REGULATED FUNCTIONS AND THE PERRON-STIELTJES INTEGRAL

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Summary. Properties of the Perron-Stieltjes integral with respect to regulated functions are investigated. It is shown that linear continuous functionals on the space $G_L(a, b)$ of functions regulated on $[a, b]$ and left-continuous on (a, b) may be represented in the form $F(x) = p\,x(a) + \int_a^b q\,dx$, where $p \in R$ and $q(t)$ is a function of bounded variation on $[a, b]$. Some basic theorems (e.g. integration-by-parts formula, substitution theorem) known for the Perron-Stieltjes integral with respect to functions of bounded variation are established.

Key words: regulated function, function of bounded variation, Perron-Stieltjes integral, left-continuous function, linear continuous functional.

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This paper deals with the space $G(a, b)$ of regulated functions on a compact interval $[a, b]$. It is known that when equipped with the supremal norm $G(a, b)$ becomes a Banach space, and linear bounded functionals on its subspace $G_L(a, b)$ of functions regulated on $[a, b]$ and left-continuous on (a, b) can be represented by means of the Dushnik-Stieltjes (interior) integral. This result is due to H. S. Kaltenborn ([7]), cf. also Ch. S. Hönig ([5]), Theorem 5.1. Together with the known relationship between the Dushnik-Stieltjes integral, the σ-Young-Stieltjes integral and the Perron-Stieltjes integral (cf. Ch. S. Hönig [6] and Š. Schwabik [12], [13]) this enables us to see that F is a linear bounded functional on $G_L(a, b)$ if and only if there exist a real number q and a function p of bounded variation on $[a, b]$ such that

$$F(x) = q\,x(a) + \int_a^b p(t)\,dx(t) \quad \text{for any} \quad x \in G_L(a, b)\,,$$

where the integral is the Perron-Stieltjes integral. We will give here the proof of this fact based only on the properties of the Perron-Stieltjes integral. To this aim, the proof of the existence of the integral

$$\int_a^b f(t)\,dg(t)$$

for any function f of bounded variation on $[a, b]$ and any function g regulated on $[a, b]$ is crucial. Furthermore, we will prove extensions of some theorems (e.g. integration-by-parts and substitution theorems) needed for dealing with generalized differential equations and Volterra-Stieltjes integral equations in the space $G(a, b)$.

1. PRELIMINARIES

Throughout the paper R_n denotes the space of real n-vectors, $R_1 = R$. Given $x \in R_n$, its components are denoted by $x_1, x_2, \ldots, x_n$ $(x = (x_1, x_2, \ldots, x_n))$. N stands for the set of all natural numbers $(N = \{1, 2, \ldots\})$. Given $M \subset R$, χ_M denotes its characteristic function ($\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$).

Let $-\infty < a < b < \infty$. The sets $d = \{t_0, t_1, \ldots, t_m\}$ of points in the closed interval $[a, b]$ such that $a = t_0 < t_1 < \ldots < t_m = b$ are called divisions of $[a, b]$. Given a division d of $[a, b]$, its elements are usually denoted by $t_0, t_1, \ldots, t_m$. The couples $D = (d, \xi)$, where $d = \{t_0, t_1, \ldots, t_m\}$ is a division of $[a, b]$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in R_m$ is such that

$$t_{j-1} \leqq \xi_j \leqq t_j \quad \text{for all} \quad j = 1, 2, \ldots, m$$

are called partitions of $[a, b]$

A function f: $[a, b] \to R$ which possesses finite limits

$$f(t+) = \lim_{\tau \to t+} f(\tau) \quad \text{and} \quad f(s-) = \lim_{\tau \to s-} f(\tau)$$

for all $t \in [a, b)$ and all $s \in (a, b]$ is said to be regulated on $[a, b]$ The set of all regulated functions on $[a, b]$ is denoted by $G(a, b)$. Given $f \in G(a, b)$, we define $f(a-) = f(a)$, $f(b+) = f(b)$,

$$\Delta^+ f(t) = f(t+) - f(t) \quad \text{if} \quad t \in [a, b), \quad \Delta^+ f(b) = 0,$$
$$\Delta^- f(t) = f(t) - f(t-) \quad \text{if} \quad t \in (a, b], \quad \Delta^- f(a) = 0$$

and

$$\Delta f(t) = f(t+) - f(t-) \quad \text{if} \quad t \in (a, b),$$
$$\Delta f(a) = \Delta^+ f(a), \quad \Delta f(b) = \Delta^- f(b).$$

It is known (cf [5], Corollary 3.2a) that if $f \in G(a, b)$, then for any $\varepsilon > 0$ the set of points $t \in [a, b]$ such that $|\Delta^+ f(t)| > \varepsilon$ or $|\Delta^- f(t)| > \varepsilon$ is finite. Consequently, for any $f \in G(a, b)$ the set of its discontinuities in $[a, b]$ is countable. The subset of $G(a, b)$ consisting of all functions regulated on $[a, b]$ and left-continuous on (a, b) will be denoted by $G_L(a, b)$.

A function f: $[a, b] \to R$ is called a finite step function on $[a, b]$ if there exists a division $\{t_0, t_1, \ldots, t_m\}$ of $[a, b]$ such that f is constant on every open interval (t_{j-1}, t_j), $j = 1, 2, \ldots, m$. The set of all finite step functions on $[a, b]$ is denoted by $S(a, b)$. A function f: $[a, b] \to R$ is called a break function on $[a, b]$ if there exist sequences $\{t_k\}_{k \in N} \subset [a, b]$, $\{c_k^-\}_{k \in N}$ and $\{c_k^+\}_{k \in N}$ such that $t_k \neq t_j$ for $k \neq j$, $c_k^- = 0$ if $t_k = a$, $c_k^+ = 0$ if $t_k = b$,

$$\sum_{k=1}^{\infty} (|c_k^-| + |c_k^+|) < \infty$$

and

(1.1) $$f(t) = \sum_{t_k \leqq t} c_k^- + \sum_{t_k < t} c_k^+$$

or equivalently

$$f(t) = \sum_{k=1}^{\infty} c_k^- \chi_{[t_k,b]}(t) + c_k^+ \chi_{(t_k,b]}(t) \quad \text{for} \quad t \in [a, b].$$

Clearly, if f is given by (1.1), then $\Delta^+ f(t_k) = c_k^+$ and $\Delta^- f(t_k) = c_k^-$ for any $k \in N$ and $f(t+) = f(t) = f(t-)$ if $t \in [a, b] \setminus \{t_k\}_{k \in N}$. Furthermore, we have $f(a) = 0$ and

$$\operatorname{var}_a^b f = \sum_{k=1}^{\infty} |c_k^-| + |c_k^+| < \infty$$

for any such function. The set of all break functions on $[a, b]$ is denoted by $B(a, b)$.

$BV(a, b)$ denotes the set of all functions with bounded variation on $[a, b]$, $\|f\|_{BV} = |f(a)| + \operatorname{var}_a^b f$ for $f \in BV(a, b)$. It is well-known that for any $f \in BV(a, b)$ there exist uniquely determined functions $f^C \in BV(a, b)$ and $f^B \in BV(a, b)$ such that f^C is continuous on $[a, b]$, f^B is a break function on $[a, b]$ and $f(t) = f^C(t) + f^B(t)$ on $[a, b]$ (the Jordan decomposition of $f \in BV(a, b)$). In particular, if $W = \{w_k\}_{k \in N}$ is the set of discontinuities of f in $[a, b]$, then

(1.2) $$f^B(t) = \sum_{k=1}^{\infty} \Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t) \quad \text{on} \quad [a, b].$$

Moreover, if we put

(1.3) $$f_n^B(t) = \sum_{k=1}^{n} \Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t) \quad \text{on} \quad [a, b]$$

for $n \in N$, then

(1.4) $$\lim_n \|f_n^B - f^B\|_{BV} = 0$$

(cf. e.g. [14], the proof of Lemma I.4.23). Obviously, $S(a, b) \subset B(a, b) \subset BV(a, b) \subset G(a, b)$.

Given $f \in G(a, b)$, we define

$$\|f\| = \sup_{t \in [a,b]} |f(t)|.$$

Clearly, $\|f\| < \infty$ for any $f \in G(a, b)$ and when endowed with this norm, $G(a, b)$ becomes a Banach space (cf. [5], Theorem 3.6). It is known that $S(a, b)$ is dense in $G(a, b)$ (cf. [5], Theorem 3.1). It means that $f\colon [a, b] \to R$ is regulated on $[a, b]$ if and only if it is a uniform limit on $[a, b]$ of a sequence of finite step functions. Obviously, $G_L(a, b)$ is closed in $G(a, b)$ and hence it is also a Banach space. (Neither $S(a, b)$ nor $BV(a, b)$ are closed in $G(a, b)$, of course.)

For some more details concerning regulated functions see the monographs by

Ch. S. Hönig [5] and by G. Aumann [1] and the papers by D. Fraňková [2] and [3].

The integrals which occur in this paper are the Perron-Stieltjes integrals. We will work with the following definition which is a special case of the definition due to J. Kurzweil [8].

Let $-\infty < a < b < \infty$. An arbitrary positive valued function $\delta: [a, b] \to (0, \infty)$ is called a gauge on $[a, b]$. Given a gauge δ on $[a, b]$, the partition (d, ξ) of $[a, b]$ is said to be δ-fine if

$$[t_{j-1}, t_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for any} \quad j = 1, 2, \ldots, m .$$

Given functions $f, g: [a, b] \to R$ and a partition $D = (d, \xi)$ of $[a, b]$, let us define

$$S_D(f \Delta g) = \sum_{j=1}^{m} f(\xi_j) \left(g(t_j) - g(t_{j-1})\right).$$

We shall say that $I \in R$ is the Kurzweil integral of f with respect to g from a to b and denote

$$I = \int_a^b f(t) \, dg(t) \quad \text{or} \quad I = \int_a^b f dg$$

if for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$|I - S_D(f \Delta g)| < \varepsilon$$

for all δ-fine partitions D of $[a, b]$.

The Perron-Stieltjes integral with respect to a function not necessarily of bounded variation was defined by A. J. Ward [15] (cf. also S. Saks [10], Chapter VI). It can be shown that the Kurzweil integral is equivalent to the Perron-Stieltjes integral (cf. [12], Theorem 2.1, where the assumption $g \in BV(a, b)$ is not used in the proof and may be omitted). Consequently, the Riemann-Stieltjes integral (both of the norm type and of the σ-type, cf. T. H. Hildebrandt [4]) is its special case. The relationship between the Kurzweil integral, the σ-Young-Stieltjes integral and the Perron-Stieltjes integral was described by Š. Schwabik (cf. [12] and [13]).

Since we will make use of some of the properties of the σ-Riemann-Stieltjes integral, let us indicate here the proof that this integral is included in the Kurzweil integral. In fact, let $f, g: [a, b] \to R$ and $I \in R$ be such that the σ-Riemann-Stieltjes integral $\sigma \int_a^b f \, dg$ exists and equals I, i.e. for any $\varepsilon > 0$ there is a division $d_0 = \{s_0, s_1, \ldots, s_{m_0}\}$ of $[a, b]$ such that for any division $d = \{t_0, t_1, \ldots, t_m\}$ which is its refinement ($d_0 \subset d$) and any $\xi \in R_m$ such that $D = (d, \xi)$ is a partition of $[a, b]$ the inequality

$$|S_D(f \Delta g) - I| < \varepsilon$$

is satisfied. Let us define $\delta_\varepsilon(\xi) = \frac{1}{2} \min_{j=0,1,\ldots,m_0} |\xi - s_j|$ for $\xi \notin d_0$ and $\delta_\varepsilon(s_j) = \varepsilon$ for $j = 1, 2, \ldots, m_0$. Then a partition $D = (d, \xi)$ of $[a, b]$ is δ_ε-fine only if for any

$j = 1, 2, \ldots, m_0$ there is an index i_j such that $s_j = \xi_{i_j}$. Furthermore,

$$S_D(f \Delta g) = \sum_{j=1}^{m} [f(\xi_j) [g(t_j) - g(\xi_j)] + f(\xi_j) [g(\xi_j) - g(t_{j-1})]]$$

for any partition $D = (d, \xi)$ of $[a, b]$. Consequently, for any δ_ε-fine partition $D = (d, \xi)$ of $[a, b]$ the corresponding integral sum $S_D(f \Delta g)$ equals the integral sum $S_{D'}(f \Delta g)$ corresponding to a partition $D' = (d', \xi')$, where d' is a division such that $d_0 \subset d'$, and hence

$$|S_{D'}(f \Delta g) - I| < \varepsilon .$$

This means that the Kurzweil integral $\int_a^b f \, dg$ exists and

$$\int_a^b f \, dg = \sigma \int_a^b f \, dg = I$$

holds.

It is well known that if $f \in G(a, b)$ and $g \in BV(a, b)$, then the integral $\int_a^b f \, dg$ exists and the inequality

$$\left| \int_a^b f \, dg \right| \leqq \|f\| \, (\operatorname{var}_a^b g) \tag{1.5}$$

holds. The Kurzweil integral is an additive function of intervals and possesses the usual linearity properties. For the proofs of these assertions and some more details concerning the Kurzweil integral with respect to functions of bounded variation see e.g. [8], [9], [11] and [14].

2. PERRON-STIELTJES INTEGRAL WITH RESPECT TO REGULATED FUNCTIONS

In this section we deal with the integrals

$$\int_a^b f(t) \, dg(t) \quad \text{and} \quad \int_a^b g(t) \, df(t) ,$$

where $f \in BV(a, b)$ and $g \in G(a, b)$. We prove some basic theorems (integration-by-parts theorem, convergence theorems, substitution theorem and unsymmetric Fubini theorem) needed in the theory of Stieltjes integral equations in the space $G(a, b)$. However, our first task is the proof of existence of the integral $\int_a^b f \, dg$ for any $f \in BV(a, b)$ and any $g \in G(a, b)$. We start with some simple special cases.

2.1. Proposition. *Let $g \in G(a, b)$ be arbitrary. Then for any $\tau \in [a, b]$ we have*

$$\int_a^b \chi_{[a,\tau]} \, dg = g(\tau+) - g(a) , \tag{2.1}$$

(2.2) $$\int_a^b \chi_{[a,\tau)}\,\mathrm{d}g = g(\tau-) - g(a)\,,$$

(2.3) $$\int_a^b \chi_{[\tau,b]}\,\mathrm{d}g = g(b) - g(\tau-)\,,$$

(2.4) $$\int_a^b \chi_{(\tau,b]}\,\mathrm{d}g = g(b) - g(\tau+)$$

and

(2.5) $$\int_a^b \chi_{[\tau]}\,\mathrm{d}g = \Delta g(\tau)\,,$$

where $\chi_{[a)}(t) \equiv \chi_{(b]}(t) \equiv 0$ *and the convention* $g(a-) = g(a)$, $g(b+) = g(b)$ *is used.*

Proof. Let $g \in G(a, b)$ and $\tau \in [a, b]$ be given.

a) Let $f = \chi_{[a,\tau]}$. It follows immediately from the definition that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau) - g(a)\,.$$

In particular, (2.1) holds in the case $\tau = b$. Let $\tau \in [a, b)$, let $\varepsilon > 0$ be given and let

$$\delta_\varepsilon(\xi) = \tfrac{1}{2}|\tau - \xi| \quad \text{for} \quad \tau < \xi \leqq b \quad \text{and} \quad \delta_\varepsilon(\tau) = \varepsilon\,.$$

It is easy to see that any δ_ε-fine partition $D = (d, \xi)$ of $[\tau, b]$ must satisfy

$$\xi_1 = t_0 = \tau\,, \quad t_1 < \tau + \varepsilon$$

and

$$S_D(f\,\Delta g) = g(t_1) - g(\tau)\,.$$

Consequently,

$$\int_\tau^b f\,\mathrm{d}g = g(\tau+) - g(\tau)$$

and

$$\int_a^b f\,\mathrm{d}g = \int_a^\tau f\,\mathrm{d}g + \int_\tau^b f\,\mathrm{d}g = g(\tau) - g(a) + g(\tau+) - g(\tau) = g(\tau+) - g(a)\,,$$

i.e., the relation (2.1) is true for every $\tau \in [a, b]$.

b) Let $f = \chi_{[a,\tau)}$. If $\tau = a$, then $f \equiv 0$, $g(\tau-) - g(a) = 0$ and (2.2) is trivial. Let $\tau \in (a, b]$. For a given $\varepsilon > 0$, let us define a gauge δ_ε on $[a, \tau]$ by

$$\delta_\varepsilon(\xi) = \tfrac{1}{2}|\tau - \xi| \quad \text{for} \quad a \leqq \xi < \tau \quad \text{and} \quad \delta_\varepsilon(\tau) = \varepsilon\,.$$

Then for any δ_ε-fine partition $D = (d, \xi)$ of $[a, \tau]$ we have

$$t_m = \xi_m = \tau\,, \quad t_{m-1} > \tau - \varepsilon$$

and hence

$$S_D(f\,\Delta g) = g(t_{m-1}) - g(a)\,.$$

It follows immediately that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau-) - g(a)$$

and in view of the obvious identity

$$\int_\tau^b f\,\mathrm{d}g = 0\,,$$

this implies (2.2).

c) The remaining relations follow from (2.1), (2.2) and the equalities $\chi_{[\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau)}$, $\chi_{(\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau]}$ and $\chi_{[\tau]} = \chi_{[a,\tau]} - \chi_{[a,\tau)}$.

2.2. Remark. Since any finite step function is a linear combination of functions $\chi_{[\tau,b]}(a \leqq \tau \leqq b)$ and $\chi_{(\tau,b]}(a \leqq \tau < b)$, it follows immediately from Proposition 2.1 that the integral $\int_a^b f\,\mathrm{d}g$ exists for any $f \in S(a, b)$ and any $g \in G(a, b)$.

Other simple cases are covered by

2.3. Proposition. *Let* $\tau \in [a, b]$. *Then an arbitrary function* $f\colon [a, b] \to R$ *satisfies*

(2.6) $$\int_a^b f\,\mathrm{d}\chi_{[a,\tau]} = \begin{cases} -f(\tau) & \text{if } \tau < b\,, \\ 0 & \text{if } \tau = b\,, \end{cases}$$

(2.7) $$\int_a^b f\,\mathrm{d}\chi_{[a,\tau)} = \begin{cases} -f(\tau) & \text{if } \tau > a\,, \\ 0 & \text{if } \tau = a\,, \end{cases}$$

(2.8) $$\int_a^b f\,\mathrm{d}\chi_{[\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau > a\,, \\ 0 & \text{if } \tau = a\,, \end{cases}$$

(2.9) $$\int_a^b f\,\mathrm{d}\chi_{(\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau < b\,, \\ 0 & \text{if } \tau = b \end{cases}$$

and

(2.10) $$\int_a^b f\mathrm{d}\chi_{[\tau]} = \begin{cases} -f(a) & \text{if } \tau = a\,, \\ 0 & \text{if } a < \tau < b\,, \\ f(b) & \text{if } \tau = b\,. \end{cases}$$

(For the proof see [14], I.4.21 and I.4.22.)

2.4. Corollary. *Let* $W = \{w_1, w_2, \ldots, w_n\} \subset [a, b]$, $c \in R$ *and* $h\colon [a, b] \to R$ *be such that*

(2.11) $$h(t) = c \quad \text{for all} \quad t \in [a, b] \setminus W.$$

Then

(2.12) $$\int_a^b f \mathrm{d}h = f(b)(h(b) - c) - f(a)(h(a) - c)$$

holds for any function $f\colon [a, b] \to R$.

Proof. A function $h\colon [a, b] \to R$ fulfils (2.11) if and only if

$$h(t) = c + \sum_{j=1}^{n} (h(w_j) - c)\, \chi_{[w_j]}(t) \quad \text{on} \quad [a, b].$$

Thus, the formula (2.12) follows from (2.6) (with $\tau = b$) and from (2.10) in Proposition 2.3.

2.5. Remark. It is well-known (cf. [14], I.4.17 or [11], Theorem 1.22) that if $g \in BV(a, b)$, $h\colon [a, b] \to R$ and $h_n\colon [a, b] \to R$, $n \in N$ are such that $\int_a^b h_n \mathrm{d}g$ exists for any $n \in N$ and $\lim_n \|h_n - h\| = 0$, then $\int_a^b h \,\mathrm{d}g$ exists and

(2.13) $$\lim_n \int_a^b h_n \mathrm{d}g = \int_a^b h \, dg$$

holds. To prove an analogous assertion for the case $g \in G(a, b)$ we need the following auxiliary assertion.

2.6. Lemma. *Let* $f \in BV(a, b)$ *and* $g \in G(a, b)$. *Then the inequality*

(2.14) $$|S_D(f\, \Delta g)| \leqq (|f(a)| + |f(b)| + \operatorname{var}_a^b f)\, \|g\|$$

holds for an arbitrary partition D *of* $[a, b]$.

Proof. For an arbitrary partition $D = (d, \xi)$ of $[a, b]$ we have (putting $\xi_0 = a$ and $\xi_{m+1} = b$)

$$|S_D(f\,\Delta g)| = |f(b)\, g(b) - f(a)\, g(a) - \sum_{j=1}^{m+1} (f(\xi_j) - f(\xi_{j-1}))\, g(t_{j-1})| \leqq$$

$$\leqq (|f(b)| + |f(a)| + \sum_{j=1}^{m+1} |f(\xi_j) - f(\xi_{j-1})|)\, \|g\| \leqq$$

$$\leqq (|f(b)| + |f(a)| + \operatorname{var}_a^b f)\, \|g\| .$$

2.7. Theorem. *Let* $g \in G(a, b)$ *and let* $h_n, h\colon [a, b] \to R$ *be such that* $\int_a^b h_n \mathrm{d}g$ *exists for an* $n \in N$ *and* $\lim_n \|h_n - h\|_{BV} = 0$. *Then* $\int_a^b h \,\mathrm{d}g$ *exists and* (2.13) *holds.*

Proof. Since $|f(b)| \leqq |f(a)| + |f(b) - f(a)| \leqq |f(a)| + \operatorname{var}_a^b f$, we have by (2.14)

$$|S_D(h_m - h_k)\, \Delta g)| \leqq 2\|h_m - h_k\|_{BV}\, \|g\|$$

for all $m, k \in N$ and all partitions D of $[a, b]$. Consequently,

$$\left|\int_a^b (h_m - h_k)\,\mathrm{d}g\right| \leqq 2\|h_m - h_k\|_{BV}\,\|g\|$$

holds for all $m, k \in N$. This immediately implies that there is $q \in R$ such that

$$\lim_n \int_a^b h_n\,\mathrm{d}g = q\,.$$

It remains to show that

$$q = \int_a^b h\,\mathrm{d}g\,.$$

For a given $\varepsilon > 0$, let $n_0 \in N$ be such that

$$\left|\int_a^b h_{n_0}\,\mathrm{d}g - q\right| < \varepsilon \quad \text{and} \quad \|h_{n_0} - h\|_{BV} < \varepsilon\,, \tag{2.16}$$

and let δ_ε be such a gauge on $[a, b]$ that

$$\left|S_D(h_{n_0}\,\Delta g) - \int_a^b h_{n_0}\,\mathrm{d}g\right| < \varepsilon \tag{2.17}$$

for all δ_ε-fine partitions D of $[a, b]$. Given an arbitrary δ_ε-partition D of $[a, b]$ we have by (2.16), (2.17) and Lemma 2.6

$$|q - S_D(h\,\Delta g)| \leqq$$

$$\leqq \left|q - \int_a^b h_{n_0}\,\mathrm{d}g\right| + \left|\int_a^b h_{n_0}\,\mathrm{d}g - S_D(h_{n_0}\,\Delta g)\right| + \left|S_D(h_{n_0}\,\Delta g) - S_D(h\,\Delta g)\right| \leqq$$

$$\leqq 2\varepsilon + |S_D((h_{n_0} - h)\,\Delta g)| \leqq 2\varepsilon + 2\|h_{n_0} - h\|_{BV}\,\|g\| \leqq 2\varepsilon(1 + \|g\|)\,.$$

This completes the proof of (2.16) and as well as of the proposition.

Now we can prove the following

2.8. Theorem. *Let $f \in BV(a, b)$ and $g \in G(a, b)$. Then the integral*

$$\int_a^b f(t)\,\mathrm{d}g(t)$$

exists and the inequality

$$\left|\int_a^b f(t)\,\mathrm{d}g(t)\right| \leqq (|f(a)| + |f(b)| + \mathrm{var}_a^b f)\,\|g\| \tag{2.18}$$

holds.

Proof. Let $f \in BV(a, b)$ and $g \in G(a, b)$ be given. Let $W = \{w_k\}_{k \in N}$ be the set of discontinuities of f in $[a, b]$ and let $f = f^C + f^B$ be the Jordan decomposition of f (f^C is continuous on $[a, b]$ and f^B is given by (1.2)). We have

$$\lim_n \|f_n^B - f^B\|_{BV} = 0$$

for f_n^B, $n \in N$ given by (1.3). By (2.3) and (2.4),

$$(2.19) \quad \int_a^b f_n^B \, dg = \sum_{k=1}^n [\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^+ f(w_k)(g(b) - g(w_k-))]$$

holds for any $n \in N$. Thus according to Theorem 2.7 the integral $\int_a^b f^B \, dg$ exists and

$$(2.20) \quad \int_a^b f^B \, dg = \lim_n \int_a^b f_n^B \, dg .$$

The integral $\int_a^b f^C \, dg$ exists as the σ-Riemann-Stieltjes integral (cf. Theorems II.13.17 and II.11.7 in [4]). This means that $\int_a^b f \, dg$ exists and

$$\int_a^b f \, dg = \int_a^b f^C \, dg + \int_a^b f^B \, dg = \int_a^b f^C \, dg + \lim_n \int_a^b f_n^B \, dg .$$

The inequality (2.18) follows immediately from Lemma 2.6.

2.9. Remark. Since

$$\sum_{k=1}^\infty |\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^- f(w_k)(g(b) - g(w_k-))| \leqq$$
$$\leqq 2\|g\| \left(\sum_{k=1}^\infty |\Delta^+ f(w_k)| + |\Delta^- f(w_k)|\right) \leqq 2\|g\| (\operatorname{var}_a^b f) < \infty ,$$

we have in virtue of (2.19) and (2.20)

$$(2.21) \quad \int_a^b f^B \, dg = \sum_{k=1}^\infty [\Delta^+ f(w_k)(g(b) - g(w_k+)) + \Delta^- f(w_k)(g(b) - g(w_k-))] .$$

As a direct consequence of Theorem 2.8 we obtain

2.10. Corollary. *Let $h_n \in G(a,b)$, $n \in N$ and $h \in G(a,b)$ be such that* $\lim_n \|h_n - h\| = 0$. *Then for any $f \in BV(a, b)$ the integrals*

$$\int_a^b f \, dh \quad \text{and} \quad \int_a^b f \, dh_n , \quad n \in N$$

exist and

$$\lim_n \int_a^b f \, dh_n = \int_a^b f \, dh .$$

2.11. Lemma. *Let* $h\colon [a, b] \to R$, $c \in R$ *and* $W = \{w_k\}_{k \in N} \subset [a, b]$ *be such that* (2.11) *and*

$$\sum_{k=1}^{\infty} |h(w_k) - c| < \infty \tag{2.22}$$

hold. Given $n \in N$, *let us define* $W_n = \{w_1, w_2, \ldots, w_n\}$ *and*

$$h_n(t) = c \quad \text{for} \quad t \in [a, b] \setminus W_n, \tag{2.23}$$
$$h_n(t) = h(t) \quad \text{for} \quad t \in W_n.$$

Then $h_n \in BV(a, b)$ *for any* $n \in N$, $h \in BV(a, b)$ *and*

$$\lim_n \|h_n - h\|_{BV} = 0. \tag{2.24}$$

Proof. The functions h_n, $n \in N$ and h evidently have a bounded variation on $[a, b]$. For a given $n \in N$, we have

$$h_n(t) - h(t) = 0 \quad \text{if} \quad t \in W_n \quad \text{or} \quad t \in [a, b] \setminus W$$

and

$$h_n(t) - h(t) = c - h(w_k) \quad \text{if} \quad t = w_k \quad \text{for some} \quad k > n.$$

Thus,

$$\lim_n h_n(t) = h(t) \quad \text{on} \quad [a, b] \tag{2.25}$$

and, moreover,

$$\sum_{j=1}^{m} |(h_n(t_j) - h(t_j)) - (h_n(t_{j-1}) - h(t_{j-1}))| \leqq 2 \sum_{k=n+1}^{\infty} |h(w_k) - c|$$

holds for any $n \in N$ and any division $\{t_0, t_1, \ldots, t_m\}$ of $[a, b]$. Consequently,

$$\operatorname{var}_a^b (h_n - h) \leqq 2 \sum_{k=n+1}^{\infty} |h(w_k) - c| \tag{2.26}$$

holds for any $n \in N$. In virtue of the assumption (2.22) the right-hand side of (2.26) tends to 0 as $n \to \infty$. Hence (2.24) follows from (2.25) and (2.26).

2.12. Proposition. *Let* $h\colon [a, b] \to R$, $c \in R$ *and* $W = \{w_k\}_{k \in N}$ *be such that* (2.11) *and* (2.22) *hold. Then*

$$\int_a^b h \, dg = \sum_{k=1}^{\infty} (h(w_k) - c) \, \Delta g(w_k) + c(g(b) - g(a))$$

holds for any $g \in G(a, b)$.

Proof. Let $g \in G(a, b)$ be given. Let $W_n = \{w_1, w_2, \ldots, w_n\}$ for $n \in N$ and let the function h_n, $n \in N$ be defined by (2.23). Given an arbitrary $n \in N$, then (2.1) (with

$\tau = b$) and (2.5) from Proposition 2.1 imply

$$\int_a^b h_n \, dg = \sum_{k=1}^{n} (h(w_k) - c) \, \Delta g(w_k) + c(g(b) - g(a)) .$$

Since (2.22) yields

$$\sum_{k=1}^{\infty} |(h(w_k) - c) \, \Delta g(w_k)| \leqq 2\|g\| \, (\sum_{k=1}^{\infty} |h(w_k) - c| < \infty$$

and Lemma 2.11 implies

$$\lim_n \|h_n - h\|_{BV} = 0 ,$$

we can use Theorem 2.7 to prove that

$$\int_a^b h \, dg = \lim_n \int_a^b h_n \, dg = \sum_{k=1}^{\infty} (h(w_k) - c) \, \Delta g(w_k) + c(g(b) - g(a)) .$$

2.13. Proposition. *Let $h \in G(a, b)$, $c \in R$ and $W = \{w_k\}_{k \in N}$ fulfil* (2.11). *Then*

$$\int_a^b f \, dh = f(b) \, (h(b) - c) - f(a) \, (h(a) - c) \tag{2.27}$$

holds for any $f \in BV(a, b)$.

Proof. Let $f \in BV(a, b)$. For a given $n \in N$, let $W_n = \{w_1, w_2, \ldots, w_n\}$ and let h_n be given by (2.23). Then

$$\lim_n \|h_n - h\| = 0 . \tag{2.28}$$

Indeed, let $\varepsilon > 0$ be given and let $n_0 \in N$ be such that $k \geqq n_0$ implies

$$|h(w_k) - c| < \varepsilon . \tag{2.29}$$

(Such an n_0 exists since $|h(w_k) - c| = |\Delta^- h(w_k)| = |\Delta^+ h(w_k)|$ for any $k \in N$ and the set of those $k \in N$ for which the inequality (2.29) does not hold may be only finite.) Now, for any $n \geqq n_0$ and any $t \in [a, b]$ such that $t = w_k$ for some $k > n$ ($t \in W \setminus W_n$) we have

$$|h_n(t) - h(t)| = |h_n(w_k) - h(w_k)| = |c - h(w_k)| < \varepsilon .$$

Since $h_n(t) = h(t)$ for all the other $t \in [a, b]$ ($t \in ([a, b] \setminus W) \cup W_n$), it follows that $|h_n(t) - h(t)| < \varepsilon$ on $[a, b]$, i.e.

$$\|h_n - h\| < \varepsilon .$$

This proves the relation (2.28).

By Corollary 2.4 we have for any $n \in N$

$$\int_a^b f \, dh_n = f(b) \, (h(b) - c) - f(a) \, (h(a) - c) .$$

Making use of (2.28) and Corollary 2.10 we obtain

$$\int_a^b f \, dh = \lim_n \int_a^b f \, dh_n = f(b)\,(h(b) - c) - f(a)\,(h(a) - c)\,.$$

2.14. Corollary. *Let $h \in BV(a, b)$, $c \in R$ and $W = \{w_k\}_{k \in N}$ fulfil* (2.11). *Then* (2.27) *holds for any $f \in G(a, b)$.*

Proof. By Proposition 2.12, (2.27) holds for any $f \in BV(a, b)$. Making use of the density of $S(a, b) \subset BV(a, b)$ in $G(a, b)$ and of the convergence theorem mentioned in Remark 2.5 we complete the proof of our assertion.

2.15. Theorem. (integration-by-parts). If $f \in BV(a, b)$ and $g \in G(a, b)$, then both the integrals $\int_a^b f \, dg$ and $\int_a^b g \, df$ exist and

$$\int_a^b f \, dg + \int_a^b g \, df = f(b)\,g(b) - f(a)\,g(a) + \sum_{a \leq t \leq b} [\Delta^- f(t)\,\Delta^- g(t) - \Delta^+ f(t)\,\Delta^+ g(t)]\,. \tag{2.30}$$

Proof. The existence of the integral $\int_a^b g \, df$ is well-known while the existence of $\int_a^b f \, dg$ is guaranteed by Theorem 2.8. Furthermore,

$$\int_a^b f \, dg + \int_a^b g \, df = \int_a^b f(t)\,[d(g(t) + \Delta^+ g(t))] + \int_a^b g(t)\,[d(f(t) - \Delta^- f(t))] - \int_a^b f(t)\,[d(\Delta^+ g(t))] + \int_b^b g(t)\,[d(\Delta^- f(t))]\,.$$

It is easy to verify that the function $h(t) = \Delta^+ g(t)$ fulfils the relation (2.11) with $c = 0$ and $h(b) = 0$. Consequently, Proposition 2.13 yields

$$\int_a^b f(t)\,[d(\Delta^+ g(t))] = -f(a)\,\Delta^+ g(a)\,.$$

Similarly, by Corollary 2.14 we have

$$\int_a^b g(t)\,[d(\Delta^- f(t))] = \Delta^- f(b)\,g(b)\,.$$

Hence

$$\int_a^b f \, dg + \int_a^b g \, df = \int_a^b f(t)\,dg(t+) + \int_a^b g(t)\,df(t-) + f(a)\,\Delta^+ g(a) + \Delta^- f(b)\,g(b)\,. \tag{2.31}$$

The first integral on the right-hand side may be modified in the following way:

$$\int_a^b f(t)\,dg(t+) = \int_a^b f(t-)\,dg(t+) + \int_a^b \Delta^- f(t)\,dg(t+)\,. \tag{2.32}$$

Making use of Proposition 2.12 and taking into account that $\Delta g_1(t) = \Delta g(t)$ on $[a, b]$ for the function g_1 defined by $g_1(t) = g(t+)$ on $[a, b]$, we further obtain

$$\int_a^b \Delta^- f(t)\, \mathrm{d}g(t+) = \sum_{a \leqq t \leqq b} \Delta^- f(t)\, \Delta g(t). \tag{2.33}$$

Similarly

$$\int_a^b g(t)\, \mathrm{d}f(t-) = \int_a^b g(t+)\, \mathrm{d}f(t-) - \int_a^b \Delta^+ g(t)\, \mathrm{d}f(t-) = \tag{2.34}$$
$$= \int_a^b g(t+)\, \mathrm{d}f(t-) - \sum_{a \leqq t \leqq b} \Delta^+ g(t)\, \Delta f(t).$$

The function $f(t-)$ is left-continuous on $(a, b]$, while $g(t+)$ is right-continuous on $[a, b)$. It means that both the integrals

$$\int_a^b f(t-)\, \mathrm{d}g(t+) \quad \text{and} \quad \int_a^b g(t+)\, \mathrm{d}f(t-)$$

exist as the σ-Riemann-Stieltjes integrals (cf. [4], II.13.17), and by the integration-by-parts theorem for these integrals (cf. [4], II.11.7) we have

$$\int_a^b f(t-)\, \mathrm{d}g(t+) + \int_a^b g(t+)\, \mathrm{d}f(t-) = f(b-)\, g(b) - f(a)\, g(a+). \tag{2.35}$$

Inserting (2.32)–(2.35) into (2.31) we get

$$\int_a^b f\, \mathrm{d}g + \int_a^b g\, \mathrm{d}f = f(b-)\, g(b) - f(a)\, g(a+) +$$
$$+ \sum_{a \leqq t \leqq b} \Delta^- f(t)\, (\Delta^- g(t) + \Delta^+ g(t)) - \sum_{a \leqq t \leqq b} (\Delta^- f(t) + \Delta^+ f(t))\, \Delta^+ g(t) +$$
$$+ f(a)\, \Delta^+ g(a) + \Delta^- f(b)\, g(b) = f(b)\, g(b) - f(a)\, g(a) +$$
$$+ \sum_{a \leqq t \leqq b} [\Delta^- f(t)\, \Delta^- g(t) - \Delta^+ f(t)\, \Delta^+ g(t)],$$

and this completes the proof.

The following proposition describes some properties of indefinite Perron-Stieltjes integrals.

2.16. Proposition. *Let* $f\colon [a, b] \to R$ *and* $g\colon [a, b] \to R$ *be such that* $\int_a^b f\, \mathrm{d}g$ *exists. Then the function*

$$h(t) = \int_a^t f\, \mathrm{d}g$$

is defined on $[a, b]$ *and*

i) *if* $g \in G(a, b)$, *then* $h \in G(a, b)$ *and*

$$\Delta^+ h(t) = f(t)\, \Delta^+ g(t)\,, \quad \Delta^- h(t) = f(t)\, \Delta^- g(t) \quad \text{on} \quad [a, b]\,; \tag{2.36}$$

ii) *if* $g \in BV(a, b)$ *and* f *is bounded on* $[a, b]$, *then* $h \in BV(a, b)$.

Proof. The former assertion follows from Theorem 1.3.5 in [8]. The latter follows immediately from the inequality

$$\sum_{j=1}^{m} \left| \int_{t_{j-1}}^{t_j} f \, dg \right| \leqq \sum_{j=1}^{m} \|f\| \, (\text{var}_{t_{j-1}}^{t_j} g) = \|f\| \, (\text{var}_a^b g)$$

which is valid for any division $\{t_0, t_1, \ldots, t_m\}$ of $[a, b]$.

In the theory of generalized differential equations the substitution formula

$$\int_a^b h(t) \left[d \int_a^t f(s) \, dg(s) \right] = \int_a^b h(t) f(t) \, dg(t) \tag{2.37}$$

is often needed. In [4], II.19.3.7 this formula is proved for the σ-Young-Stieltjes integral under the assumption that $g \in G(a, b)$, h is bounded on $[a, b]$, and the integral $\int_a^b f \, dg$ as well as one of the integrals in (2.37) exists. In [14], Theorem I.4.25 this assertion was proved for the Kurzweil integral. Though it was assumed there that $g \in BV(a, b)$, this assumption was not used in the proof. We will give here a slightly different proof based on the Saks-Henstock lemma (cf. e.g. [11], Lemma 1.11).

2.17. Lemma. (Saks, Henstock). *Let* $f, g\colon [a, b] \to R$ *be such that the integral* $\int_a^b f \, dg$ *exists. Let* $\varepsilon > 0$ *be given and let* δ *be a gauge on* $[a, b]$ *such that*

$$\left| S_D(f \, \Delta g) - \int_a^b f \, dg \right| < \varepsilon$$

holds for any δ*-fine partition* D *of* $[a, b]$. *Then for an arbitrary system* $\{([\beta_i, \gamma_i], \sigma_i),\ i = 1, 2, \ldots, k\}$ *of intervals and points such that*

$$a \leqq \beta_1 \leqq \sigma_1 \leqq \gamma_1 \leqq \beta_2 \leqq \ldots \leqq \beta_k \leqq \sigma_k \leqq \gamma_k \leqq b \tag{2.38}$$

and

$$[\beta_i, \gamma_i] \subset [\sigma_i - \delta(\sigma_i), \sigma_i + \delta(\sigma_i)]\,, \quad i = 1, 2, \ldots, k\,,$$

the inequality

$$\left| \sum_{i=1}^{k} f(\sigma_i) \, [g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f \, dg \right| < \varepsilon \tag{2.39}$$

holds.

Making use of Lemma 2.17 we can prove the following useful assertion

2.18. Lemma. *If $f: [a, b] \to R$ and $g: [a, b] \to R$ are such that $\int_a^b f\, dg$ exists, then for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that*

$$\sum_{j=1}^{m} \left| f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\, dg \right| < \varepsilon \tag{2.40}$$

holds for any δ-fine partition (d, ξ) of $[a, b]$.

Proof. Let $\delta: [a, b] \to (0, \infty)$ be such that

$$\left| S_D(f\, \Delta g) - \int_a^b f\, dg \right| = \left| \sum_{j=1}^{m} f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\, dg \right| < \frac{\varepsilon}{2}$$

for all δ-fine partitions $D = (d, \xi)$ of $[a, b]$. Let us choose an arbitrary δ-fine partition $D = (d, \xi)$ of $[a, b]$. Let $\gamma_i = t_{p_i}$ and $\beta_i = t_{p_i - 1}$, $i = 1, 2, \ldots, k$ be all points of the division d such that

$$f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\, dg \geqq 0 .$$

Then the system $\{([\beta_i, \gamma_i], \sigma_i),\ i = 1, 2, \ldots, k\}$, where $\sigma_i = \xi_{p_i}$, fulfils (2.38) and (2.39) and hence we can use Lemma 2.17 to prove that the inequality

$$\sum_{i=1}^{k} \left| f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\, dg \right| < \frac{\varepsilon}{2}$$

is true. Similarly, if $\omega_i = t_{q_i}$ and $\vartheta_i = t_{q_i - 1}$, $i = 1, 2, \ldots, r$ are all points of the division d such that

$$f(\xi_{q_i})[g(\omega_i) - g(\vartheta_i)] - \int_{\vartheta_i}^{\omega_i} f\, dg \leqq 0 ,$$

then the inequality

$$\sum_{i=1}^{r} \left| f(\xi_{q_i})[g(\omega_i) - g(\vartheta_i)] - \int_{\vartheta_i}^{\omega_i} f\, dg \right| < \frac{\varepsilon}{2}$$

holds. Summarizing, we conclude that

$$\sum_{j=1}^{m} \left| f(\xi_j)[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\, dg \right| =$$

$$= \sum_{i=1}^{k} \left| f(\xi_{p_i})[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\, dg \right| +$$

$$+ \sum_{i=1}^{r} \left| f(\xi_{q_i})[g(\omega_i) - g(\vartheta_i)] - \int_{\vartheta_i}^{\omega_i} f\, dg \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This completes the proof.

2.19. Theorem (Substitution). *Let $f, g, h\colon [a, b] \to R$ be such that h is bounded on $[a, b]$ and the integral*

$$\int_a^b f \, dg$$

exists. Then the integral

$$\int_a^b h(t) f(t) \, dg(t)$$

exists if and only if the integral

$$\int_a^b h(t) \left[d \int_a^t f(s) \, dg(s) \right]$$

exists, and in this case the relation (2.37) *holds.*

Proof. Let $|h(t)| \leqq C < \infty$ on $[a, b]$. Let us assume that the integral $\int_a^b hf \, dg$ exists and let $\varepsilon > 0$ be given. There exists a gauge δ_1 on $[a, b]$ such that

$$\left| \sum_{j=1}^{m} h(\xi_j) f(\xi_j) [g(t_j) - g(t_{j-1})] - \int_a^b hf \, dg \right| < \frac{\varepsilon}{2}$$

is satisfied for any δ_1-fine partition (d, ξ) of $[a, b]$. By Lemma 2.18 there exists a gauge δ on $[a, b]$ such that $\delta(t) \leqq \delta_1(t)$ on $[a, b]$ and

$$\sum_{j=1}^{m} \left| f(\xi_j) [g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f \, dg \right| < \frac{\varepsilon}{2C}$$

holds for any δ-fine partition (d, ξ) of $[a, b]$. Let us denote

$$k(t) = \int_a^t f \, dg \quad \text{for} \quad t \in [a, b] .$$

Then for any δ-fine partition $D = (d, \xi)$ of $[a, b]$ we have

$$\left| S_D(h \, \Delta k) - \int_a^b hf \, dg \right| =$$

$$= \left| \sum_{j=1}^{m} h(\xi_j) \int_{t_{j-1}}^{t_j} f \, dg - \sum_{j=1}^{m} h(\xi_j) f(\xi_j) [g(t_j) - g(t_{j-1})] + \right.$$

$$\left. + \sum_{j=1}^{m} h(\xi_j) f(\xi_j) [g(t_j) - g(t_{j-1})] - \int_D hf \, dg \right| \leqq$$

$$\leqq \left| \sum_{j=1}^{m} h(\xi_j) \left[\int_{t_{j-1}}^{t_j} f \, dg - f(\xi_j) [g(t_j) - g(t_{j-1})] \right] \right| +$$

$$+ \left| \sum_{j=1}^{m} h(\xi_j) f(\xi_j) [g(t_j) - g(t_{j-1})] - \int_a^b hf \, dg \right| < \varepsilon .$$

This implies the existence of the integral $\int_a^b h \, dk$ and the relation (2.37). The second implication can be proved in an analogous way.

The convergence result 2.10 enables us to extend the known theorems on the change of the integration order in iterated integrals

$$(2.38) \qquad \int_c^d g(t) \left[d \int_a^b h(t, s) \, df(s) \right], \quad \int_a^b \left(\int_c^d g(t) \, d_t h(t, s) \right) df(s),$$

where $-\infty < c < d < \infty$ and h is of strongly bounded variation on $[c, d] \times [a, b]$ (cf. Theorem I.6.20 in [14]). In what follows $v(h)$ denotes the Vitali variation of the function h on $[c, d] \times [a, b]$ (cf. [4], Definition III.4.1 or [14], I.6.1). For a given $t \in [c, d]$, $\operatorname{var}_a^b h(t, \cdot)$ denotes the variation of the function $s \in [a, b] \to h(t, s) \in R$ on $[a, b]$. Similarly, for $s \in [a, b]$ fixed, $\operatorname{var}_c^d h(\cdot, s)$ stands for the variation of the function $t \in [c, d] \to h(t, s) \in R$ on $[c, d]$.

2.20. Theorem. *Let $h: [c, d] \times [a, b] \to R$ be such that*

$$v(h) + \operatorname{var}_c^d h(\cdot, a) + \operatorname{var}_a^b h(c, \cdot) < \infty .$$

Then for any $f \in BV(a, b)$ and any $g \in G(c, d)$ both the integrals (2.38) *exist and*

$$(2.39) \qquad \int_c^d g(t) \left[d \int_a^b h(t, s) \, df(s) \right] = \int_a^b \left(\int_c^d g(t) \, d_t h(t, s) \right) df(s).$$

Proof. Let us notice that by Theorem I.6.20 from [14] our assertion is true if g is also supposed to be of bounded variation. In the general case of $g \in G(c, d)$ there exists a sequence $\{g_n\}_{n \in N} \subset S(c, d)$ such that $\lim_n \|g_n - g\| = 0$. Then, since the function

$$v(t) = \int_a^b h(t, s) \, df(s)$$

is of bounded variation on $[c, d]$ (cf. the first part of the proof of Theorem I.6.20 in [14]), the integral on the left-hand side of (2.39) exists and by Corollary 2.10 and Theorem I.6.20 in [14] we have

$$(2.40) \qquad \int_c^d g(t) \left[d \int_a^b h(t, s) \, df(s) \right] = \lim_n \int_c^d g_n(t) \left[d \int_a^b h(t, s) \, df(s) \right] =$$
$$= \lim_n \int_a^b \left(\int_c^d g_n(t) \, d_t h(t, s) \right) df(s).$$

Let us denote

$$w_n(s) = \int_c^d g_n(t) \, d_t h(t, s) \quad \text{for} \quad s \in [a, b] \quad \text{and} \quad n = 1, 2, \ldots .$$

Then $w_n \in BV(a, b)$ for any $n \in N$ (cf. [14], Theorem I.6.18) and by Theorem I.4.17 from [14] mentioned here in Remark 2.5 we obtain

$$\lim_n w_n(s) = \int_c^d g(t)\, \mathrm{d}_t h(t, s) := w(s) \quad \text{on} \quad [a, b]\,.$$

As

$$|w_n(s) - w(s)| \leqq \|g_n - g\|\,(\operatorname{var}_c^d h(\cdot, s)) \leqq \|g_n - g\|\,(\operatorname{var}_c^d h(\cdot, a) + \mathrm{v}(h))$$

for any $s \in [a, b]$ (cf. [14], Lemma I.6.6), we have

$$\lim_n \|w_n - w\| = 0\,.$$

It means that $w \in G(a, b)$ and by Theorem 2.8 the integral

$$\int_a^b w(s)\, \mathrm{d}f(s) = \int_a^b \left(\int_c^d g(t)\, \mathrm{d}_t h(t, s) \right) \mathrm{d}f(s)$$

exists as well. Since obviously

$$\lim_n \int_a^b \left(\int_c^d g_n(t)\, \mathrm{d}_t h(t, s) \right) \mathrm{d}f(s) = \lim_n \int_a^b w_n(s)\, \mathrm{d}f(s) =$$
$$= \int_a^b w(s)\, \mathrm{d}f(s) = \int_a^b \left(\int_c^d g(t)\, \mathrm{d}_t h(t, s) \right) \mathrm{d}f(s)\,,$$

the relation (2.39) follows from (2.40).

3. LINEAR BOUNDED FUNCTIONALS ON $G_L(a, b)$

By Theorem 2.8 the expression

(3.1) $$F_\eta(x) = q\, x(a) + \int_a^b p\, \mathrm{d}x$$

is defined for any $x \in G(a, b)$ and any $\eta = (p, q) \in BV(a, b) \times R$. Moreover, for any $\eta \in BV(a, b) \times R$ (3.1) defines a linear bounded functional on $G(a, b)$.

Proposition 2.3 immediately implies

3.1. Lemma. *Let $\eta = (p, q) \in BV(a, b) \times R$ be given. Then*

(3.2) $$F_\eta(\chi_{[a,b]}) = q\,,$$
$$F_\eta(\chi_{[b]}) = p(b)\,,$$
$$F_\eta(\chi_{(\tau,b]}) = p(\tau) \quad \textit{for any} \quad \tau \in [a, b)\,.$$

3.2. Corollary. *If $\eta = (p, q) \in BV(a, b) \times R$ and $F_\eta(x) = 0$ for all $x \in S(a, b)$ which are left-continuous on (a, b), then $p(t) \equiv 0$ on $[a, b]$ and $q = 0$.*

3.3. Lemma. *Let $x \in G(a,b)$ be given. Then for a given $\eta = (p,q) \in BV(a,b) \times R$,*

$$F_\eta(x) = x(a) \quad \text{if} \quad p \equiv 0 \quad \text{and} \quad q = 1\,, \tag{3.3}$$
$$F_\eta(x) = x(b) \quad \text{if} \quad p \equiv 1 \quad \text{and} \quad q = 1\,,$$
$$F_\eta(x) = x(\tau-) \quad \text{if} \quad p = \chi_{[a,\tau]},\ \tau \in (a, b] \quad \text{and} \quad q = 1\,,$$
$$F_\eta(x) = x(\tau+) \quad \text{if} \quad p = \chi_{[a,\tau]},\ \tau \in [a, b) \quad \text{and} \quad q = 1\,.$$

Proof follows from Proposition 2.1.

3.4. Corollary. *If $x \in G(a, b)$ and $F_\eta(x) = 0$ for all $\eta = (p, q) \in BV(a, b) \times R$, then*

$$x(a) = x(a+) = x(\tau-) = x(\tau+) = x(b-) = x(b) = 0 \tag{3.4}$$

holds for any $\tau \in (a, b)$. In particular, if $x \in G_L(a, b)$ (x is left-continuous on (a, b)) and $F_\eta(x) = 0$ for all $\eta \in BV(a, b) \times R$, then $x(t) \equiv 0$ on $[a, b]$.

3.5. Remark. The space $BV(a, b) \times R$ is supposed to be equipped with the usual norm ($\|\eta\|_{BV\times R} = \|p\|_{BV} + |q|$ for $\eta = (p, q) \in BV(a, b) \times R$). Obviously, it is a Banach space with respect to this norm.

3.6. Proposition. *The spaces $G_L(a, b)$ and $BV(a, b) \times R$ form a dual pair with respect to the bilinear form*

$$x \in G_L(a, b)\,, \quad \eta \in BV(a, b) \times R \to F_\eta(x) \in R\,. \tag{3.5}$$

Proof follows from Corollaries 3.2 and 3.4

On the other hand, we have

3.7. Lemma. *If F is a linear bounded functional on $G_L(a, b)$ and*

$$p(t) = F(\chi_{(t,b]}) \quad \text{if} \quad t \in [a, b)\,, \quad p(b) = F(\chi_{[b]})\,, \tag{3.6}$$

then $p \in BV(a, b)$ and

$$|p(a)| + |p(b)| + \operatorname{var}_a^b p \leqq 2\|F\|\,, \tag{3.7}$$

where $\|F\| = \sup\{|F(x)|;\ x \in G_L(a, b),\ \|x\| \leqq 1\}$.

Proof is analogous to that of part $c(i)$ of Theorem 5.1 in [5]. Indeed, for an arbitrary division $\{t_0, t_1, \ldots, t_m\}$ of $[a, b]$ we have

$$\sup_{|c_j| \leqq 1, c_j \in R} \left|p(a)\,c_0 + p(b)\,c_1 + \sum_{j=1}^{m} [p(t_j) - p(t_{j-1})]\,c_j\right| =$$
$$= \sup_{|c_j| \leqq 1, c_j \in R} \left|F\left(c_0\chi_{(a,b]} + c_1\chi_{[b]} + \sum_{j=1}^{m-1} c_j\chi_{(t_{j-1},t_j]} + c_m\chi_{(t_{m-1},b)}\right)\right| \leqq$$
$$\leqq \sup_{\|h\| \leqq 2, h \in G_L(a,b)} |F(h)| = 2\|F\|\,.$$

In particular, for $c_0 = \text{sign}\, p(a)$, $c_1 = \text{sing}\, p(b)$ and $c_j = \text{sign}\,(p(t_j) - p(t_{j-1}))$, $j = 1, 2, \ldots, m$, we get

$$|p(a)| + |p(b)| + \sum_{j=1}^{m} |p(t_j) - p(t_{j-1})| \leqq 2\|F\|,$$

and the inequality (3.7) immediately follows.

Using the ideas from the proof of Theorem 5.1 in [5] we may now prove the following representation theorem.

3.8. Theorem. *F is a linear bounded functional on $G_L(a, b)$ $(F \in G_L^*(a, b))$ if and only if there is an $\eta = (p, q) \in BV(a, b) \times R$ such that*

$$F(x) = F_\eta(x) \left(:= q\, x(a) + \int_a^b p\, \mathrm{d}x\right) \quad \text{for any} \quad x \in G_L(a, b). \tag{3.8}$$

The mapping

$$\Phi: \eta \in BV(a, b) \times R \to F_\eta \in G_L^*(a, b)$$

is an isomorphism.

Proof. Let a linear bounded functional F on $G_L(a, b)$ be given and let us put

$$q = F(\chi_{[a,b]}), \quad p(t) = F(\chi_{(t,b]}) \quad \text{for} \quad t \in [a, b) \quad \text{and} \quad p(b) = F(\chi_{[b]}). \tag{3.9}$$

Then Lemma 3.6 implies $\eta = (p, q) \in BV(a, b) \times R$ and by Lemma 3.1 we have

$$F(\chi_{[a,b]}) = F_\eta(\chi_{[a,b]}), \quad F(\chi_{(t,b]}) = F_\eta(\chi_{(t,b]}) \quad \text{for any} \quad t \in [a, b)$$

and

$$F(\chi_{[b]}) = F_\eta(\chi_{[b]}).$$

Since all functions from $S(a, b) \cap G_L(a, b)$ obviously are finite linear combinations of the functions $\chi_{[a,b]}$, $\chi_{(\tau,b]}$, $\tau \in [a, b)$ and $\chi_{[b]}$, it follows that $F(x) = F_\eta(x)$ holds for any $x \in S(a, b) \cap G_L(a, b)$. Now, the density of $S(a, b) \cap G_L(a, b)$ in $G_L(a, b)$ mplies that

$$F(x) = F_\eta(x) \quad \text{for all} \quad x \in G_L(a, b).$$

This completes the proof of the first assertion of the theorem.

Given an $x \in G_L(a, b)$, then Lemma 2.6 yields

$$|F_\eta(x)| \leqq (|p(a)| + |p(b)| + \text{var}_a^b\, p + |q|)\, \|x\|$$

and consequently,

$$\|F_\eta\| \leqq |p(a)| + |p(b)| + \text{var}_a^b\, p + |q| \leqq 2(\|p\|_{BV} + |q|) = 2\|\eta\|_{BV \times R}.$$

On the other hand, according to Lemma 3.7 we have

$$\|p\|_{BV} \leqq (|p(a)| + |p(b)| + \text{var}_a^b\, p) \leqq 2\|F\|.$$

Furthermore, in virtue of (3.9) we have $|q| \leqq \|F\|$ and hence

$$\|\eta\|_{BV\times R} = \|p\|_{BV} + |q| \leqq 2\|F\| .$$

It means that

$$\tfrac{1}{2}\|F\| \leqq \|\eta\|_{BV\times R} \leqq 3\|F\|$$

and this completes the proof of the theorem.

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Souhrn

REGULOVANÉ FUNKCE A PERON-STIELTJESŮV INTEGRÁL

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Studují se vlastnosti Perronova-Stieltjesova integrálu při integraci vzhledem k „regulovaným" funkcím (tj. funkcím, které mají v každém vnitřním bodě vyšetřovaného intervalu konečné obě jednostranné limity, v levém krajním bodě mají konečnou limitu zprava a v pravém krajním bodě mají konečnou limitu zleva). Ukazuje se, že lineární spojité funkcionály na prostoru $G_L(a, b)$ funkcí regulovaných na $[a, b]$ a zleva spojitých na (a, b) $(-\infty < a < b < \infty)$ mohou být representovány ve tvaru $F(x) = p\, x(a) + \int_a^b q \, dx$, kde $p \in R$ a $q(t)$ je funkce konečné variace na $[a, b]$. Některé věty známé pro integraci vzhledem k funkcím s konečnou variací jsou zobecněny na případ integrace vzhledem k regulovaným funkcím.

Резюме

ПРЕРЫВИСТЫЕ ФУНКЦИИ И ИНТЕГРАЛ ПЕРРОНА-СТИЛТЬЕСА

MILAN TVRDÝ

Изучаются свойства интеграла Перрона-Стилтьеса при интегрировании относительно прерывистых функций (т.е. функций, обладающих в каждой точке рассматриваемого интервала конечными односторонними пределами). Оказывается, что лихейные непрерывные функционалы на пространстве $G(a, b)$ прерывистых на $[a, b]$ и непрерывных слева на (a, b) функций $(-\infty < a < b < \infty)$ могут быть представлены в виде $F(x) = p\, x(a) + \int_a^b q \, dx$, где $p \in R$ и $q(t)$ — функция с конечной вариацией на $[a, b]$. Некоторые теоремы, известные для интегрирования относительно функций с конечной вариацией, обобщены на случае интегрирования относительно прерывистых функций.

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