) - v)$$

is a regular graph of the degree n. Hence, $T^{n+1} - v$ has an n-factor.

2.2.3. $u \in W' \cup W''$ and $|W' \cup W''| = n + 1$. In accordance with Remark 1, the set $W' \cup W''$ can be arranged into a sequence $w_1, \ldots, w_{n+1}$ such that $w_k = u$ and $1 \leqq k \leqq (n + 2)/2$. For every i, $1 \leqq i \leqq n + 1 - k$, and for every j, $1 \leqq j \leqq$ $\leqq k - 1$, we have $d_T(u, w_{k+i}) \leqq i$ and $d_T(u, w_{k-j}) \leqq j$.

In accordance with Remark 1, the set $((V(T) - (W' \cup W'')) - \{v\})$ can be arranged into a sequence $z_1, z_2, \ldots, z_n$ such that $d_T(z_g, u) \leqq g + 1$ for every g, $1 \leqq$ $\leqq g \leqq n$. The graph obtained from the graphs

$$K(\{w_1, \ldots, w_{n+1}\}) - \{w_1 w_2, w_3 w_4, \ldots, w_{n-1} w_n\} \quad \text{and} \quad K(\{z_1, \ldots, z_n\})$$

by adding the edges

$$z_n w_k, z_{n-1} w_{k-1}, \ldots, z_{n-k+1} w_1, z_{n-k} w_{k+1}, \ldots, z_1 w_n$$

is an n-factor of $T^{n+1} - v$.

2.3. $|V(T) - (W' \cup W'')| \leqq n$.

2.3.1. There exist disjoint u-sets W_1 and W_2 such that $|W_1| \leqq |W_2| \leqq n$ and $W_1 \cup W_2 = V(T) - \{u\}$. It follows from Lemma 2 that $T^{n+1} - v$ has an n-factor.

2.3.2. For arbitrary disjoint u-sets W_1 and W_2 such that $|W_1| \leqq n$ and $|W_2| \leqq n$ we have $W_1 \cup W_2 \neq V(T) - \{u\}$. Since $|W'| \leqq n$, $|W''| \leqq n$ and $|V(T) - (W' \cup$ $\cup W'')| \leqq n$, we conclude that there exist disjoint u-sets A, B and C such that $|A| \leqq n$, $|B| \leqq n$, $|C| \leqq n$, $|A \cup B| > n$, $|A \cup C| > n$, $|B \cup C| > n$ and $A \cup B \cup C =$ $= V(T) - \{u\}$. It follows from Lemma 3 that $T^{n+1} - v$ has an n-factor.

3. For every component R of $T - v$ we have $|V(R)| \leqq n$. Since $p > n + 2$, it follows from Lemma 1 that there exist disjoint v-sets W' and W'' in T which fulfil (1)–(4).

3.1. $|V(T) - (W' \cup W'')| \geqq n + 2$. From the fact that $T - (W' \cup W'')$ is a tree and $|V(R)| \leqq n$ for every component R of $T - v$ it follows that $v \in (V(T) -$ $-(W' \cup W''))$. The induction hypothesis implies that $(T - (W' \cup W''))^{n+1} - v$ has an n-factor. The set $W' \cup W''$ can be arranged into a sequence $w_1, \ldots, w_m$ described in Remark 1. From Remark 2 it follows that there exists an n-factor of the graph $\langle W' \cup W'' \rangle_{T^{n+1}}$. Hence, $T^{n+1} - v$ has an n-factor.

3.2. $|V(T) - (W' \cup W'')| = n + 1$. Then $v \in (V(T) - (W' \cup W''))$. Denote $A =$ $= W'$, $B = W''$, $C = (V(T) - (W' \cup W'')) - \{v\}$. Then $|A| \leqq n$, $|B| \leqq n$, $|C| = n$, $|A \cup B| > n$, $|A \cup C| > n$, $|B \cup C| > n$ and $A \cup B \cup C = V(T) - \{v\}$. It follows from Lemma 3 that $T^{n+1} - v$ has an n-factor.

3.3. $|V(T) - (W' \cup W'')| < n + 1$.

3.3.1. There exist disjoint v-sets W_1 and W_2 such that $|W_1| \leqq |W_2| \leqq n$ and $W_1 \cup W_2 = V(T) - \{v\}$. It follows from Lemma 2 that $T^{n+1} - v$ has an n-factor.

3.3.2. For arbitrary disjoint v-sets W_1 and W_2 such that $|W_1| \leqq n$ and $|W_2| \leqq n$ we have $W_1 \cup W_2 \neq V(T) - \{v\}$. Since $|W'| \leqq n$, $|W''| \leqq n$ and $|V(T) - (W' \cup W'')| \leqq n$, we conclude that there exist disjoint v-sets A, B and C such that $|A| \leqq n$, $|B| \leqq n$, $|C| \leqq n$, $|A \cup B| > n$, $|A \cup C| > n$, $|B \cup C| > n$ and $A \cup B \cup C = V(T) - \{v\}$. It follows from Lemma 3 that $T^{n+1} - v$ has an n-factor.

Thus the lemma is proved.

Proof of Theorem 1. Let G be a graph satisfying the conditions of Theorem 1. Then G is connected, and thus there exists a spanning tree of G, say T. According to Lemma 4, $T^{n+1} - v$ has an n-factor. Thus $G^{n+1} - v$ has an n-factor, which completes the proof.

Let G be a tree of an order $p \geqq n + 3$ which is given in Fig. 1. It is obvious that $G^{n+1} - v_1 - v_2$ has not n-factor. Hence Theorem 1 cannot be improved in the sense of removing more vertices than one.

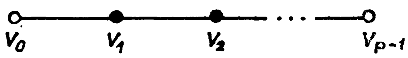

Fig. 1.

Note that for $n = 2$ a stronger result is known. Chartrand and Kapoor [2] proved that if G is a connected graph, then G^3 is 1-hamiltonian.

References

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Súhrn

EXISTENCIA N-FAKTOROV V MOCNINÁCH SÚVISLÝCH GRAFOV

ELENA WISZTOVÁ

V článku je dokázaná nasledovná veta: Nech G je súvislý graf s p vrcholmi, kde $p \geqq n + 2$ a $n \geqq 1$. Predpokladajme, že ak n je nepárne, tak p je tiež nepárne. Potom pre ľubovoľný vrchol $v \in V(G)$, graf $G^{n+1} - v$ má n-faktor.

Резюме

СУЩЕСТВОВАНИЕ N-ФАКТОРОВ В СТЕПЕНЯХ СВЯЗНЫХ ГРАФОВ

ELENA WISZTOVÁ

В статье доказана следующая теорема: Пусть G-связный граф с $p \geqq n + 2$ вершинами, где $n \geqq 1$, и предположим, что p нечетно, если n нечетно. Тогда для произвольной вершины $v \in V(G)$ граф $G^{n+1} - v$ имеет n-фактор.

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