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# ON ENDOMORPHISMS OF THE DIRECT SUM OF TWO MODULES

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**1.** In this note we give conditions for an endomorphism $A : E \to E$ of the direct sum $E$ of modules $E_1$, $E_2$ to be monomorphic or/and epimorphic.

We start with the following classical theorem on the inverse of a partitioned matrix (see [1], p. 189, for some historical comments).

**2. Theorem.** *Let $F$ be a field and let $n$, $r$, $s$ be positive integers such that $n = r + s$. Let $A$, $n \times n$, be a regular matrix over $F$ partitioned in the following way:*

$$A = \begin{array}{c} \\ r \\ s \end{array}\!\!\begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \left[\begin{array}{c|c} \alpha_{11} & \alpha_{12} \\ \hline \alpha_{21} & \alpha_{22} \end{array}\right] \end{array}$$

*Suppose that $\alpha_{11}$ is regular and put*

$$X = \alpha_{11}^{-1}\alpha_{12}\,, \quad Y = \alpha_{21}\alpha_{11}^{-1}\,, \quad Z = \alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}\,.$$

*Then*

(2.1) *$Z$ is regular*

(2.2)
$$A^{-1} = \left[\begin{array}{c|c} \alpha_{11}^{-1} + XZ^{-1}Y & -XZ^{-1} \\ \hline -Z^{-1}Y & Z^{-1} \end{array}\right].$$

Proof. Let the letter $I$ resp. $O$ stand for the unit resp. zero matrix of a corresponding dimension. Using

(2.3)
$$\left[\begin{array}{c|c} \alpha_{11}^{-1} & O \\ \hline -\alpha_{21}\alpha_{11}^{-1} & I \end{array}\right] A = \left[\begin{array}{c|c} I & \alpha_{11}^{-1}\alpha_{12} \\ \hline O & Z \end{array}\right]$$

we see immediately that (2.1) is true. Now, (2.2) is a matter of a simple computation.

**3.** From (2.3) we also see that conversely the regularity of $Z$ implies that of $A$. It is of interest to decide, whether these results may be given a more general setting; in particular, whether they may be extended to infinite dimensional linear spaces.

Let $R$ be a ring, let $E_1, E_2$ be modules over $R$, and let $E$ denote the direct sum of $E_1, E_2$. In what follows, an element of $E$ will be denoted by $[x_1, x_2]$ or $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$.

It is easy to see that each endomorphism $A : E \to E$ is described by well-determined homomorphisms $\alpha_{ij} : E_j \to E_i$, $i, j = 1, 2$ so that

(31) $$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + \alpha_{22}x_2 \end{bmatrix}.$$

Provided $\alpha_{11}$ is isomorphic, we may define an endomorphism $Z : E_2 \to E_2$ as follows:

(3.2) $$Z = \alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}\,.$$

**4. Theorem.** *Let*

(4.1) $\alpha_{11}$ *be isomorphic*

*Then*

(4.2) *$A$ is monomorphic iff $Z$ is monomorphic*

(4.3) *$A$ is epimorphic iff $Z$ is epimorphic*

Proof. (4.2) $\Rightarrow$: Let $A$ be monomorphic. Suppose that there exists $\xi_2 \neq 0$ such that $Z\xi_2 = 0$. Then $A[0, \xi_2] = [\alpha_{12}\xi_2, \alpha_{22}\xi_2] \neq 0$, hence evidently $\alpha_{12}\xi_2 \neq 0$. Put $\xi_1 = \alpha_{11}^{-1}\alpha_{12}\xi_2$; then $\xi_1 \neq 0$, as the image of the non-zero element $\alpha_{12}\xi_2$ by the isomorphism $\alpha_{11}^{-1}$. Then $[-\xi_1, \xi_2] \neq 0$, but from (3.1), (3.2) we see that $A[-\xi_1, \xi_2] = [0, Z\xi_2] = 0$; this is a contradiction.

(4.2) $\Leftarrow$: Let $Z$ be monomorphic and suppose that, for some $[\xi_1, \xi_2] \neq 0$,

(4.4) $$A[\xi_1, \xi_2] = [0, 0]\,.$$

Then, in virtue of $\alpha_{11}\xi_1 + \alpha_{12}\xi_2 = 0$ and (4.1), $\xi_2$ is a non-zero element. Further we see that $\xi_1 = -\alpha_{11}^{-1}\alpha_{12}\xi_2$. Now, from (4.4) we get that $-\alpha_{21}\alpha_{11}^{-1}\alpha_{12}\xi_2 + \alpha_{22}\xi_2 = Z\xi_2 = 0$, which is a contradiction.

(4.3) $\Rightarrow$: Suppose that $A$ is epimorphic, and let $\eta \in E_2$. Then there exists $[\xi_1, \xi_2] \in E$ such that $A[\xi_1, \xi_2] = [0, \eta]$, i.e. $\alpha_{11}\xi_1 + \alpha_{12}\xi_2 = 0$, $\alpha_{21}\xi_1 + \alpha_{22}\xi_2 = \eta$. Thus, $\xi_1 = -\alpha_{11}^{-1}\alpha_{12}\xi_2$; hence $-\alpha_{21}\alpha_{11}^{-1}\alpha_{12}\xi_2 + \alpha_{22}\xi_2 = \eta$, i.e. $Z\xi_2 = \eta$, which shows that $Z$ is epimorphic.

(4.3) $\Leftarrow$: Suppose that $Z$ is epimorphic. We show that each $[\xi_1, \xi_2] \in E$ is the image by $A$ of an element of $E$. For this it is sufficient to solve successively the equations

(4.5) $$A[x_1, x_2] = [\xi_1, 0]\,, \quad A[x_1, x_2] = [0, \xi_2]\,.$$

Firstly, let us choose $\xi \in E_2$ such that $Z\xi = -\alpha_{21}\alpha_{11}^{-1}\xi_1$. Then

$$A[\alpha_{11}^{-1}\xi_1 - \alpha_{11}^{-1}\alpha_{12}\xi, \xi] = [\xi_1, 0]$$

as is easy to prove.

As for the second equation in (4.5), let $\eta \in E_2$ be such that $Z\eta = \xi_2$. Now a direct computation shows that

$$A[-\alpha_{11}^{-1}\alpha_{12}\eta, \eta] = [0, \xi_2] .$$

This concludes the proof of the theorem.

**5. Theorem.** *Let $\alpha_{11}$ be isomorphic. Then $A$ is isomorphic iff $Z$ is isomorphic. If this is the case, $A^{-1}$ is given by* (2.2).

Proof. The first assertion is a direct consequence of theorem 4. The second one can be verified easily, using

$$A^{-1}\begin{bmatrix}\alpha_{11}x_1 + \alpha_{12}x_2\\ \alpha_{21}x_1 + \alpha_{22}x_2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

**6.** As an application, we state a result on systems of linear equations, which we formulate for the infinite dimensional case.

Let $F$ be a field, $n$ a positive integer. Let $E_1$ be the linear space of all $n$-tuples $[x_1, \ldots, x_n]$, and $E_2$ be the linear space of all sequences $[x_{n+1}, x_{n+2}, \ldots]$, with $x_i \in F$ for $i = 1, 2, \ldots$

Further, let

$$\alpha_{11} = \begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{bmatrix} \quad \text{be a regular matrix, } n \times n$$

$$\alpha_{12} = \begin{bmatrix}a_{1,n+1}, & a_{1,n+2}, & \cdots\\ \vdots & \vdots & \\ a_{n,n+1}, & a_{n,n+2}, & \cdots\end{bmatrix} \quad \text{be a row-finite matrix, } n \times \infty$$

$$\alpha_{21} = \begin{bmatrix}a_{n+1,1}, & \ldots, & a_{n+1,n}\\ a_{n+2,1}, & \ldots, & a_{n+2,n}\\ \vdots & & \vdots\end{bmatrix} \quad \text{be a matrix, } \infty \times n$$

$$\alpha_{22} = \begin{bmatrix}a_{n+1,n+1}, & a_{n+1,n+2}, & \cdots\\ a_{n+2,n+1}, & \ldots, & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix} \quad \text{be a row-finite matrix, } \infty \times \infty$$

with $a_{ij} \in F$ for all $i, j = 1, 2, \ldots$

Now we have the following direct consequence of theorems 4 and 5.

**7. Theorem.** *Suppose that the infinite linear system*

$$\begin{aligned} a_{n+1,n+1}x_{n+1} + a_{n+1,n+2}x_{n+2} + \ldots &= b_{n+1} \\ a_{n+2,n+1}x_{n+1} + a_{n+2,n+2}x_{n+2} + \ldots &= b_{n+2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

*has at most one (at least one) {exactly one} solution* $[x_{n+1}, x_{n+2}, \ldots] \in E_2$, *for each* $[b_{n+1}, b_{n+2}, \ldots] \in E_2$. *Suppose further that* $\alpha_{21}\alpha_{11}^{-1}\alpha_{12}$ *is the zero matrix. Then the infinite linear system*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

*has, for each* $[b_1, b_2, \ldots] \in E$, *at most one (at least one) {exactly one} solution* $[x_1, x_2, \ldots] \in E$.

### *Reference*


[1] *E. Bodewig:* Matrix calculus, Amsterdam 1956.



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