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ON A CERTAIN MAPPING ON THE SET WITH ORTHOGONALITY

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Summary. We consider a set with orthogonality $(\Omega, \perp)$ and the corresponding complete lattice with orthogonality $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. We investigate the mapping $T: \exp \Omega \to \exp \Omega$ defined as $T(A) = \Omega - A^\perp$ for $\emptyset \neq A \subset \Omega$ and $T(\emptyset) = \emptyset$. As an application, we have used the mapping T for the characterization of the maximality of an independent set $M \subset \Omega$. At the end, we have used the mapping T for the construction of an isomorphism of the center of some orthomodular lattice to the family of all subsets of a given set.

Keywords: set with orthogonality, lattice with orthogonality, mapping $T(A) = \Omega - A^\perp$, independent set, center of an orthomodular lattice.

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1. This paper is devoted to a study of the mapping $T: \exp \Omega \to \exp \Omega$ where $(\Omega, \perp)$ is a given set with an orthogonality relation and we put $T(A) = \Omega - A^\perp$ for $A \subset \Omega$, $A \neq \emptyset$ and $T(\emptyset) = \emptyset$. First of all, we summarize some properties of the mapping T. Then we state some of its applications.

2. In addition to the pair $(\Omega, \perp)$, we shall consider the generated complete lattice with orthogonality $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ with the support $S = \{A^\perp; \emptyset \neq A \subset \Omega\} = \{A \subset \Omega;\ \emptyset \neq A = A^{\perp\perp}\}$, where $A^\perp = \{y \in \Omega;\ y \perp x \text{ for all } x \in A\}$. Here, the set Ω plays the role of the unit element of $\mathscr{S}$ and the set $\{o\}$ plays the role of the nought element of $\mathscr{S}$.

First of all, the mapping T has the following properties which one can easily prove.

2.1. $T(\{o\}) = \emptyset$, $T(\Omega) = \Omega - \{o\}$.

2.2. If $A \subset B$, then $T(A) \subset T(B)$.

2.3. If $\emptyset \neq A$, then $A \subset A^{\perp\perp} \subset T(A) \cup \{o\}$.

2.4. $T(A \cup B) = T(A) \cup T(B)$.

2.5. $T(A \cap B) \subset T(A) \cap T(B)$.

2.6. $T(A^\perp) = \Omega - A^{\perp\perp}$.

2.7. If $A, B \in S$, then $T(A \vee B) = T(A) \cup T(B)$.

2.8. If $A, B \in S$, $A^\perp \vee B^\perp = A^\perp \cup B^\perp$, then $T(A \cap B) = T(A) \cap T(B)$.

2.9. If $A \in S$, then $T(A^\perp) = \Omega - A$.

2.10. If $A, B \in S$, $A \neq B$, then $T(A) \neq T(B)$.

2.11. If $A, B \in S$, $A \subset B$, then $T(B) = T(A) \cup [T(A^{\perp}) \cap T(B)]$.

2.12. If A is an atom of $\mathscr{S}$, then $\emptyset \subset T(B) \subset T(A)$ for some $B \in S$ implies that either $T(B) = \emptyset$ or $T(B) = T(A)$.

2.13. If $A, B \in S$ and the set B covers the set A $(A \prec B)$, then $T(A) \subset T(C) \subset T(B)$ for some $C \in S$ implies that either $T(C) = T(A)$ or $T(C) = T(B)$.

For $x \in \Omega$, $x \neq o$, let us write $T(x)$ instead of $T(\{x\})$.

2.14. If $x, y \in \Omega$, $x \neq o \neq y$, $x \perp y$, then $T(x) \neq T(y)$.

Let us recall that we say that the lattice $\mathscr{S}$ satisfies axiom A when, for every $x \in \Omega$, $x \neq o$, the set $\{x\}^{\perp\perp}$ is an atom of the lattice $\mathscr{S}$.

2.15. If the lattice $\mathscr{S}$ contains more than two elements and satisfies axiom A, then $\bigcap_{x \in \Omega - \{o\}} T(x) = \emptyset$.

Proof. If $\bigcap_{x \in \Omega - \{o\}} T(x) \neq \emptyset$, then there is an element $p \in \Omega - \{o\}$ such that $p \not\perp x$ for any $x \in \Omega - \{o\}$. Hence $\{p\}^{\perp} = \{o\}$ which implies $\{p\}^{\perp\perp} = \Omega$, a contradiction.

We say that the lattice $\mathscr{S}$ satisfies axiom P when, for every $x \in \Omega$, $x \notin A$, $x \notin A^{\perp}$ (where $A \in S$, A arbitrary), there is an atom $A_1 \subset A$ and an atom $A_2 \subset A^{\perp}$ such that $x \in A_1 \vee A_2$. If, moreover, the lattice $\mathscr{S}$ is orthomodular and satisfies axiom A, then $A_1 = A \cap (A^{\perp} \vee \{x\}^{\perp\perp})$, $A_2 = A^{\perp} \cap (A \vee \{x\}^{\perp\perp})$.

Let $\emptyset \neq M \subset \Omega, o \notin M$ and let the set M contain at least two points. We call the set M j-independent if $\bigcap_{x \in M} (M - \{x\})^{\perp\perp} = \{o\}$. We call the set M k-independent if $A^{\perp\perp} \cap B^{\perp\perp} = \{o\}$ whenever $M = A \cup B$, $A \neq \emptyset \neq B$, $A \cap B = \emptyset$. We call the set M l-independent if $x \notin (M - \{x\})^{\perp\perp}$ for all $x \in M$.

2.16. Let the lattice $\mathscr{S}$ be orthomodular and let $\mathscr{S}$ satisfy axiom A and axiom P. Let M be an i-independent set. Then it is maximal i-independent if and only if $\bigcup_{x \in M} T(x) = \Omega - \{o\}$, $i = j, k, l$.

Proof. a) Let M be maximal. If $\bigcup_{x \in M} T(x) \neq \Omega - \{o\}$, then there is an element $z \in \Omega - \{o\}$ such that $z \perp x$ for all $x \in M$. According to Lemma 2.7 of [3], the set $M \cup \{z\}$ is i-independent as well, $i = j, k, l$, contrary to the maximality of M.

b) Let us suppose that $\bigcup_{x \in M} T(x) = \Omega - \{o\}$. If M is not maximal, then in accordance with Lemma 2.8 of [3] there is $z \in \Omega$, $z \neq o$, such that $z \perp x$ for all $x \in M$. Hence $z \notin T(x)$ for all $x \in M$, and consequently $\bigcup_{x \in M} T(x) \neq \Omega - \{o\}$.

This assertion immediately implies: For every $y \in \Omega - \{o\}$ there is an element $x \in M$ such that $y \in T(x)$.

Combining the last assertion and Lemma 2.8 of [3] we have the following

2.17. **Theorem.** *Let the lattice $\mathcal{S}$ be orthomodular and let it satisfy axiom A and axiom P. If $M \subset \Omega$ is an i-independent set, $i = j, k, l$, then the following assertions are equivalent.*

a) M is maximal.

b) $M^{\perp\perp} = \bigvee_{x \in M} \{x\}^{\perp\perp} = \Omega$.

c) $\bigcup_{x \in M} T(x) = \Omega - \{o\}$.

3. Now, we shall deal with some applications of the mapping T. First we shall prove the following lemma.

3.1. **Lemma.** *Let the lattice $\mathcal{S}$ be orthomodular and let $\mathcal{S}$ satisfy axiom A and axiom P. If M_1, M_2 are maximal i-independent sets, $i = j, k, l$, and if one of them is finite then the other set is finite as well and both of them have the same number of elements.*

Proof. Let $M_1 = \{x_1, \ldots, x_n\}$ and let card $M_2 \geqq n$. For each element $y_1 \in M_2$ there exists an element $x_{i_1} \in M_1$ such that $y_1 \notin (M_1 - \{x_{i_1}\})^{\perp\perp}$ because otherwise we should have $y_1 \in \bigcap_{x \in M_1} (M_1 - \{x\})^{\perp\perp} = \{o\}$. The last identity follows from Theorem 2.12 of [1]. Let us denote $x_{i_1} = x_1$. According to Theorem 2.10 of [3], it is true that $\{y_1\} \cup (M_1 - \{x_1\})$ is an i-independent set, $i = j, k, l$. In accordance with Theorem 2.10 of [2], we have $(M_1 - \{x_1\})^{\perp\perp} \prec \{x_1\}^{\perp\perp} \vee (M_1 - \{x_1\})^{\perp\perp} = \Omega$. Of course, $(M_1 - \{x_1\})^{\perp\perp} \subset \{y_1\}^{\perp\perp} \vee (M_1 - \{x_1\})^{\perp\perp} \subset \Omega$, hence $\{y_1\}^{\perp\perp} \vee (M_1 - \{x_1\})^{\perp\perp} = \Omega$ because the identity $(M_1 - \{x_1\})^{\perp\perp} = \{y_1\}^{\perp\perp} \vee (M_1 - \{x_1\})^{\perp\perp}$ is not true.

There exists an element $x_{i_2} \in M_1 - \{x_1\}$ such that, for $y_2 \in M_2$, $y_2 \neq y_1$, we have $y_2 \notin (\{y_1\} \cup M_1 - \{x_1\} - \{x_{i_2}\})^{\perp\perp}$. Indeed, otherwise we should have $y_2 \in \bigcap_{x \in M_1 - \{x_1\}} (\{y_1\} \cup M_1 - \{x_1, x\})^{\perp\perp} = \{y_1\}^{\perp\perp}$ where the last identity follows from Theorem 2.12 of [1]. However, the relation $y_2 \in \{y_1\}^{\perp\perp}$ is not true. Let us denote $x_{i_2} = x_2$. The set $\{y_1, y_2\} \cup (M_1 - \{x_1, x_2\})$ is i-independent, $i = j, k, l$, and the identity $\{y_1, y_2\}^{\perp\perp} \vee (M_1 - \{x_1, x_2\})^{\perp\perp} = \Omega$ is valid.

Let us suppose that we already know that the set $\{y_1, \ldots, y_{n-1}\} \cup \{x_n\}$, where $y_1, \ldots, y_{n-1} \in M_2$, $x_n \in M_1$, is i-independent, $i = j, k, l$, and that $\{y_1, \ldots, y_{n-1}\}^{\perp\perp} \vee \{x_n\}^{\perp\perp} = \Omega$. We take $y_n \in M_2 - \{y_1, \ldots, y_{n-1}\}$. At the same time, $y_n \notin \{y_1, \ldots, y_{n-1}\}^{\perp\perp}$, hence $\{y_1, \ldots, y_n\}$ is an i-independent set, $i = j, k, l$, and $\{y_1, \ldots, y_n\}^{\perp\perp} = \Omega$. If $M_2 = \{y_1, \ldots, y_n, y_{n+1}, \ldots\}$, then we have $\Omega = \{y_1, \ldots, y_n\}^{\perp\perp} = \{y_1, \ldots, y_n, y_{n+1}, \ldots\}^{\perp\perp} = \Omega \vee \{y_{n+1}, \ldots\}^{\perp\perp}$ hence $y_{n+1} \in \Omega = \{y_1, \ldots, y_n\}^{\perp\perp}$, a contradiction. Thus, card $M_1 = n =$ card M_2.

3.2. **Theorem.** *Let the lattice $\mathcal{S}$ be orthomodular and let $\mathcal{S}$ satisfy axiom A and axiom P. Let us suppose that there exists at least one infinite maximal i-independent set, $i = j, k, l$, in the set Ω. For every two maximal i-independent* *sets* M_1, M_2, $i = j, k, l$, *let the following assertion hold*: *For every* $x \in M_1$, *it is true that* card $[T(x) \cap M_2] \leqq$ card M_1. *Then and only then* card $M_1 =$ card M_2.

Proof. According to Lemma 3.1, every maximal i-independent set is infinite.

a) If card $M_1 =$ card M_2, then, for every $x \in M_1$, we have card $[T(x) \cap M_2] \leqq$ $\leqq$ card $M_2 =$ card M_1.

b) Let us suppose that card $[T(x) \cap M_2] \leqq$ card M_1 for every $x \in M_1$. According to 2.16 we have $\bigcup_{x \in M_1} [T(x) \cap M_2] = M_2$. Hence card $M_2 \leqq$ card M_1 . card $M_1 =$ $=$ card M_1. If we replace M_1 by M_2 and M_2 by M_1, we have card $M_1 \leqq$ card M_2 which yields card $M_1 =$ card M_2.

At the end of the paper, we shall construct an example of an orthomodular lattice whose center is isomorphic to the family of all subsets of a given set and which has two different blocks (a block is the maximal set of pairwise compatible elements).

3.3. Example. Let D stand for the given set which has at least two points. Suppose that $d \in D$. We put $A = D - \{d\}$, $\Omega = A \cup \{o, u, v, x, y\}$. Let us define the orthogonality relation $\perp$ as follows: $o \perp z$ for every $z \in \Omega$; $a \perp b$ for every different $a, b \in$ $\in A$; $u \perp v$, $x \perp y$; $u \perp a$, $v \perp a$, $x \perp a$, $y \perp a$ for every $a \in A$. The support of this orthogonality generated lattice $\mathscr{S}$ consists just of the following subsets of the set Ω: $A_0 \cup \{o\}, A_1 \cup \{o, u\}, A_2 \cup \{o, v\}, A_3 \cup \{o, x\}, A_4 \cup \{o, y\}, A_5 \cup \{o, u, v, x, y\}$ where A_i, for $i = 0, 1, \ldots, 5$, are arbitrary subsets of the set A. The lattice $\mathscr{S}$ is orthomodular and satisfies axiom A and axiom P. There are just two maximal orthogonal sets in the set Ω, namely $M_1 = A \cup \{u, v\}$ and $M_2 = A \cup \{x, y\}$. The set M_1 generates the block $B_1 = \{A_0 \cup \{o\}, A_1 \cup \{o, u\}, A_2 \cup \{o, v\}, A_3 \cup$ $\cup \{o, u, v, x, y\}; A_i \subset A, i = 0, 1, 2, 3\}$. The set M_2 generates the block $B_2 = \{A_0 \cup$ $\cup \{o\}, A_1 \cup \{o, x\}, A_2 \cup \{o, y\}, A_3 \cup \{o, u, v, x, y\}; A_i \subset A, i = 0, 1, 2, 3\}$. Thus, the center C of the lattice $\mathscr{S}$ is $C = B_1 \cap B_2 = \{A_0 \cup \{o\}, A_3 \cup \{o, u, v, x, y\}; A_0 \subset$ $\subset A, A_3 \subset A\}$. We define an isomorphism $i: C \to \exp D$ as follows: $i(A_0 \cup \{o\}) =$ $= T(A_0 \cup \{o\}) = A_0$, $i(A_3 \cup \{o, u, v, x, y\}) = [T(A_3 \cup \{o, u, v, x, y\}) - \{u, v,$ $x, y\}] \cup \{d\} = A_3 \cup \{d\}$.

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Souhrn

O JISTÉM ZOBRAZENÍ V MNOŽINĚ S ORTOGONALITOU

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Uvažujeme množinu s ortogonalitou $(\Omega, \perp)$ a odpovídající úplný svaz s ortogonalitou $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$. Vyšetřuje se zobrazení $T: \exp \Omega \to \exp \Omega$ definované jako $T(A) = \Omega - A^{\perp}$ pro $\emptyset \neq A \subset \Omega$ a $T(\emptyset) = \emptyset$. Jako aplikace se využívá zobrazení T k charakterizaci maximality nezávislých podmnožin $M \subset \Omega$. Nakonec se zobrazení T využije ke konstrukci izomorfismu centra jistého ortomoduálního svazu se systémem všech podmnožin dané množiny.

Резюме

ОБ ОДНОМ ОТОБРАЖЕНИИ НА МНОЖЕСТВЕ С ОРТОГОНАЛЬНОСТЬЮ

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Рассматривается множество с отношением ортогональности $(\Omega, \perp)$ и порожденная им полная решетка с ортогональностью $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ и исследуется отображение T: $\exp \Omega \to \exp \Omega$ определенное формулами $T(A) = \Omega - A^{\perp}$ для $\emptyset \neq A \subset \Omega$ и $T(\emptyset) = \emptyset$. В качестве приложения отображение применено к характеризации максимальности независимых множеств $M \subset \Omega$. Кроме того отображение T использовано для конструкции изоморфизма центра некоторой ортомодулятной решетки на систему всех подмножеств данного множества.

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