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ON THE NONHOMOGENEOUS ALGEBRAIC INTEGRAL EQUATION

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In this paper we shall deal with the nonhomogeneous algebraic integral equation

$$(1)\qquad \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \mu^\alpha\, y^\alpha(s)\, L_j[y^{\alpha_1} \dots y^{\alpha_\nu}] = f(s)\,, \quad s \in \langle a, b\rangle$$

where

$$L_j[\;] = \sum_{(\alpha_1 + \dots + \alpha_\nu = j - \alpha)} \int_a^b \dots \int_a^b L_{\alpha\alpha_1 \dots \alpha_\nu}(st_1 \dots t_\nu)\,[y^{\alpha_1}(t_1) \dots y^{\alpha_\nu}(t_\nu)]\, dt_1 \dots dt_\nu\,;$$

μ is a real or complex parameter, $L_{\alpha\alpha_1\dots\alpha_\nu}(st_1 \dots t_\nu)$ and $f(s)$ are given real or complex functions. For equation (1) we shall study the problem of existence of "small" solutions in $C[a, b]$, i.e., of such solutions absolute values of which assume sufficiently small values, analogically as in [1].

Let us assume that the functions $L_{\alpha\alpha_1\dots\alpha_\nu}(st_1 \dots t_\nu)$ are continuous in $\langle a, b\rangle \times \dots \times \langle a, b\rangle$ ($(\nu + 1)$ factors) and that $f(s)$ is continuous in $\langle a, b\rangle$. After excluding the term linearly depending on $y(s)$ we can write equation (1), under the assumption

$$(2)\qquad g(s) \equiv \int_a^b \dots \int_a^b L_{10\dots0}(st_1 \dots t_\nu)\, dt_1 \dots dt_\nu \neq 0\,,$$

in the form

$$(3)\qquad \mu\, y(s) - \int_a^b L(s, t)\, y(t)\, dt = V_0(s) - V(y; \mu)$$

where

$$L(s, t) = \frac{-1}{g(s)} \int_a^b \dots \int_a^b [L_{010\dots0}(stt_2 \dots t_\nu) + L_{001\dots0}(st_2 t \dots t_\nu) + \dots + L_{0\dots01}(st_\nu t_2 \dots t_{\nu-1} t)]\, dt_2 \dots dt_\nu\,,$$

$$V_0(s) = \frac{f(s)}{g(s)}\,, \quad V(y; \mu) = \frac{1}{g(s)} \sum_{j=2}^{n} \sum_{\alpha=0}^{j} \mu^\alpha\, y^\alpha(s)\, L_j[y^{\alpha_1} \dots y^{\alpha_\nu}]\,.$$

First we shall prove two inequalities. Assume that

$$\|y(s)\| = \max_s |y(s)| \leqq w < d\,, \quad \|V_0(s)\| \leqq V_0\,, \tag{4}$$

$$\sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)} \int_a^b \ldots \int_a^b \left\| \frac{1}{g(s)} L_{\alpha\alpha_1\ldots\alpha_\nu}(st_1 \ldots t_\nu) \right\| \mathrm{d}t_1 \ldots \mathrm{d}t_\nu \leqq B_{j\alpha}\,.$$

Then

$$\|V(y;\mu)\| \leqq \sum_{j=2}^{n} \sum_{\alpha=0}^{j} |\mu|^\alpha \|y\|^j B_{j\alpha} \leqq B(w,\mu)$$

where

$$B(w,\mu) = \sum_{j=2}^{n} \sum_{\alpha=0}^{j} |\mu|^\alpha w^j B_{j\alpha}\,.$$

From the expression for $B(w, \mu)$ there follows

$$B(0,\mu) = 0\,, \quad B'(0,\mu) = 0\,, \quad B''(0,\mu) = 2\sum_{\alpha=0}^{2} |\mu|^\alpha B_{2\alpha}\,.$$

If $\sum_{\alpha=0}^{2} |\mu|^\alpha B_{2\alpha} \neq 0$ then we can write

$$B(w,\mu) = \tfrac{1}{2} w^2 B''(\theta w, \mu)\,, \quad 0 < \theta < 1$$

in a neighbourhood of the point $w = 0$ and for $\frac{1}{2}B''(\theta w, \mu) \leqq P(\mu)$ there is

$$\|V(y;\mu)\| \leqq w^2 P(\mu)\,; \tag{5}$$

this is the first inequality mentioned above.

Now consider such a function $u(s) \in C[a, b]$ for which $\|u\| \leqq w < d$. After some rearranging we obtain

$$\|y^\alpha(s)\, y^{\alpha_1}(t_1) \ldots y^{\alpha_\nu}(t_\nu) - u^\alpha(s)\, u^{\alpha_1}(t_1) \ldots u^{\alpha_\nu}(t_\nu)\| \leqq j w^{j-1} \|y - u\|\,;$$

from this expression there follows that

$$\|V(y;\mu) - V(u;\mu)\| \leqq \sum_{j=2}^{n} \sum_{\alpha=0}^{j} j w^{j-1} |\mu|^\alpha B_{j\alpha} \|y - u\| =$$
$$= w B''(\theta_1 w, \mu)\, \|y - u\|\; (0 < \theta_1 < 1)\,.$$

Hence, there is

$$\|V(y;\mu) - V(u;\mu)\| \leqq 2w\, P(\mu)\, \|y - u\|\,; \tag{6}$$

this is the second inequality mentioned above.

In this paper we shall study the case when the homogeneous linear integral equation

$$\mu\, y(s) - \int_a^b L(s,t)\, y(t)\, \mathrm{d}t = 0 \tag{7}$$

has only the trivial solution. Then there exists the continuous resolving kernel $R(s, t; \mu)$ of the kernel $L(s, t)$ and equation (3) can be written in the form

$$y(s) = W_0(s; \mu) - W(y; \mu) \tag{8}$$

where

$$W_0(s; \mu) = V_0(s) + \int_a^b R(s, t; \mu)\, V_0(t)\, dt,$$

$$W(y; \mu) = V(y; \mu) + \int_a^b R(s, t; \mu)\, V(y; \mu)\, dt.$$

Under the assumption

$$\left\| 1 + \int_a^b R(s, t; \mu)\, dt \right\| \leqq R(\mu)$$

and on the basis of relations (4), (5) and (6) we get the following inequalities

$$\|W_0(s; \mu)\| \leqq V_0\, R(\mu) = R^*(\mu), \quad \|W(y; \mu)\| \leqq w^2\, R(\mu)\, P(\mu) = w^2\, P^*(\mu), \tag{9}$$

$$\|W(y; \mu) - W(u; \mu)\| \leqq 2w\, P(\mu)\, R(\mu)\, \|y - u\| = 2w\, P^*(\mu)\, \|y - u\|. \tag{10}$$

Let us denote $\max(R^*(\mu), P^*(\mu)) = T^*(\mu)$ and study the quadratic equation

$$\tau^2\, T^*(\mu) - \tau + T^*(\mu) = 0. \tag{11}$$

For $T^*(\mu) < \frac{1}{2}$ both roots are positive and the smaller one

$$\tau = \frac{1 - \sqrt{(1 - 4T^{2*}(\mu))}}{2T^*(\mu)} = T^*(\mu) + (T^*(\mu))^3 + \ldots$$

tends to zero simultaneously with $T^*(\mu)$. Then $\tau < d$ holds for $T^*(\mu)$ small enough.

Theorem. *Let the following assumptions hold*

a) *the homogeneous linear integral equation* (7) *has only the trivial solution,*
b) *for the smaller root* τ *of equation* (11) *there is* $\tau < d$,
c) $\tau\, P^*(\mu) < \frac{1}{2}$.

Then equation (1) *has only one solution* $y(s) \in C[a, b]$ *for which* $\|y\| < d$ *holds.*

Proof. Let us find a solution of equation (8) be means of the method of successive approximations

$$y_0(s; \mu) = W_0(s; \mu), \tag{12}$$

$$y_k(s; \mu) = W_0(s; \mu) - W(y_{k-1}; \mu), \quad k = \overline{1, \infty}.$$

Using the inequalities (9) and taking the condition b) of the theorem into account we obtain

$$\|y_0\| \leqq R^*(\mu) < \tau ,$$
$$\|y_1\| \leqq \|W_0(s; \mu)\| + \|W(y_0; \mu)\| < T^*(\mu) + \tau^2 T^*(\mu) = \tau ,$$
$$\|y_k\| \leqq \|W_0(s; \mu)\| + \|W(y_{k-1}; \mu)\| < T^*(\mu) + \tau^2 T^*(\mu) = \tau , \quad k = \overline{2, \infty} ,$$

that is

$$\|y_k(s; \mu)\| < \tau , \quad k = \overline{0, \infty} .$$

Using inequality (10) we get

$$\|y_k - y_{k-1}\| \leqq (2\tau P^*(\mu))^{k-1} \tau^2 P^*(\mu) , \quad k = \overline{1, \infty} .$$

This implies

$$\|y_{k+l} - y_k\| \leqq \|y_{k+l} - y_{k+l-1}\| + \ldots + \|y_{k+1} - y_k\| \leqq$$
$$\leqq (2\tau P^*(\mu))^k \frac{1 - (2\tau P^*(\mu))^l}{1 - 2\tau P^*(\mu)} \tau^2 P^*(\mu)$$

and that means, according to the condition c) of the theorem, that the sequence $\{y_k(s; \mu)\}$ is fundamental. Because of the completeness of the space $C[a, b]$ the sequence $\{y_k(s; \mu)\}$ converges to a function $y(s; \mu)$. If we pass on to the limit $k \to \infty$ in relation (12) the limit function satisfies equation (8) and so, under condition (2), it is a solution of equation (1).

Let us suppose that there exists another solution $y^*(s)$ of equation (8) with $\|y^*\| < d$. Then the difference of these solutions satisfies the relation

$$y(s) - y^*(s) = W(y; \mu) - W(y^*; \mu) .$$

Using inequality (10) we get

$$\|y - y^*\| \leqq 2\tau P^*(\mu) \|y - y^*\| .$$

From this relation $2\tau P^*(\mu) \geqq 1$ follows for $y \neq y^*$, which is in contradiction with the assumption c) of the theorem. Hence, the theorem is proved.

If equation (7) has a non-trivial solution branching of the solution appears. Branching of solutions of equation (1) has been studied from a certain point of view in [2].

References

[1] *L. Lichtenstein:* Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-Differentialgleichungen nebst Anwendungen, Berlin 1931.

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