Hung Tzaw Hu
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SEMIGROUPS ON D-SPACES

HUNG-TZAW HU[1]), Gainesville

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In this paper we are seeking sufficient topological conditions on the underlying space of a semigroup S under which if $S^2 = S \neq K$, then S must have either a left or a right identity. We show that S be a D-space with a nondegenerated maximal level set M (to be defined) is such a condition. A point p of a continuum space is a *D-point* if and only if for any two subcontinuua C_1 and C_2 with p as a common point then either $C_1 \subset C_2$ or $C_2 \subset C_1$. A continuum X is called a *D-space* if X contains a D-point p and the naturally induced relation on X is a closed relation. Also we prove that a locally connected D-space is an arc.

Throughout this work a space will always be a Hausdorff topological space. A *continuum* is a compact connected space. An *arc* is a continuum with exactly two non-cutpoints. For standard semigroup-theoretic definitions and results we refer to [1] and [2]. It is well known that a compact semigroup has unique minimal ideal and is denoted by K [2]. If S is a semigroup and b is an element of S, the smallest ideal containing a is denoted by $J(b)$. Clearly we have the identity $J(b) = b \cup Sb \cup bS \cup SbS$. Green's relation $\leqq_{\mathscr{J}}$ are defined on S as $x \leqq_{\mathscr{J}} Y$ if $J(x) \subset J(y)$. An element x in a semigroup S is $\mathscr{J}$-maximal, if it is maximal *relative* to the quasi-ordering $\leqq_{\mathscr{J}}$. It is well known that if S is compact, then each element of S is below a $\mathscr{J}$-maximal element; in particular maximal element exists [2].

Definition 1. Let X be a continuum. A point p of X is a *D-point* iff for any two subcontinuua C_1 and C_2 with $p \in C_1 \cap C_2$, either $C_1 \subset C_2$ or $C_2 \subset C_1$.

Let X be a continuum with D-point p. For each point a of X, let $\mathscr{F}_a$ be the collection of all subcontinuua of X which contains both a and p. Then $\mathscr{F}_a$ is a non-empty collection of compact connected subsets of X and $\mathscr{F}_a$ is totally ordered by inclusion. If $L[a] = \bigcap \mathscr{F}_a$, then $L[a]$ is the unique minimal subcontinuum containing a and p. It is easy to see that $L[p] = \{p\}$ since $\{p\}$ is itself a subcontinuum which contains p.

[1]) The result of this work was contained in the author's doctoral dissertation written at the University of Florida under Professor K. N. SIGMON and Professor A. D. WALLACE.

We define a relation $\leqq$ on X as $a \leqq b$ iff $L[a] \subset L[b]$. Since $L[a]$ and $L[b]$ are always comparable under inclusion for any pair of elements a and b of X, $\leqq$ is a total quasi-order on X. By the definition of $\leqq$ and construction of $L[a]$ we have $L[a] = \{b \mid b \leqq a\}$, and call $L[a]$ the *lower set* of a in X. The sets $U[a] = \{b \mid a \leqq b\}$ and $L_a = L[a] \cap U[a]$ are called the *upper set* and *level set* of a respectively. For convenience, we write $a \approx b$ iff $a \leqq b$ and $b \leqq a$, and $a < b$ iff $a \leqq b$ and $b \nleqq a$. Recall that a quasi-order $\leqq$ on a set X is called *order dense* if and only if for any pair of elements a and b of X satisfying $a < b$, there exists a point c of X such that $a < c < b$.

Lemma 2. *Let X be a continuum with a D-point p. If $U[a]$ is closed for each point a of X, then $\leqq$ is order dense.*

Proof. Suppose not; i.e., suppose there exists a pair of elements a and b in X, with $a < b$ and no point c in X satisfies $a < c < b$. Since $\leqq$ is a total quasi-order on X we have $L[a] \cup U[b] = X$ and $L[a] \cap U[b] = \square$. Then X is a union of a pair of disjoint nonempty closed subsets $L[a]$ and $U[b]$, which contradicts the assumption that X is connected. Hence the proof is complete.

Proposition 3. *Let X be a continuum with a D-point p and $\leqq$ be the induced quasi-order. Then the following two statements are equivalent.*

(1) *"$\leqq$" is a closed relation on X, and*
(2) *$U[a]$ is closed for each point a of X.*

Proof. (2) $\Rightarrow$ (1). Let b and c be a pair of elements of X such that $(b, c) \notin \leqq$. Since $\leqq$ is a total quasi-order on X we have $c < b$. By Lemma 2 there is a point d in X satisfying $c < d < b$. If $U = X \setminus L[d]$ and $V = X \setminus U[d]$, then U is an open set containing b while V is an open set containing c. It is claimed that $(U \times V) \cap \leqq = \square$. Suppose not; i.e., suppose there exists a pair of elements x and y of X such that $(x, y) \in (U \times V) \cap \leqq$. Then $d < x$, $y < d$ and $x \leqq y$, which implies that $d < d$. But this is a contradiction. Hence the proof is complete.

(1) $\Rightarrow$ (2) We omit the proof because it is trivial.

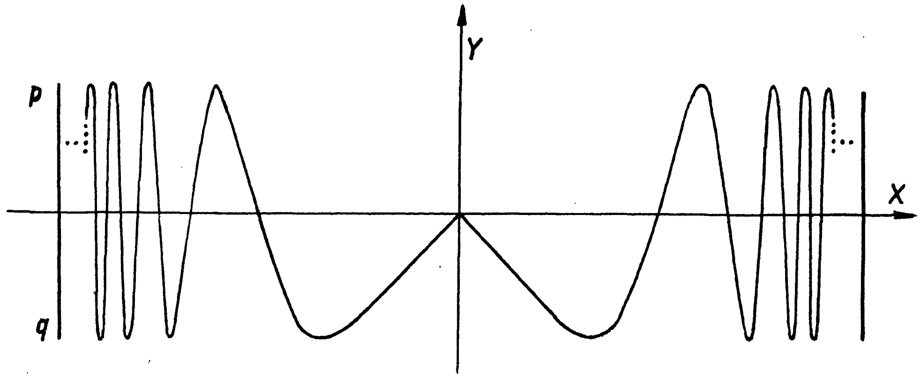
Remark 4. There exists a continuum with a D-point whose induced quasi-order $\leqq$ is not closed. Let X be a space defined by

$$X = \left\{(x, y) \mid y = \sin\left(\frac{1}{x + 1/\pi}\right), \quad -1/\pi < x \leqq 0\right\} \cup$$

$$\left\{(x, y) \mid y = \sin\left(\frac{1}{1/\pi - x}\right), \ 0 \leqq x < 1/\pi\right\} \cup$$

$$\{(x, y) \mid -1 \leqq y \leqq 1, \ x = 1/\pi \text{ or } x = -1/\pi\},$$

with the usual topology as seen in the following figure.



Then X is a continuum with a D-point $p = (-1/\pi, 1)$. But the induced quasi-order $\leqq$ is not closed, since $U[q] = X \setminus \{(x, y) \mid x = -1/\pi, -1 \leqq y \leqq 1\}$ is not a closed set in X where $q = (-1/\pi, -1)$. By Proposition 3 we know that $\leqq$ is not a closed relation.

Definition 5. A continuum X with a D-point p is called a *D-space* if the induced quasi-order $\leqq$ is a closed relation on X.

We have seen in Remark 4 that X is not locally connected at p, while the induced quasi-order $\leqq$ is not closed. But if X is a D-space, then X is locally connected at its D-point p.

Proposition 6. *If X is a D-space with D-point p, then X is locally connected at p.*

Proof. Let V be an open proper subset containing the D-point p. If $\mathscr{L}$ is the collection of all $L[x] \cap (X \setminus V)$ for each x of $X \setminus V$, then $\mathscr{L}$ is a nonempty collection of nonempty closed subsets of X and $\mathscr{L}$ is totally ordered by inclusion, so that $\bigcap \mathscr{L} \neq \square$. If d is a point of $\bigcap \mathscr{L}$, then $d \leqq x$ for all x of $X \setminus V$, since $d \in L[x]$ for all x of $X \setminus V$. This implies that $X \setminus V \subset U[d]$. It is not difficult to verify that $X \setminus U[d] = \bigcup \{L[c] \mid c < d\}$. Then $X \setminus U[d] \subset V$ and is an open connected set containing p. Hence the proof is complete.

Suppose X is a D-space with a D-point p. We let $\mathscr{U}$ be the collection of all $U[b]$ for each point b of X. Then $\mathscr{U}$ is a collection of closed subsets of X which is totally ordered by inclusion. If we define $M = \bigcap \mathscr{U}$, then M is a nonempty closed subset of X, and is called *the maximal level set* of X. Recall a point d in a continuum X is a *weak cut point between a and b* if a and b are points in X different from d, and any subcontinuum of X containing a and b also contain d. The point d is simply *a weak cut point* if there exist a and b such that d is a weak cut point between a and b.

Theorem 7. *Let X be a D-space with D-point p. Then the following statements hold.*

(1) *X is irreducible between its D-point p and any point m of M.*

(2) *Each point $z \in X \setminus (M \cup \{p\})$ is a weak cut point of X. Furthermore, if M contains more than one point, then every point z, except D-point p, is a weak cut point of X.*

(3) *Each level set L_d, except $L_p = \{p\}$ and the maximal level set M, cuts X.*

(4) *If, furthermore, X is locally connected, then X is an arc.*

Proof. (1) Let A be a subcontinuum in X which contains p and a point m of M. Since $m \in M$ we have $L[m] = X$. But $L[m]$ is the minimal continuum which contains p and m so that $L[m] \subset A$ and hence $A = X$.

(2) From (1) we know that each point $z \in X \setminus (M \cup \{p\})$ is a weak cut point between p and m a point of M. In the case that M contains more than one point, it is sufficient to prove that each point m of M is also a weak cut point. Let n be a point of M which is different from m. Then it can be easily verified that m is a weak cut point between p and n, since $L[n] = X$ is the minimal continuum which contains p and n.

(3) If we let $P = L[d] \setminus L_d$ and $Q = U[d] \setminus L_d$, then $X \setminus L_d = P \cup Q$ and both P and Q are nonemty sets since $M \subset Q$ and $p \in P$. Also $P^* \cap Q = \square$, since $P^* \cap \cap Q \subset L[d] \cap Q = \square$. Similarly it is true that $P \cap Q^* = \square$. Thus $X \setminus L_d$ is a disconnected set, which completes the proof.

(4) We prove this part by steps.

(i) We show that for each $a \neq p$, there exists some point $a_0 \in L_a$ such that if V is an open neighborhood of a_0, then there exists a point $b \in V$ with $b < a_0$. Suppose not and let a be a point such that for each $l \in L_a$ there exists an open neighborhood V_l of l for which $l \leqq b$ for all $b \in V_l$. Then $U[a] = (X \setminus L[a]) \cup (\bigcup_{l \in L_a} V_l)$ is both open and closed in X. Since $a \neq p$, $U[a]$ is a proper subset of X which is both open and closed in X, which is impossible because X is connected.

(ii) We prove that each level set L_a is a singleton set. Recall that $L_p = \{p\}$, since $L[p] = \{p\}$. So we assume that $a \neq p$ and L_a contains more than one point. By (i) there exists a point $a_0 \in L_a$ which satisfies the statement mentioned in (i). Since L_a contains more than one point, we let a_1 be a point in L_a which is different from a_0 and let V_{a_0} be a connected neighborhood of a_0 which is small enough that $a_1 \notin V^*_{a_0}$. By (i) there is a point b in V_{a_0} such that $b < a_0$. Then $L[b] \cup V^*_{a_0}$ is a closed connected set containing p and a_0. Since p is a D-point, we have $L[a_0] \subset L[b] \cup V^*_{a_0}$, which implies that $a_1 \in L[b]$. But this is a contradiction since $b < a_0$ and $a_0 \approx a_1$.

(iii) From (3) and (ii) we know that each point $b \in X \setminus M \cup \{p\}$ is a cut point of X. From (ii) we know that M is a singleton set, hence let $M = \{m\}$. In order to prove that X is an arc, it is sufficient to prove that p and m are noncut points. So suppose p is a cut point, i.e., $X \setminus \{p\} = P \cup Q$, where P and Q are disjoint nonempty open sets, and $P^* \cap Q^* = \{p\}$. Since $P^* \cap Q^* = \{p\}$ and $P^* \cup Q^* = X$ is a connected set it is not difficult to prove that both P^* and Q^* are connected. Then both P^* and Q^* contain the D-point p, but neither of them contains the other as a subset, which contradicts the fact that p is a D-point. On the other hand we have $X \setminus \{m\} = \bigcup\{L[d] \mid d \in X \text{ and } d \neq m\}$ is a connected set, which implies that m is not a cut point. Hence the proof is complete.

Remark 8. (1) The converse of part (1) in Theorem 7 is not true. The space X in Remark 4 is irreducible between points $(-1/\pi, 1)$ and $(1/\pi, 1)$ but is not a D-space.

(2) In application 11, there are two D-spaces which are not locally connected.

It has been shown by McCharen that if S is a compact semigroup satisfying $S^2 = S$ and b is a $\mathcal{J}$-maximal element of S, then there exist idempotents u and v such that $b = ubv$ [3]. We can derive easily from this result that if S is compact and $S^2 = S$, then there exists a $\mathcal{J}$-maximal idempotent. Also it has been shown by McCharen that if S is a continuum semigroup satisfying $S^2 = S$ and e is a $\mathcal{J}$-maximal idempotent of S which is a weak cut point of S, then $S = K$ [3].

Theorem 9. *Let S be a D-space with a D-point p whose maximal level set M contains more than one point. If S satisfies $S^2 = S \neq K$, then the D-point p is either a left or a right identity for S.*

Proof. We know there exists an idempotent e which is a $\mathcal{J}$-maximal element of S since S is compact and $S^2 = S$. Then e is not a weak cut point of S since, by assumption $S \neq K$. Thus e must be the only point of S which is not a weak cut point of, namely, $e = p$. Since by part (2) of Theorem 7 we know that every point, except the D-point p, is a weak cut point of S. Therefore e is the only $\mathcal{J}$-maximal element S. This implies that $SeS = S$, since e is an idempotent and hence $J(e) = SeS$.

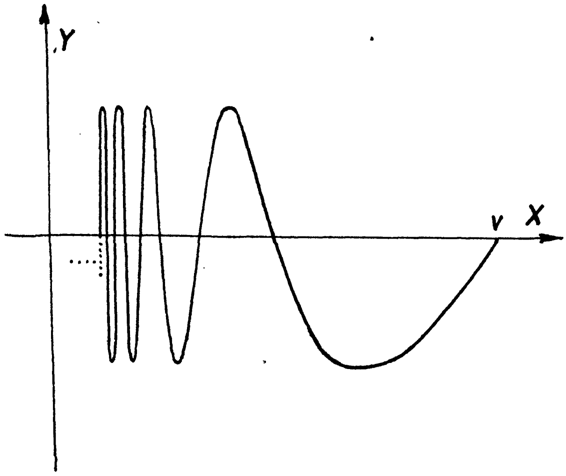
On the other hand we know that each eS and Se is a subcontinuum and contains the D-point p, so that either $eS \subset Se$ or $Se \subset eS$. Without loss of generality we may assume $eS \subset Se$. Then we have

$$S = SeS \subset S(Se) = S^2e = Se\,.$$

In this case e is a right identity. Similarly if $Se \subset eS$ then e is a left identity.

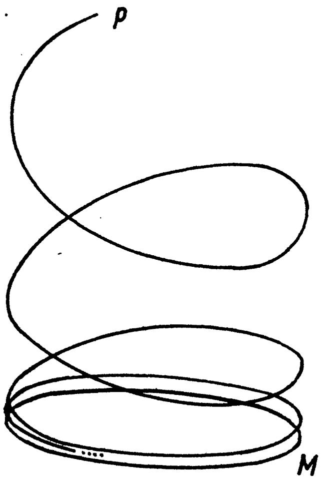
Remark 10. The closed unit interval $I = [0, 1]$ is a D-space with one end point as a D-point and the other end point as the maximal level set. It is possible to construct a semigroup S on I which satisfies condition $S^2 = S \neq K$ but S has neither a left nor a right identity.

Application 11. (1) Let the underlying space of S be defined as $S = \{(x, y) \mid y = \sin(1/x),\ 0 < x \leqq 1/\pi\} \cup \{(x, y) \mid x = 0,\ -1 \leqq y \leqq 1$, with the usual topology as seen in the following figure.



If S satisfies $S^2 = S \mp K$, then $v = (1/\pi, 0)$ is either a left or a right identity for S. Since S is a D-space with D-point $v = (1/\pi, 0)$ its corresponding maximal level set is $M = \{(x, y) \mid x = 0, \ -1 \leqq y \leqq 1\}$.

(2) Let $S = \{(e^{2\pi it}, e^{-t}) \mid t \in [0, \infty]\} \cup [C \times \{0\}]$ where C is a unit circle, with the usual topology as seen in the following figure.



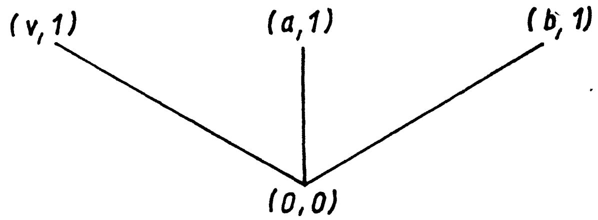
If S satisfies $S^2 = S \mp K$ then p is either a left or a right identity for S. Since S is a D-space with a D-point p, its corresponding maximal level set is $M = C \times \{0\}$.

We end this paper with an example. This is an example of a continuum semigroup on a triod, probably the simplest continuum one can find which is not a D-space, which satisfies $S^2 = S \mp K$ and has neither a left nor a right identity.

Example 12. This example is constructed as follows. Let $T = \{v, a, b, 0\}$ with multiplication defined in the following table

$\cdot$	v	a	b	0
v	v	a	0	0
a	0	0	0	0
b	b	0	0	0
0	0	0	0	0

Then T is a semigroup, with discrete topology on it, with an idempotent v such that $vTv = \{v, 0\}$, $vT = \{v, a, 0\}$, $Tv = \{v, b, 0\}$ and $TvT = T$. Let $I = [0, 1]$ denote the closed real unit interval with the usual multiplication. Let $S_0 = T \times I$ with product topology and coordinewise multiplication, then S_0 is a semigroup. If we let $S_1 = \{(v, 0), (a, 0), (b, 0)\} \cup [\{0\} \times I]$, then S_1 is a closed ideal in S_0. Then the Rees quotient $S = S_0/S_1$ is a semigroup with zero and $(v, 1)$ as their only two idempotents, and it is easy to check that $(v, 1)\, S \neq S$, $S(v, 1) \neq S$ but $S^2 = S$. The underlying space of S is homeomorphic to a triod [5] as in the following figure



It is not difficult to see that this space is not a D-space, because none of the points in S can be a D-point.

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Author's address: Department of Mathematics, University of Florida, Gainesville, Florida 32611, U.S.A.