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LOCALLY PATH-LIKE GRAPHS

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Summary. If G is a graph and x its vertex then $N_G(x)$ is the subgraph of G induced by the set of all vertices adjacent to x in G. A graph G is said to be locally path-like if $N_G(x)$ is a path for each vertex x of G.
In the paper the upper bound of the number of edges of locally path-like graphs is determinated.

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Let G be a finite graph without loops and multiple edges. The neighbourhood of a vertex x in the graph G is understood to be the subgraph of G induced by all vertices adjacent to x.

A. A. Zykov [4] suggested a problem concerning the characterization of graphs with a given neighbourhood. Denote by $N_G(x)$ the neighbourhood of x in G. If for each vertex x, $N_G(x)$ is isomorphic to a given graph H then H is called the realizable graph and G is called the realization of H. The set of all realizations of H will be denoted by $\mathscr{R}(H)$.

B. Zelinka [3] studied the class $\mathscr{R}(T)$ of the locally tree-like graphs where T is any tree. He proposed the following problem: to find the upper bound of the number of edges of a finite connected locally tree-like graph with n vertices.

In this paper we study a certain subclass of $\mathscr{R}(T)$. The graph G is called locally path-like if $N_G(x)$ is a path for each vertex x of G. Denote the class of all locally path-like graphs by $\mathscr{R}(P)$.

We will also give the upper bound for the number of edges of the locally path-like graphs. Zelinka [2] has shown that the maximal number of edges of the polytopic locally path-like graph (called locally snake-like graph) with n vertices is $[11n/4 - 6]$. He also constructed the maximal graphs of this class.

Lemma 1. *Let G be a locally path-like graph. Then every edge of G belongs to at least one triangle of G.*

Proof. Let an edge $e = x_1x_2$ belong to no triangle of G. Then x_2 is an isolated vertex in $N_G(x_1)$ and $N_G(x_1)$ is not a path, which is a contradiction.

Lemma 2. *Every edge of a locally path-like graph G belongs to at most two triangles of G.*

Proof. Let an edge $e = x_1x_2$ belong to the triangles $T_1, T_2, \ldots, T_k$ with vertex sets $V(T_i) = \{x_1, x_2, y_i\}$ for $i = 1, 2, \ldots, k$ $(k \geqq 3)$. Then $N_G(x_1)$ contains the subgraph $K_{1,k}$ with the vertices $x_2, y_1, y_2, \ldots, y_k$, which is a contradiction.

If an edge e belongs to exactly one triangle of G then we call this edge a boundary edge of G.

Lemma 3. *Let G be a locally path-like graph with n vertices. Then G contains exactly n boundary edges.*

Proof. Let x be any vertex of G. Then $N_G(x) \cong P_k$ with the vertex set $\{y_1, y_2, \ldots, y_k\}$ $(k \geqq 2)$. If y_1 and y_2 are the end vertices of P_k then the edge $e_1 = xy_1$ belongs to exactly one triangle T_1 and $e_k = xy_k$ belongs to exactly one triangle T_{k-1}. The other edges $e_i = xy_i$ $(i = 2, 3, \ldots, k - 1)$ belong to two triangles T_{i-1} and T_i. Hence every vertex $x \in V(G)$ is incident to exactly two boundary edges and thus G contains exactly n boundary edges.

It is evident that every boundary edge belongs to exactly one circuit which consists of boundary edges only.

The following theorem is a corollary of the above lemmas.

Theorem 1. *Let G be a locally path-like graph with n vertices. Then every edge of G is either a boundary edge or belongs to exactly two triangles and the number of boundary edges is n.*

Zelinka [3] proved that the minimal number of edges of a connected locally tree-like graph with n vertices is $2n - 3$. Now we determine the upper bound for the number of edges of the locally path-like graphs.

The following simple lemma is proved in [1].

Lemma 4. *Let G be a graph with n vertices and let d_i be the degree of a vertex x_i. Let*

$$\sum_{i=1}^{n} d_i = nk\,. \tag{1}$$

Then

$$\sum_{i=1}^{n} d_i^2 \geqq nk^2\,. \tag{2}$$

Theorem 2 (Zelinka [3]). *Let G be a locally tree-like graph n vertices, m edges and t triangles. Then*

$$t = \frac{2m - n}{3}\,. \tag{3}$$

As $\mathscr{R}(P) \subset \mathscr{R}(T)$, it is evident that the assertion mentioned above holds also for locally path-like graphs.

Lemma 5. *Let G be a locally path-like graph with n vertices and m edges. Let*

$$2m = nk\,. \tag{4}$$

Then exactly one of the following assertions holds:

(i) *G contains a triangle T with vertices x_1, x_2, x_3 for which*

$$d_1 + d_2 + d_3 > 3k\,; \tag{5}$$

(ii) *for each triangle T_j ($j = 1, 2, \ldots, t$) we have*

$$\sum_{x_i \in T_j} d_i = 3k\,. \tag{6}$$

Proof. Let $\sum_{x_i \in T_j} d_i = 3k + r_j$ for $j = 1, 2, \ldots, t$. Then

$$\sum_{j=1}^{t} \sum_{x_i \in T_j} d_i = 3kt + \sum_{j=1}^{t} r_j\,.$$

It follows from (3) and (4) that $t = (nk - n)/3$ and hence

$$\sum_{j=1}^{t} \sum_{x_i \in T_j} d_i = nk^2 - nk + \sum_{j=1}^{t} r_j\,. \tag{7}$$

As each vertex x_i belongs to $d_i - 1$ triangles then the degree d_i of x_i is in (7) included $(d_i - 1)$ times and thus, in order to obtain $\sum_{i=1}^{n} d_i$ on the left-hand side of (7), we have to subtract the expression $d_i(d_i - 2)$ from the right-hand side of (7).

Hence

$$\sum_{i=1}^{n} d_i = nk^2 - nk + \sum_{j=1}^{t} r_j - \sum_{i=1}^{n} d_i(d_i - 2) =$$

$$= nk^2 - nk + \sum_{j=1}^{t} r_j - \sum_{i=1}^{n} d_i^2 + 2\sum_{i=1}^{n} d_i\,.$$

This yields

$$\sum_{i=1}^{n} d_i^2 - nk^2 = -nk + \sum_{j=1}^{t} r_j + \sum_{i=1}^{n} d_i\,.$$

Using the inequality (2) from Lemma 4 we get

$$0 \leqq -nk + \sum_{i=1}^{n} d_i + \sum_{j=1}^{t} r_j\,.$$

As we assumed that $\sum_{i=1}^{n} d_i = nk$, we can see that

$$\sum_{j=1}^{t} r_j \geqq 0\,.$$

If $r_j = 0$ for each $j \in \{1, 2, \ldots, t\}$ then $\sum_{x_i \in T_j} d_i = 3k$. In the opposite case at least one r_j is positive and T_j is the triangle T from the assertion of our lemma.

Now we can determine the upper bound of the number of edges of locally path-like graphs.

Theorem 3. *Let G be a locally path-like graph with n vertices and m edges. Then*

$$m \leqq \frac{n(n + 6)}{6}.$$

Proof. Suppose that $m > n(n + 6)/6$. If we substitute $k = (n + 6)/3$ in (4), Lemma 5 implies that G contains a triangle T with vertices x_1, x_2, x_3 which satisfy

$$\sum_{i=1}^{3} d_i > n + 6.$$

As there exists no vertex x adjacent to all the vertices x_1, x_2, x_3 (in the opposite case $N_G(x)$ would contain C_3) thus there exist at least two vertices x_4, x_5 adjacent to both end vertices of an edge $e \in T$. Without loss of generality we can suppose that it is the edge x_1x_2. Then x_2 is of degree 3 in the graph $N_G(x_1)$, which is a contradiction.

References

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Souhrn

LOKÁLNĚ CESTOVITÉ GRAFY

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Jestliže G je graf a x jeho vrchol, potom $N_G(x)$ je podgraf grafu G indukovaný na množině všech vrcholů G, sousedních s x. Graf G nazveme lokálně cestovitým, je-li $M_G(x)$ cesta pro každý vrchol x z G.

V článku je stanovena horní hranice počtu hran lokálně cestovitých grafů.

Резюме

ЛОКАЛЬНО ЦЕПНЫЕ ГРАФЫ

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Для графа G и его вершины x пусть $N_G(x)$ обозначает подграф графа G, порожденный множеством всех вершин смежных с x в G. В статье анализируются графы, для которых $N_G(x)$ является простой цепью для всех x из G, и найдена верхняя грань числа ребер таких графов.

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