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Jiří Sedláček

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DIRECTED GRAPHS AND THEIR INCIDENCE MATRICES

JIŘÍ SEDLÁČEK, Praha

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The article presents several remarks concerning a relationship between nonnegative matrices and directed graphs.

I. A directed graph $\mathscr{G} = [U, H]$, where U is the vertex set and H the edge set, will sometimes be called simply a graph. First, let us examine the primitive matrices (in FROBENIUS'sense). It is known that these matrices can easily be characterized by geometrical properties of their graphs – cf. [6]. If the degree of a primitive matrix is at least two, then the corresponding graph has at least two cycles. Let us consider more in detail a primitive matrix with just two cycles. It is known that the lengths of these cycles are relative prime numbers.

Theorem 1. *Let A be a primitive matrix of the n-th degree $(n \geqq 2)$, whose graph contains exactly two cycles with lengths a, b, and let $a < b$. Let k be the least positive integer such that $A^k > 0$. Then we have $k = n + a(b - 2)$.*

Proof: Let $\mathscr{G} = [U, H]$ be the graph of the matrix A and $\mathscr{C}_a = [U_a, H_a]$, $\mathscr{C}_b = [U_b, H_b]$ its two cycles with $|U_a| = a$, $|U_b| = b$. Construct $\mathscr{G}_0 = [U_0, H_0]$, where $U_0 = U_a \cap U_b$, $H_0 = H_a \cap H_b$ and put $|U_0| = c$. If $c > 1$, the vertices of the graph $\mathscr{G}_0$ may be denoted so that $U_0 = \{x_1, x_2, \ldots, x_c\}$, $H_0 = \{\overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \ldots, \overrightarrow{x_{c-1}x_c}\}$; if $c = 1$, $H_0 = \emptyset$. Let the remaining vertices of $\mathscr{G}$ be denoted as follows: On $\mathscr{C}_a$ (provided $c \neq a$), following its orientation, as $x'_{c+1}, x'_{c+2}, \ldots, x'_a$. On $\mathscr{C}_b$, following again its orientation, as $x_{c+1}, x_{c+2}, \ldots, x_b$.

First, we are going to show that in $\mathscr{G}$ there is no path with length $n + a(b - 2) - 1$ which begins at x_{c+1} and ends at x_b. If such a path existed, it could be composed of the cycle $\mathscr{C}_a$ (passed r-times), of the cycle $\mathscr{C}_b$ (passed s-times) and of a portion of the cycle $\mathscr{C}_b$ between x_{c+1} and x_b which has the length $b - c - 1$. Then we should have

$$n + a(b - 2) - 1 = ar + bs + (b - c - 1).$$

Since $n = a + b - c$, we can write $ab - a = ar + bs$, i.e. $bs \equiv 0 \pmod{a}$. As a, b are relative prime, it follows that $s \equiv 0 \pmod{a}$. The case that $s = 0$ is to be rejected, as otherwise $\mathscr{C}_a$ would have been passed through (b-1)-times although $\mathscr{C}_a$ and the portion between x_{c+1} and x_b mentioned above have no common vertex. Thus put $s = ta$ where t is a positive integer. Consequently, we have the equation $-a = ar + (t - 1) ab$ the left hand side of which is negative and the right one positive which is a contradiction.

It remains to prove that between any two vertices of the graph $\mathscr{G}$ there is a path with length $n + a(b - 2)$. However, let us prove first the following auxiliary assertion: For x_1 and any arbitrarily chosen vertex of the graph $\mathscr{G}$ there is a path with length $a(b - 1)$ joining them. It is convenient to extend the definition of x_i to all integers i by putting $x_i = x_j$ exactly then if $i \equiv j \pmod{b}$. It can be easily verified that vertices x_{1+ma} (for $m = 0, 1, \ldots, b - 1$) constitute exactly the set U_b. As any path beginning at x_1 can be "prolonged" by a multiple of the number a, provided the cycle $\mathscr{C}_a$ circuited several times is placed at the beginning, it follows that any vertex of $\mathscr{C}_b$ can be reached from x_1 by a path with length $a(b - 1)$. If $c < a$, such an assertion can be verified even for every vertex $x'_i \in U_a - U_0$. Indeed, it suffices to step back in the preceding path from x'_i, first along $\mathscr{C}_a$ and then along $\mathscr{C}_b$, a path with total length a, thus getting at the vertex $x_{1+ma} \neq x_1$. Consequently, there is a path with length $a(b - 1)$ between x_1 and x'_i. Hence the auxiliary assertion is proved.

If $c > 1$, all paths beginning at x_1, which were considered above, pass consecutively through the vertices $x_1, x_2, \ldots, x_c$. If $c = 1$, the assertion is trivial. From this it follows that for the i-th vertex $x_i \in U_0 (1 \leqq i \leqq c)$ we have: For x_i and any vertex of the graph $\mathscr{G}$ there is a path of the length $a(b - 1) - i + 1$ that joins them. Finally, consider the set $U - U_0$. For any element $x \in U - U_0$ we have: between x and x_1 there is a path the length of which does not exceed the number $b - c$. Consequently, to any $x \in U - U_0$ a positive integer[1]) d can be assigned such that between x and any arbitrary vertex of U there is a path of length d. Moreover, evidently $d \leqq (b - c) + a(b - 1) = n + a(b - 2)$. From this our assertion[2]) follows immediately.[3])

II. In the next paragraph we will continue to consider the primitive matrices. Let a positive integer n be given. Denote by $M(n)$ the set of all square nonnegative matrices A of degree n such that $A^2 > 0$.

Obviously, $M(n) \neq \emptyset$. It can be seen that all elements of the set $M(n)$ are special primitive matrices. Denote further by $p(A)$ the number of positive elements of a matrix A. We shall establish the minimum of the numbers $p(A)$ for $A \in M(n)$.

[1]) Note that in the paper [8] the least of such positive integers was called the "Zeiger" in German. Dulmage and Mendelsohn [2] use the term "reach" for this concept.

[2]) For example, the Lemma 1 in the paper [8] p. 305, may be used. Lemma 1 reads: Let the primitive vertex x have a reach $d_{\min}$; let $d \geqq d_{\min}$ (d-integer). Then for every vertex y a path joining x and y exists the length of which is d. (A vertex x is called primitive if there exists a positive integer m such that for every vertex y a path between x and y of the length m can be constructed.)

[3]) If $a = n - 1$, $b = n$ is set in Th. 1, an extreme case of a matrix known from [3], [5], [9] is obtained. See also Th. 1 and 5 in [2].

This investigation may also be formulated in terms of the graph theory. Instead of the matrix A a finite directed graph[4]) $\mathcal{G} = [U, \Gamma]$ with $U = \{x_1, x_2, \ldots, x_n\}$ may be considered such that for any two vertices of $\mathcal{G}$ (not necessarily distinct) there is a path of the length 2 joining them. Then we are to find a graph with properties already stated, which possesses the least number of edges, and to establish this number.

Theorem 2. *For any positive integer n we have* $\min\limits_{A \in M(n)} p(A) = 2n - 1$.

Proof. For the sake of brevity introduce the notation $\alpha(n) = \min\limits_{A \in M(n)} p(A)$.

We are going to prove the theorem by induction with respect to n. For $n = 1$ and $n = 2$ the assertion is obvious. Thus assume that $n > 2$ and $\alpha(n - 1) = 2n - 3$. Choose a matrix $A_0 \in M(n)$ such that $p(A_0) = \alpha(n)$ and assume that $\alpha(n) \leqq 2n - 2$. Construct the graph of the matrix A_0 and denote it by $\mathcal{G}_0 = [U_0, \Gamma_0]$. Obviously, the graph $\mathcal{G}_0$ possesses at most $2n - 2$ edges. The question is whether $|\Gamma_0(x)| \geqq 2$ can be true for any $x \in U_0$. If this were true, $\mathcal{G}_0$ would have at least $2n$ edges. Consequently, there is an x_1 in U_0 such that $|\Gamma_0(x_1)| \leqq 1$. The case that $|\Gamma_0(x_1)| = 0$ is not possible so that $\Gamma_0(x_1) = \{x_2\}$ may be put, x_1 and x_2 being different. As there is a path with length 2 between them, x_2 must be equipped by a loop. As a path of the length 2 beginning and ending at x_1 can be found, the edge $\overrightarrow{x_2 x_1}$ must exist in $\mathcal{G}_0$.

Next, let us construct a new graph $\mathcal{G}_1 = [U_1, \Gamma_1]$ as follows[5]): Put $U_1 = (U_0 - \{x_1, x_2\}) \cup \{y\}$, where y non $\in U_0$ and let us define the mapping Γ_1 by: a) $\Gamma_1(y) = (\Gamma_0(x_1) \cup \Gamma_0(x_2) \cup \{y\}) - \{x_1 x_2\}$; b) for $z \in U_1 - \{y\}$, $\Gamma_0(z) \cap \{x_1, x_2\} = \emptyset$ let $\Gamma_1(z) = \Gamma_0(z)$ whereas for $z \in U_1 - \{y\}$, $\Gamma_0(z) \cap \{x_1, x_2\} \neq \emptyset$ let $\Gamma_1(z) = (\Gamma_0(z) \cup \{y\}) - \{x_1, x_2\}$. It can be easily verified that by this construction at least two edges drop away so that $\mathcal{G}_1$ has at most $2n - 4$ edges. We have $|U_1| = n - 1$.

If A_1 is the incidence matrix of the graph $\mathcal{G}_1$, then $p(A_1) \leqq 2n - 4$ so that by the induction assumption $A_1^2 > 0$ is not true. However, it can be ascertained that between any pair of vertices x_a, x_b of $\mathcal{G}_1$ there is a path of the length 2. If $x_a \neq y \neq x_b$, such a path $\boldsymbol{S}$ can be found in $\mathcal{G}_0$; if $\boldsymbol{S}$ does not contain any of vertices x_1, x_2, then $\boldsymbol{S}$ can be found even in $\mathcal{G}_1$. The path $\boldsymbol{S}$ cannot contain x_1 and x_2 simultaneously. Thus let $\boldsymbol{S}$ pass through the vertex x_1 (or x_2) and, consequently, have the form $x_a, \overrightarrow{x_a x_1}, x_1, \overrightarrow{x_1 x_b}, x_b$ (or $x_a, \overrightarrow{x_a x_2}, x_2, \overrightarrow{x_2 x_b}, x_b$); then a path $x_a, \overrightarrow{x_a y}, y, \overrightarrow{y x_b}, x_b$ can be found in $\mathcal{G}_1$. It remains to discuss the following cases for $\mathcal{G}_1$: a) $x_a = y$, $x_b \neq y$; b) $x_a \neq y$, $x_b = y$; c) $x_a = y = x_b$. We shall not prove in detail that a path between x_a and x_b

4) The significance of the mapping Γ for the graph definition is known from the book [1]

5) The ensuing construction can be visualised as follows: In the graph $\mathcal{G}_0$ we let the vertices x_1, x_2 coincide into a new vertex y equipped with a loop and the both edges $\overrightarrow{x_1 x_2}$, $\overrightarrow{x_2 x_1}$ will be dropped; if a pair of edges beginning at y and ending at another vertex arises, only one edge of the pair will be retained. In a similar manner any other pair of edges beginning at some vertex and ending at y will be treated.

of the length 2 also exists in $\mathscr{G}_1$. This result yields a contradiction thus proving that $\alpha(n) \geqq 2n - 1$. It remains to construct a square nonnegative matrix belonging to $M(n)$ which has exactly $2n - 1$ positive elements. Such a matrix is obtained if the elements of the first row and of the first column are equalled to one whereas the remaining elements are zero. By this the proof is finished.

Let us examine again the considered extremal matrix from $M(n)$. We shall show that this matrix is determined uniquely with respect to its graph by the relation $p(A) = 2n - 1$.

Theorem 3. *Let $A_1 \in M(n)$, $A_2 \in M(n)$, $p(A_1) = p(A_2) = 2n - 1$ and let $\mathscr{D}_1, \mathscr{D}_2$ be the graphs of the matrices A_1 and A_2, respectively. Then the graphs $\mathscr{D}_1$ and $\mathscr{D}_2$ are isomorph.*

Proof: The cases that $n = 1$ and $n = 2$ are trivial. Thus, let $n > 2$. Choosing a matrix $A \in M(n)$ with $p(A) = 2n - 1$ construct the graph $\mathscr{D} = [U, \Gamma]$ of A. As $|\Gamma(x)| \geqq 2$ cannot be true for any $x \in U$, a vertex $x_1 \in U$ can be found such that $\Gamma(x_1) = \{x_2\}$, $x_1 \neq x_2$. As a path of the length 2 must exist between x_1 and x_2, the vertex x_2 is provided with a loop; for the same reason we find that the edge $\overrightarrow{x_2x_1}$ exists in $\mathscr{D}$. Next, put $U_0 = U - \{x_1, x_2\} = \{x_3, \ldots, x_n\}$. For any $x_i \in U_0$ there is a path of the length 2 between x_1 and x_i so that there are edges $\overrightarrow{x_2x_i}$ in $\mathscr{D}$ for $i = 3, 4, \ldots, n$. In $\mathscr{D}$ we have so far described $n + 1$ edges so that $n - 2$ edges remain each of which begins at some vertex belonging to U_0. Since $|U_0| = n - 2$, we have $|\Gamma(x_i)| = 1$ for any $x_i \in U_0$. Obviously, none of the vertices x_i is provided with a loop and moreover, the edge $\overrightarrow{x_ix_1}$ cannot exist since a path of the length 2 between x_i and x_1 should exist. For similar reasons the existence of the edge $\overrightarrow{x_ix_j}$ with $x_i \in U_0$, $x_j \in U_0$, $x_i \neq x_j$ must be rejected. Thus, for every $x_i \in U_0$ we have $\Gamma(x_i) = \{x_2\}$. Thereby all edges have been considered and the uniqueness of the construction of $\mathscr{D}$ is obvious. Hence the proof.

Hitherto only the second power of matrices has been considered and in the extreme case the unicity has been proved. However, for the third power the unicity of a similar problem is no more true. For example, in Fig. 1 four distinct graphs are sketched each of which has three vertices and five edges. If A_1, A_2, A_3, A_4 are the corresponding incidence matrices, then obviously $A_i^3 > 0$ $(1 \leqq i \leqq 4)$; it is also evident that for any matrix $A \geqq 0$ of the third degree with $A^3 > 0$ the inequality $p(A) < 5$ is not fulfilled.

Now we can formulate a more general problem. Let be given positive integers n, k with $k > 1$. Denote $M(n, k)$ the set of all square nonnegative matrices A of degree n such that $A^k > 0$. We are looking for $\min\limits_{A \in M(n,k)} p(A)$. The problem of establishing it in its full generality is still open; it seems, however, that the following conjecture is true:[6])

[6]) The author is aware of the fact that the given formulae present an upper bound for the required minimum.

If k is even, then the considered minimum equals to

$$n - \left[- \frac{2n - 2}{k} \right];$$

if k is odd, then the minimum equals to

$$n - \left[- \frac{2n - 2}{k - 1} \right].$$

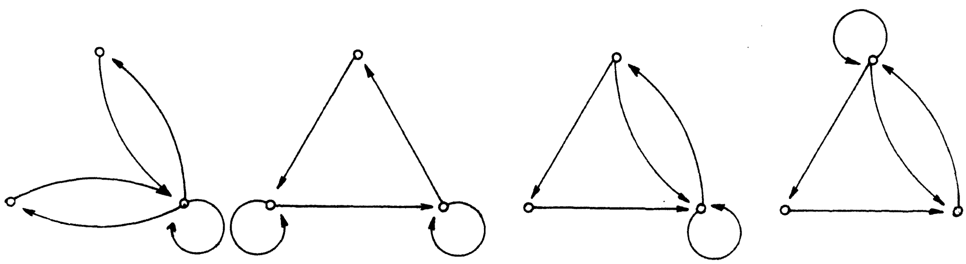

Fig. 1.

III. Let us consider a relationship between primitive matrices and a certain concept from the "pure" graph theory. O. Ore [4] has considered nondirected graphs such that each pair of distinct vertices can be joined by a "way" which passes through each vertex of the graph exactly once. Let us extend these problems to directed graphs.

Let a finite graph $\mathcal{G} = [U, \Gamma]$ be given. A path passing through each vertex of $\mathcal{G}$ exactly once will be called the *Hamilton arc*. If for any pair $x \in U$, $y \in U$, $x \neq y$ there is a Hamilton arc in $\mathcal{G}$ beginning at x and ending at y, then $\mathcal{G}$ will be called simply an H-graph. A trivial case of an H-graph is obtained if a complete graph is taken for $\mathcal{G}$. In the following theorem we shall consider nontrivial H-graphs which have only simple edges[7]).

Theorem 4. *Let an integer $n \geqq 7$ be given. Then there is always an H-graph $\mathcal{G} = [U, \Gamma]$ with $|U| = n$ which has only simple edges.*

Proof. We shall construct one of the possible H-graphs. Putting $U = \{x_1, \ldots, x_n\}$ define the mapping Γ as follows: For $i \in \{1, 2, \ldots, n\}$ put $\Gamma(x_i) = \{x_{i+1}, x_{i+2}, x_{i+3}\}$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, $x_{n+3} = x_3$.[8]) Obviously, it suffices to prove that for any index $j \in \{2, 3, \ldots, n\}$ there is a Hamilton arc $\boldsymbol{H}_{1j}$ which begins at x_1 and ends at x_j. For the sake of brevity construct first the cycle $\mathcal{C} = [U, \Gamma_0]$ where $\Gamma_0(x_i) = \{x_{i+1}\}$ for $i \in \{1, 2, \ldots, n\}$. If $j = n$, then $\boldsymbol{H}_{1j}$ will be constructed directly on $\mathcal{C}$. Thus, let $1 < j < n$.

[7]) If for two distinct vertices x and y there is an edge $\overrightarrow{xy}$ but the edge $\overrightarrow{yx}$ does not exist, then $\overrightarrow{xy}$ will be called a simple edge.

[8]) In Fig. 2 the case with $n = 7$ is plotted; this case will be referred to later on.

If j is even, let H_{1j} be constructed as follows; we pass consecutively through the vertices $x_1, x_3, x_5, \ldots, x_{j-1}, x_{j+1}$ with odd indices, then along $\mathscr{C}$ we pass all vertices up to x_n across the edge $\overrightarrow{x_n x_2}$, we enter x_2 and (provided j does not equal 2) pass vertices $x_4, x_6, \ldots, x_{j-2}, x_j$ with even indices.

If j is odd, then the construction of H_{1j}, for example, runs as follows: Leaving x_1 we enter x_2 and then through vertices $x_4, x_6, x_8, \ldots$ with even indices we get to x_{j+1}; we continue along $\mathscr{C}$ to x_n, then along the edge $\overrightarrow{x_n x_3}$ to x_3 and (if j does not equal 3) through vertices $x_5, x_7, \ldots$ with odd indices up to x_j.

Thereby H_{1j} is constructed and the proof is given.

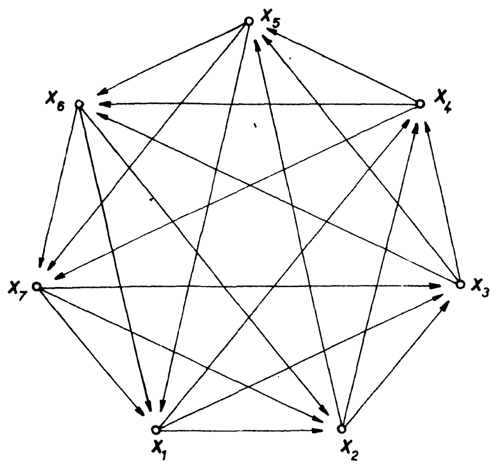

Fig. 2.

Let us remain a little more at the introduced concept of an H-graph. Leaving aside the case of $[U, \Gamma]$ with $U = \{x_1, x_2\}$, $\Gamma(x_1) = \{x_2\}$, $\Gamma(x_2) = \{x_1\}$, we can easily verify that the following assertion is true:

Theorem 5. *Let $\mathscr{G} = [U, \Gamma]$ be an H-graph with $|U| \geqq 3$. Then the incidence matrix of the graph $\mathscr{G}$ is primitive.*

Proof. It suffices to prove that there is a positive integer d with the following property: For any pair $(x, y) \in U \times U$ there is a path of the length d in $\mathscr{G}$. Verify this property for $d = |U|$. The equality $|\Gamma^{-1}(y)| = 1$ cannot be true, since otherwise a Hamilton arc beginning at the vertex $z \in \Gamma^{-1}(y)$ and ending at y would not exist. Consequently, $|\Gamma^{-1}(y)| \geqq 2$ so that a vertex $t \neq x$ can be found in $\Gamma^{-1}(y)$. Now it suffices to complete the Hamilton arc between x and t by the edge $\overrightarrow{ty}$ and by the vertex y, which completes the proof.

In conclusion let us add several remarks. We did not try to establish the least positive integer d which can be assigned to an H-graph (or to the corresponding incidence matrix). For example, denoting the incidence matrix of the graph in Fig. 2 by A, we have $A^7 > 0$ as it has been shown above. However, it can be verified that even $A^4 > 0$ whereas A^3 is not a positive matrix.

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Výtah

ORIENTOVANÉ GRAFY A JEJICH INCIDENČNÍ MATICE

JIŘÍ SEDLÁČEK, Praha

I. Nechť A je primitivní matice n-tého stupně $(n \geqq 2)$, jejíž diagram obsahuje právě dva cykly. Nechť a, b jsou délky těchto cyklů, přičemž $a < b$. Nechť k je nejmenší přirozené číslo, pro něž platí $A^k > 0$. Dokazuje se, že platí $k = n + a(b - 2)$.

II. Dále se ukazuje, že minimální počet kladných prvků nezáporné matice A stupně n-tého, pro níž je $A^2 > 0$, je $2n - 1$. Diagram této extrémní matice je určen jednoznačně.

III. Poslední část souvisící s prací [4] zavádí pojem tzv. H-grafu. Tím se problematika ze [4] přenáší na orientované grafy a ukazuje se souvislost s primitivními maticemi.

Резюме

ОРИЕНТИРОВАННЫЕ ГРАФЫ И ИХ МАТРИЦЫ ИНЦИДЕНТНОСТИ

ЙИРЖИ СЕДЛАЧЕК (Jiří Sedláček), Прага

I. Пусть A – примитивная матрица степени $n(n \geqq 2)$, диаграмма которой содержит точно два цикла. Пусть a, b – длины этих циклов, причем $a < b$. Пусть k – наименьшее натуральное число, для которого $A^k > 0$. Доказывается, что $k = n + a(b - 2)$.

II. Далее показано, что минимальное число положительных элементов неотрицательной матрицы A степени n, для которой $A^2 > 0$, равно $2n - 1$. Диаграмма этой экстремальной матрицы определена однозначно.

III. Последняя часть, примыкающая к работе [4], посвящена понятию т. наз. H-графа. Таким образом, проблематика из [4] распространяется на ориентированные графы, и указывается связь с примитивными матрицами.