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BRANCHING OF SOLUTION OF ALGEBRAIC INTEGRAL EQUATION

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In this paper we shall deal with the general algebraic integral equation for the function $y(s)$

$$F[\mu, y] \equiv \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \mu^\alpha y^\alpha(s)\, L_j[y^{\alpha_1} \ldots y^{\alpha_\nu}] = f(s) \tag{1}$$

where

$$L_j[y^{\alpha_1} \ldots y^{\alpha_\nu}] = \sum_{(\alpha_1 + \ldots + \alpha_\nu = j - \alpha)} \int_a^b (\nu) \int_a^b L_{\alpha\alpha_1\ldots\alpha_\nu}(st_1 \ldots t_\nu)\, [y^{\alpha_1}(t_1) \ldots y^{\alpha_\nu}(t_\nu)]\, dt_1 \ldots dt_\nu\,;$$

$L_{\alpha\alpha_1\ldots\alpha_\nu}(st_1 \ldots t_\nu)$ and $f(s)$ are given functions and μ is a real parameter.

The type of non-linear integral equations which are called algebraic integral equations was first introduced by W. SCHMEIDLER in [1].

For equation (1) we shall study branching of a solution which occurs for a certain value of the parameter μ. The pair $(\mu_0, y_0(s))$ which satisfies equation (1) is called the branch point if for every $\varepsilon > 0$ there exists such μ that $|\mu - \mu_0| < \varepsilon$ and (1) has for this μ at least two solutions which lie in the ε-neighbourhood of the solution $y_0(s)$.

Theorem. *Let $L_{\alpha\alpha_1\ldots\alpha_\nu}(st_1 \ldots t_\nu)$ be real functions continuous in all variables in the region $\langle a, b\rangle \times \ldots \times \langle a, b\rangle$ ((ν + 1) factors) for all suitable non-negative numbers $\alpha, \alpha_1, \ldots, \alpha_\nu$ and let $f(s)$ be a real function continuous in $\langle a, b\rangle$. Let $y_0(s)$ be a solution of equation (1) for $\mu = \mu_0$ continuous in $\langle a, b\rangle$ and let the discriminant of the polynomial $F[\mu_0, y]$ be different from zero in $\langle a, b\rangle$ for $y_0(s)$. Then for equation (1) the following assertions are valid:*

a) *If number 1 is not an eigenvalue of the kernel*

$$L(s, t) = \frac{-1}{p(s)} \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \mu_0^\alpha y_0^\alpha(s)\,. \tag{2}$$

$$\cdot \sum_{(\alpha_1 + \ldots + \alpha_\nu = j - \alpha)} \int_a^b (\nu) \int_a^b \{\alpha_1 L_{\alpha\alpha_1\ldots\alpha_\nu}(stt_2 \ldots t_\nu)\, y_0^{\alpha_1 - 1}(t)\, y_0^{\alpha_2}(t_2) \ldots y_0^{\alpha_\nu}(t_\nu) +$$

$$+ \sum_{k=2}^{\nu} \alpha_k L_{\alpha\alpha_1\ldots\alpha_\nu}(st_k t_2 \ldots t_{k-1} t t_{k+1} \ldots t_\nu)\, y_0^{\alpha_k - 1}(t)\, y_0^{\alpha_1}(t_k) \prod_{\substack{i=2\\ i \neq k}}^{\nu} y_0^{\alpha_i}(t_i)\}\, dt_2 \ldots dt_\nu$$

where

$$p(s) = \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \alpha \mu_0^{\alpha} y_0^{\alpha-1}(s)\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}],$$

then in a neighbourhood of the point μ_0 *there exists the unique solution of* (1) *in the form*

$$y(s) = \sum_{l=0}^{\infty} (\mu - \mu_0)^l\, y_l(s). \tag{3}$$

b) *If number* 1 *is a p-multiple eigenvalue of the kernel* (2) *and* b1) *if*

$$\int_a^b \frac{\partial F[\mu, y_0]/\partial\mu|_{\mu=\mu_0}}{\partial F[\mu_0, y]/\partial y(s)|_{y=y_0}}\, \alpha_i(s)\,\mathrm{d}s = 0, \quad i = \overline{1, p}, \tag{4}$$

is valid for the associated eigenfunctions $\alpha_i(s)$ *then in a neighbourhood of* μ_0 *there exist, in general,* 2^p *solutions of* (1) *in the form* (3);

b2) *if any of the conditions* (4) *is not fulfilled then in a neighbourhood of* μ_0 *there exist, in general,* 2^p *solutions of* (1) *in the form*

$$y(s) = \sum_{l=0}^{\infty} (\mu - \mu_0)^{l/2}\, y_l(s). \tag{5}$$

All the solutions are continuous in $\langle a, b \rangle$ *and tend to* $y_0(s)$ *for* $\mu \to \mu_0$.

Proof. If we denote

$$\lambda = \mu - \mu_0, \quad \psi(s) = y(s) - y_0(s),$$

equation (1) can be rewritten in the form

$$p(s)\,\psi(s) = p(s) \int_a^b L(s, t)\, \psi(t)\, \mathrm{d}t - \tag{6}$$

$$- \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \Big\{ \mu_0^{\alpha} y_0^{\alpha}(s)\, L_j'' \Big[\sum_{l_1=0}^{\alpha_1} \binom{\alpha_1}{l_1} \psi^{l_1} y_0^{\alpha_1 - l_1} \dots \sum_{l_\nu=0}^{\alpha_\nu} \binom{\alpha_\nu}{l_\nu} \psi^{l_\nu} y_0^{\alpha_\nu - l_\nu} \Big] +$$

$$+ \mu_0^{\alpha} \sum_{m=1}^{\alpha} \binom{\alpha}{m} \psi^m(s) y_0^{\alpha-m}(s)\, L_j' \Big[\sum_{l_1=0}^{\alpha_1} \binom{\alpha_1}{l_1} \psi^{l_1} y_0^{\alpha_1 - l_1} \dots \sum_{l_\nu=0}^{\alpha_\nu} \binom{\alpha_\nu}{l_\nu} \psi^{l_\nu} y_0^{\alpha_\nu - l_\nu} \Big] +$$

$$+ \sum_{k=1}^{\alpha} \binom{\alpha}{k} \lambda^k \mu_0^{\alpha-k} \sum_{m=0}^{\alpha} \binom{\alpha}{m} \psi^m(s)\, y_0^{\alpha-m}(s)\, L_j \Big[\sum_{l_1=0}^{\alpha_1} \binom{\alpha_1}{l_1} \psi^{l_1} y_0^{\alpha_1 - l_1} \dots \sum_{l_\nu=0}^{\alpha_\nu} \binom{\alpha_\nu}{l_\nu} \psi^{l_\nu} y_0^{\alpha_\nu - l_\nu} \Big] +$$

$$+ \mu_0^{\alpha} \sum_{m=2}^{\alpha} \binom{\alpha}{m} \psi^m(s)\, y_0^{\alpha-m}(s)\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] \Big\};$$

L'_j, L''_j have the same meaning as L_j under the condition that $\sum_{i=1}^{\nu} l_i \neq 0$, $\sum_{i=1}^{\nu} l_i \neq 0,1$ respectively.

Let us look for a solution of (6) in the form

$$\psi(s) = \sum_{l=1}^{\infty} \lambda^l y_l(s) . \tag{7}$$

If we substitute (7) into (6) and compare coefficients of the same powers of λ we obtain for $y_l(s)$ a system of nonhomogeneous linear integral equations. From the assumption on the discriminant of the polynomial $F[\mu_0, y]$ there follows that

$$p(s) = \left.\frac{\partial F[\mu_0, y]}{\partial y(s)}\right|_{y=y_0}$$

is different from zero in $\langle a, b\rangle$. Then the system of equations for $y_l(s)$ can be written in the form

$$y_l(s) = \int_a^b L(s, t)\, y_l(t)\, \mathrm{d}t + f_l(s)\,, \quad l = \overline{1, \infty} \tag{8}$$

where

$$f_1(s) = \frac{-1}{p(s)} \left.\frac{\partial F[\mu, y_0]}{\partial \mu}\right|_{\mu=\mu_0}, \tag{9}$$

$$f_2(s) = \frac{-1}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{s[y_1^2, y_1]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] + \mu_0^\alpha\, y_0^\alpha(s) \left(Q[y_1^2] + R[y_1, y_1]\right) +$$

$$+ \alpha\mu_0^{\alpha-1}\, y_0^{\alpha-1}(s) \left(y_0(s) + \mu_0\, y_1(s)\right) S[y_1]\} + M_2[y_0]\rangle\,,$$

$$f_{l+1}(s) = \frac{-1}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{t[y_1 y_l, y_l]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] +$$

$$+ \mu_0^\alpha\, y_0^\alpha(s) \left(2Q[y_1 y_l] + T[y_1, y_l]\right) + \alpha\mu_0^{\alpha-1}\, y_0^{\alpha-1}(s) \left(\mu_0\, y_1(s) + y_0(s)\right) S[y_l] +$$

$$+ \alpha\mu_0^\alpha\, y_0^{\alpha-1}(s)\, y_l(s)\, S[y_1]\} + M_{l+1}[y_0, y_1, \dots, y_{l-1}]\rangle =$$

$$= K[y_l] - \frac{1}{p(s)} M_{l+1}[y_0, y_1, \dots, y_{l-1}]\,, \quad l = \overline{2, \infty}$$

where the notation

$$Q[y_1 y_l] = L_j\left[\sum_{k=1}^{\nu} \binom{\alpha_k}{2} y_0^{\alpha_k-2}(t_k)\, y_1(t_k)\, y_l(t_k) \prod_{\substack{i=1\\ i\neq k}}^{\nu} y_0^{\alpha_i}(t_i)\right], \tag{10}$$

$$R[y_1, y_1] = L_j[\sum_{k=1}^{\nu-1} \sum_{m=k+1}^{\nu} \alpha_k\alpha_m\, y_0^{\alpha_k-1}(t_k)\, y_1(t_k)\, y_0^{\alpha_m-1}(t_m)\, y_1(t_m) \prod_{\substack{i=1\\ i\neq k\\ i\neq m}}^{\nu} y_0^{\alpha_i}(t_i)]\,,$$

$$S[y_l] = L_j[\sum_{k=1}^{v} \alpha_k\, y_0^{\alpha_k - 1}(t_k)\, y_l(t_k) \prod_{\substack{i=1\\ i \neq k}}^{v} y_0^{\alpha_i}(t_i)]\,,$$

$$T[y_1, y_{l+1}] = L_j[\sum_{k=1}^{v} \sum_{\substack{m=1\\ m \neq k}}^{v} \alpha_k \alpha_m\, y_0^{\alpha_k - 1}(t_k)\, y_1(t_k)\, y_0^{\alpha_m - 1}(t_m)\, y_{l+1}(t_m) \prod_{\substack{i=1\\ i \neq k\\ i \neq m}}^{v} y_0^{\alpha_i}(t_i)]\,,$$

$$q[y_1 y_l] = \binom{\alpha}{2} \mu_0^{\alpha} y_0^{\alpha-2}(s) y_1(s) y_l(s)\,, \quad r[y_l] = \alpha^2 \mu_0^{\alpha-1} y_0^{\alpha-1}(s) y_l(s)\,,$$

$$s[y_1^2, y_1] = q[y_1^2] + r[y_1]\,, \quad t[y_1 y_{l+1}, y_{l+1}] = 2q[y_1 y_{l+1}] + r[y_{l+1}]\,, \quad l = \overline{1, \infty}\,,$$

has been introduced.

Solving system (8) it is necessary to distinguish whether number 1 is or is not an eigenvalue of the kernel $L(s, t)$.

a) If number 1 is not the eigenvalue of $L(s, t)$ it is possible to solve all equations of system (8) uniquely and to write the solutions continuous in $\langle a, b \rangle$ in the form

$$y_l(s) = \int_a^b \Gamma(s, t) f_l(t)\, \mathrm{d}t + f_l(s)\,, \quad l = \overline{1, \infty} \tag{11}$$

where $\Gamma(s, t)$ is the continuous resolving kernel of $L(s, t)$. So it is possible to construct the series (7) formally.

Now we shall prove that the constructed series converges absolutely and uniformly according to s and λ in $\langle a, b \rangle$ for λ sufficiently small. Let us choose such numbers A, B, C and D that for $s \in \langle a, b \rangle$

$$|p(s)|^{-1} < A\,, \quad \int_a^b |\Gamma(s, t)|\, \mathrm{d}t < B\,, \quad |y_0(s)| < C\,, \tag{12}$$

$$\int_a^b (v) \int_a^b |L_{\alpha\alpha_1 \ldots \alpha_v}(s t_1 \ldots t_v)|\, \mathrm{d}t_1 \ldots \mathrm{d}t_v \leqq D_{\alpha\alpha_1 \ldots \alpha_v}\,, \quad \max_{\alpha, \ldots, \alpha_v} D_{\alpha\alpha_1 \ldots \alpha_v} = D$$

is valid and denote

$$\max_s |\psi(s)| = \Psi\,, \quad \max(1, B) = L\,.$$

Then for Ψ the following equation can be obtained from (6) and (12):

$$F[\lambda, \Psi] \equiv \Psi - 2LAD \sum_{j=1}^{n} \sum_{\alpha=0}^{j} |\mu_0|^{\alpha} \Big\{ C^j H_2[\Psi] + G_2[\Psi] + G_1[\Psi]\, H_1[\Psi] + (\Psi + C)^j \sum_{k=1}^{\alpha} \binom{\alpha}{k} \Big(\frac{\lambda}{|\mu_0|}\Big)^k \Big\} = 0 \tag{13}$$

where

$$H_m[\Psi] = \sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)} \sum_{l_1=0}^{\alpha_1} \binom{\alpha_1}{l_1} (\Psi C^{-1})^{l_1} \ldots \sum_{l_\nu=0}^{\alpha_\nu} \binom{\alpha_\nu}{l_\nu} (\Psi C^{-1})^{l_\nu},$$

$m = 1$ if $\sum_{i=1}^{\nu} l_i \neq 0$, $m = 2$ if $\sum_{i=1}^{\nu} l_i \neq 0, 1$,

$$G_m[\Psi] = \sum_{k=m}^{\alpha} \binom{\alpha}{k} \Psi^k C^{j-k}, \quad m = 1, 2.$$

Let us look for a solution of (13) in the form of the power series expansion

$$\Psi = \sum_{l=1}^{\infty} \lambda^l k_l. \tag{14}$$

Substituting (14) into (13) and comparing coefficients of the same powers of λ we have the following system of equations for k_l

$$k_1 = 2LAD \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \alpha |\mu_0|^{\alpha-1} C^j, \tag{15}$$

$$k_2 = 2LA \Big\langle D \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \Big\{ |\mu_0|^\alpha C^{j-2} k_1^2 \Big[\binom{\alpha}{2} + \alpha(j-\alpha) + $$

$$+ \sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)} \Big(\sum_{k=1}^{\nu} \binom{\alpha_k}{2} + \sum_{k=1}^{\nu-1} \sum_{m=k+1}^{\nu} \alpha_k \alpha_m \Big) \Big] + \alpha j |\mu_0|^{\alpha-1} C^{j-1} k_1 \Big\} + \overline{M}_2[C] \Big\rangle,$$

$$k_l = 2LA \Big\langle D \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \Big\{ |\mu_0|^\alpha C^{j-2} k_1 k_{l-1} \Big[2 \binom{\alpha}{2} + 2\alpha(j-\alpha) + \alpha j |\mu_0|^{\alpha-1} C^{j-1} k_{l-1} + $$

$$+ \sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)} \Big(2 \sum_{k=1}^{\nu} \binom{\alpha_k}{2} + \sum_{k=1}^{\nu} \sum_{\substack{m=1 \\ m \neq k}}^{\nu} \alpha_k \alpha_m \Big) \Big] \Big\} + \overline{M}_l[C, k_1, k_2, \ldots, k_{l-2}] \Big\rangle$$

where $\overline{M}_l[C, k_1, k_2, \ldots, k_{l-2}]$ are upper bounds for $M_l[y_0, y_1, \ldots, y_{l-2}]$. From these relations and from (9), (10), (11) and (12) it is obvious that

$$|y_l(s)| < k_l \quad \text{for} \quad s \in \langle a, b \rangle, \quad l = \overline{1, \infty}.$$

This implies that the region of convergence of (14) is the region of convergence of (7). From the implicit function theorem according to

$$\frac{\partial F[\lambda, \Psi]}{\partial \Psi} = 1 \quad \text{for} \quad \lambda = \Psi = 0$$

there follows that from (13) it is possible to determine Ψ as an unambiguous and continuous function of λ so that the series (14) has a finite radius of convergence. As the series (14) is a majorant for (7), the series (7) converges absolutely and uniformly according to s and λ in $\langle a, b \rangle$ and in a neighbourhood of the point $\lambda = 0$. Hence, the series (3) represents the unique solution of (1) in the neighbourhood of $\mu = \mu_0$ continuous in $\langle a, b \rangle$ which tends to $y_0(s)$ for $\mu \to \mu_0$.

b) Let number 1 be a p-multiple eigenvalue of the kernel $L(s, t)$ with continuous eigenfunctions $\varphi_i(s)$ $(i = \overline{1, p})$ and with continuous associated eigenfunctions $\alpha_i(s)$ $(i = \overline{1, p})$. If equations (8) are to have solutions it is necessary and sufficient that

$$\int_a^b f_l(s)\, \alpha_i(s)\, ds = 0\,, \quad l = \overline{1, \infty}\,, \quad i = \overline{1, p}\,, \tag{16}$$

be valid.

b1) Let us assume that (16) is valid for $l = 1$. Then the solution of the first equation from (8) can be written according to the third Fredholm's theorem

$$y_1(s) = g_1(s) + \sum_{i=1}^{p} C_i^1\, \varphi_i(s) \tag{17}$$

where

$$g_l(s) = \int_a^b \Phi(s, t) f_l(t)\, dt + f_l(s)\,, \quad (l = \overline{1, \infty}, \text{ for } l = \overline{2, \infty} \text{ see further})\,;$$

$\Phi(s, t)$ is the continuous resolving kernel of the kernel

$$L(s, t) - \sum_{i=1}^{p} \varphi_i(s)\, \alpha_i(t)\,.$$

For the determination of constants C_i^1 $(i = \overline{1, p})$ we obtain from conditions (16) by solving the second equation from (8) after substituting (17) into $f_2(s)$ the following system of p nonlinear equations

$$\begin{aligned}\int_a^b \frac{\alpha_i(s)}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{ s[(g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m)^2\,,\ \ g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] + \\ + \mu_0^\alpha\, y_0^\alpha(s)\, (Q[(g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m)^2] + R[g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m, g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m]) + \\ + \alpha \mu_0^{\alpha-1} y_0^{\alpha-1}(s)\, [y_0(s) + \mu_0(g_1(s) + \sum_{m=1}^{p} C_m^1\, \varphi_m(s))]\, S[g_1 + \sum_{m=1}^{p} C_m^1 \varphi_m]\} \\ + M_2[y_0] \rangle\, ds = 0\,, \quad i = \overline{1, p}\,.\end{aligned} \tag{18}$$

From system (18) we obtain, in general, 2^p systems $C_i^1 (i = \overline{1, p})$. So we generally determine 2^p functions $y_1(s)$

$$y_{1j}(s) = g_1(s) + \sum_{i=1}^{p} C_{ij}^1\, \varphi_i(s)\,, \quad j = \overline{1, 2^p}\,.$$

The solution of the l-th ($l \geqq 2$) equation of system (8) can be written in the form

$$y_l(s) = g_l(s) + \sum_{i=1}^{p} C_i^l \varphi_i(s) . \tag{19}$$

From conditions (16) by solving the ($l + 1$)-st equation from (8) after substituting (19) into $f_{l+1}(s)$ we obtain the nonhomogeneous system of p linear equations for C_i^l

$$\int_a^b \alpha_i(s) K[\sum_{m=1}^{p} C_m^l \varphi_m] \, ds = m_i^l , \quad i = \overline{1, p} \tag{20}$$

where

$$m_i^l = \int_a^b \alpha_i(s) \left(\frac{1}{p(s)} M_{l+1}(y_0, y_1, \ldots, y_{l-1}) - K[g_l] \right) ds .$$

Under the assumption that the determinant of system (20) is different from zero it is possible to determine C_i^l ($i = \overline{1, p}$) uniquely in the form

$$C_i^l = \sum_{k=1}^{p} F_{ik} m_k^l , \quad i = \overline{1, p} , \tag{21}$$

and so to determine the functions $y_{lj}(s)$ ($j = \overline{1, 2^p}$) uniquely.

Therefore it is possible to construct 2^p series of the type (7). The convergence of these series may be proved in the following way. Let us consider two sequences $\{u_i\}_0^\infty$, $\{v_i\}_0^\infty$ of such numbers u_i, v_i that

$$\left.\begin{aligned} |y_1(s)| &\leqq |g_1(s)| + \sum_{i=1}^{p} |C_i^1| \, |\varphi_i(s)| < u_0 + v_0 \\ |y_{l+1}(s)| &\leqq |g_{l+1}(s)| + \sum_{i=1}^{p} |C_i^{l+1}| \, |\varphi_i(s)| < u_l + v_l \end{aligned}\right\} s \in \langle a, b \rangle , \quad l = \overline{1, \infty}$$

holds. Let us choose constants u_0, v_0 so that the inequality mentioned above is valid and for determination of constants u_l, v_l ($l = \overline{1, \infty}$) consider the function

$$E(z) = AD \sum_{j=1}^{n} \sum_{\alpha=0}^{j} |\mu_0|^\alpha \left\{ C^j H_2[z] + G_2[z] + \sum_{k=1}^{\alpha} \binom{\alpha}{k} \left(\frac{\lambda}{|\mu_0|} \right)^k (z + C)^j + \right. \tag{22}$$

$$\left. + G_1[z] \, H_1[z] - \frac{\alpha \lambda}{|\mu_0|} C^j \right\}$$

where the notations $G_1[z]$, $H_1[z]$, $G_2[z]$ and $H_2[z]$ have the same meaning as in (13). If we put instead of z

$$z = \sum_{l=0}^{\infty} \lambda^{l+1} (u_l + v_l) \tag{23}$$

into (22) and if we expand the expression obtained by powers of λ then

$$(24) \qquad E(z) = \sum_{l=2}^{\infty} \lambda^l E_l$$

where

$$E_2 = A\left\langle D\sum_{j=1}^{n}\sum_{\alpha=0}^{j}\left\{|\mu_0|^\alpha C^{j-2}(u_0+v_0)^2\left[\sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)}\left(\sum_{k=1}^{\nu}\binom{\alpha_k}{2}+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\sum_{k=1}^{\nu-1}\sum_{m=k+1}^{\nu}\alpha_k\alpha_m\right)+\binom{\alpha}{2}+\alpha(j-\alpha)\right]+\alpha j|\mu_0|^{\alpha-1}C^{j-1}(u_0+v_0)\right\}+\overline{M}_2[C]\right\rangle,$$

$$E_{l+2} = \overline{K}(u_l+v_l) + A\overline{M}_{l+2}[C, u_0+v_0, \ldots, u_{l-1}+v_{l-1}], \quad l = \overline{1, \infty}$$

where

$$\overline{K} = AD\sum_{j=1}^{n}\sum_{\alpha=0}^{j}\left\{|\mu_0|^\alpha C^{j-2}(u_0+v_0)\left[\sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)}\left(2\sum_{k=1}^{\nu}\binom{\alpha_k}{2}+\sum_{k=1}^{\nu}\sum_{\substack{m=1\\m\neq k}}^{\nu}\alpha_k\alpha_m\right)+\right.\right.$$

$$\left.\left.+2\binom{\alpha}{2}+2\alpha(j-\alpha)+\alpha j|\mu_0|^{\alpha-1}C^{j-1}\right\}\right.$$

and $\overline{M}_{l+2}$ are upper bounds for M_{l+2}. From relations (9) and (12) there follows that E_l are upper bounds for the functions $f_l(s)$.

Further, let us choose such numbers N, Φ, β and β_1 that

$$(25) \qquad \max_{i,k} |F_{ik}| = N,$$

$$\left.\begin{aligned} &\int_a^b |\Phi(s,t)|\, dt < \Phi, \\ &\max_i |\varphi_i(s)| < \beta, \end{aligned}\right\} \quad s \in \langle a, b\rangle$$

$$\max_i \int_a^b |\alpha_i(s)|\, ds = \beta_1$$

is valid and designate $\max(1, \Phi) = K$. Then we determine u_l ($l = \overline{1, \infty}$) from the equation

$$(26) \qquad u_l = 2KE_{l+1}.$$

If we take into account that

$$|g_{l+1}(s)| < \max_s |f_{l+1}(s)|\left(1+\int_a^b|\Phi(s,t)|\, dt\right) < 2KE_{l+1}$$

then $|g_{l+1}(s)| < u_l$.

The constants v_l ($l = \overline{1, \infty}$) can be determined from the equation

$$(1 + d\overline{K})\, v_l = dE_{l+2} \tag{27}$$

where

$$d = p^2 \beta \beta_1 N\,.$$

As there is

$$|C_i^{l+1}| \leqq N \sum_{k=1}^{p} |m_k^{l+1}| < N\beta_1 pQ\,, \quad \sum_{i=1}^{p} |C_i^{l+1}|\,|\varphi_i(s)| < dQ$$

where

$$Q = \overline{K}u_l + A\overline{M}_{l+2}[C, u_0 + v_0, \ldots, u_{l-1} + v_{l-1}]$$

and $v_l = dQ$ according to equation (27), we have $\sum_{i=1}^{p} |C_i^{l+1}|\,|\varphi_i(s)| < v_l$ and so $|y_{l+1}(s)| < u_l + v_l$.

If we introduce the notation

$$u = \sum_{l=1}^{\infty} \lambda^{l+1} u_l\,, \quad v = \sum_{l=1}^{\infty} \lambda^{l+1} v_l\,, \tag{28}$$

then $z = \lambda(u_0 + v_0) + u + v$ and the determination of u_l, v_l from (26) and (27) is equivalent to solving the following system for u, v:

$$\begin{gathered} u = 2KE(\lambda(u_0 + v_0) + u + v)\,, \\ (1 + d\overline{K})\,\lambda v = d(E(\lambda(u_0 + v_0) + u + v) - \lambda^2 E_2) \end{gathered} \tag{29}$$

in the form (28). If we perform the substitution

$$u = \lambda U\,, \quad v = \lambda V$$

in (29) and devide the first equation by λ and the second one by λ^2 we obtain for U, V the system

$$\begin{gathered} \Phi_1 \equiv U - \frac{2K}{\lambda} E(\lambda(u_0 + v_0 + U + V)) = 0\,, \\ \Phi_2 \equiv (1 + d\overline{K})\, V - d\left(\frac{1}{\lambda^2} E(\lambda(u_0 + v_0 + U + V)) - E_2\right) = 0\,. \end{gathered} \tag{30}$$

For system (30) we use the implicit function theorem. If system (30) is to determine unambiguous continuous functions $U(\lambda)$, $V(\lambda)$ in a neighbourhood of the point $\lambda = 0$ it is necessary and sufficient that

$$\Delta = \frac{D(\Phi_1, \Phi_2)}{D(U, V)} \neq 0 \quad \text{for} \quad \lambda = U = V = 0\,. \tag{31}$$

As for $\lambda = U = V = 0$,

$$\frac{\partial \Phi_1}{\partial U} = 1\,, \quad \frac{\partial \Phi_2}{\partial V} = 0\,, \quad \frac{\partial \Phi_2}{\partial U} = -d\bar{K}\,, \quad \frac{\partial \Phi_2}{\partial V} = 1$$

holds and so $\Delta = 1$, the assumptions of the theorem mentioned above are fulfilled and system (30) has only one continuous solution U, V in the form of the series

$$U = \sum_{l=1}^{\infty} \lambda^l u_l\,, \quad V = \sum_{l=1}^{\infty} \lambda^l v_l$$

which have a finite radius of convergence in a neighbourhood of the point $\lambda = 0$. The same is valid for series (23). As this series is a majorant for (7), series (7) converges absolutely and uniformly according to s and λ in $\langle a, b\rangle$ and in a neighbourhood of the point μ_0 and because of the continuity of its terms the limit functions $\psi_j(s)$ are continuous in $\langle a, b\rangle$. Hence, in the neighbourhood of the point $\mu = \mu_0$ there exist, in general, 2^p solutions of equation (1) in the form (3) which tend to $y_0(s)$ for $\mu \to \mu_0$.

b2) If any of conditions (16) for $l = 1$ is not fulfilled it is not possible to solve equations (8) and the problem of determination of the number of solutions of equation (6) for μ from a neighbourhood of μ_0 becomes more complicated. Such solutions can be sought in the form

(32)
$$\psi(s) = \sum_{l=1}^{\infty} (\mu - \mu_0)^{l/k}\, y_l(s)$$

where k is a positive integer. The functions $y_l(s)$ can be determined from a system of linear integral equations obtained with the aid of substitution (32) in (6) and by comparison of coefficients of the same powers of $(\mu - \mu_0)^{1/k}$. For example, for $k = 2$, i.e.

(33)
$$\psi(s) = \sum_{l=1}^{\infty} v^l\, y_l(s)\,, \quad v = (\mu - \mu_0)^{1/2}$$

we obtain the system

(34)
$$y_l(s) - \int_a^b L(s, t)\, y_l(t)\, \mathrm{d}t = h_l(s)\,, \quad l = \overline{1, \infty}$$

where

(35)
$$h_1(s) = 0\,,$$

$$h_2(s) = \frac{-1}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{\mu_0^{\alpha} y_0^{\alpha}(s)\,(Q[y_1^2] + R[y_1, y_1]) +$$

$$+ q[y_1^2]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_v}] + r'[y_1]\, S[y_1]\} + M_2[y_0]\rangle\,,$$

$$h_{l+1}(s) = \frac{-1}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{\mu_0^\alpha y_0^\alpha(s)(2Q[y_1 y_l] + T[y_1, y_l]) +$$

$$+ 2q[y_1 y_l]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] + r'[y_1]\, S[y_l] + r'[y_l]\, S[y_1]\} +$$

$$+ M_{l+1}[y_0, y_1, \dots, y_{l-1}]\rangle = G[y_l] - \frac{1}{p(s)} M_{l+1}[y_0, y_1, \dots, y_{l-1}]\,, \quad l = \overline{2, \infty}\,,$$

$$r'[y_l] = \alpha \mu_0^\alpha\, y_0^{\alpha-1}(s)\, y_l(s)\,, \quad l = \overline{1, \infty}$$

and other notations have the same meaning as in (10).

The solution of the first equation from (34) can be written in the form

$$y_1(s) = \sum_{i=1}^{p} D_i^1\, \varphi_i(s)\,. \tag{36}$$

If other equations from (34) are to have solutions it is necessary and sufficient to fulfil the conditions

$$\int_a^b h_l(s)\, \alpha_i(s)\, \mathrm{d}s = 0\,, \quad i = \overline{1, p}\,, \quad l = \overline{2, \infty}\,. \tag{37}$$

If we substitute (36) into (37) when $l = 2$ we obtain the following system of p non-linear equations for D_i^1 $(i = \overline{1, p})$

$$\int_a^b \frac{\alpha_i(s)}{p(s)} \langle \sum_{j=1}^{n} \sum_{\alpha=0}^{j} \{\mu_0^\alpha y_0^\alpha(s)\, (Q[(\sum_{m=1}^{p} D_m^1 \varphi_m)^2] + R[\sum_{m=1}^{p} D_m^1 \varphi_m, \sum_{m=1}^{p} D_m^1 \varphi_m]) + \tag{38}$$

$$+ q[(\sum_{m=1}^{p} D_m^1 \varphi_m)^2]\, L_j[y_0^{\alpha_1} \dots y_0^{\alpha_\nu}] + r'[\sum_{m=1}^{p} D_m^1 \varphi_m]\, S[\sum_{m=1}^{p} D_m^1 \varphi_m]\} +$$

$$+ M_2[y_0]\rangle\, \mathrm{d}s = 0\,.$$

From (38) we obtain, in general, 2^p systems D_i^1 $(i = \overline{1, p})$ and so we have 2^p functions $y_1(s)$

$$y_{1j}(s) = \sum_{i=1}^{p} D_{ij}^1\, \varphi_i(s)\,, \quad j = \overline{1, 2^p}\,.$$

The solution of the l-th ($l \geqq 3$) equation from (34) can be written

$$y_l(s) = g_l(s) + \sum_{i=1}^{p} D_i^l\, \varphi_i(s) \tag{39}$$

where

$$g_l(s) = \int_a^b \Phi(s, t)\, h_l(t)\, \mathrm{d}t + h_l(s)\,;$$

$\Phi(s, t)$ has the same meaning as in the section b1). For D_i^l $(i = \overline{1, p})$ we obtain from conditions (37) after substituting (39) into $h_{l+1}(s)$ the system of p linear equations

$$(40) \qquad \sum_{m=1}^{p} D_m^l \int_a^b \alpha_i(s)\, G[\varphi_m]\, ds = d_i^l, \quad i = \overline{1, p}$$

where

$$d_i^l = \int_a^b \alpha_i(s) \left(\frac{1}{p(s)} M_{l+1}[y_0, y_1, \ldots, y_{l-1}] - G[g_l] \right) ds .$$

Under the assumption that the determinant of the system (40) is different from zero it is possible to determine D_i^l uniquely in the form

$$(41) \qquad D_i^l = \sum_{k=1}^{p} H_{ik} d_k^l$$

and so to determine the functions $y_{lj}(s)$ $(j = \overline{1, 2^p}$ uniquely.

Therefore it is possible to construct, in general, 2^p series of the type (33). The proof of convergence of these series in a neighbourhood of the point $v = 0$ will be carried out analogically as that in the section b1). Let us choose such a constant v_0 that

$$|y_1(s)| \leqq \sum_{i=1}^{p} |D_i^1|\, |\varphi_i(s)| < v_0 \quad \text{for} \quad s \in \langle a, b \rangle$$

is valid. Further, let us consider the function

(42)

$$S(z) = AD \sum_{j=1}^{n} \sum_{\alpha=0}^{j} |\mu_0|^\alpha \left\{ C^j H_2[z] + G_2[z] + \sum_{k=1}^{\alpha} \binom{\alpha}{k} \left(\frac{v^2}{|\mu_0|} \right)^k (z + C)^j + \right.$$
$$\left. + G_1[z]\, H_1[z] \right\}$$

where the symbols $G_1[z]$, $G_2[z]$, $H_1[z]$, $H_2[z]$ have the same meaning as in (13). If we put instead of z

$$(43) \qquad z = v v_0 + \sum_{l=1}^{\infty} v^{l+1}(u_l + v_l)$$

in $S(z)$ and expand the expression obtained in powers of v, we obtain

$$(44) \qquad S(z) = \sum_{l=2}^{\infty} v^l S_l$$

where S_l are upper bounds for the functions $h_l(s)$ if $(u_{l-1} + v_{l-1})$ are upper bounds for $y_l(s)$ $(l = \overline{2, \infty})$.

Let us choose such a number H that

$$\max_{i,k} |H_{ik}| = H$$

holds. Then we determine u_l and v_l $(l = \overline{1, \infty})$ from the equations

$$u_l = 2KS_{l+1}, \quad (1 + e\bar{G})\, v_l = eS_{l+2} \tag{45}$$

where $e = p^2\beta\beta_1 H$ and

$$\bar{G} = AD\sum_{j=1}^{n}\sum_{\alpha=0}^{j}|\mu_0|^\alpha C^{j-2}v_0\left\{\sum_{(\alpha_1+\ldots+\alpha_\nu=j-\alpha)}\left(2\sum_{k=1}^{\nu}\binom{\alpha_k}{2}+\sum_{k=1}^{\nu}\sum_{\substack{m=1\\ m\neq k}}^{\nu}\alpha_k\alpha_m\right)+\right.$$

$$\left.+\,2\binom{\alpha}{2}+2\alpha(j-\alpha)\right\}.$$

We can easily see that

$$|y_{l+1}(s)| < u_l + v_l\,.$$

If we introduce the notation

$$U = \sum_{l=1}^{\infty} \nu^l u_l\,, \quad V = \sum_{l=1}^{\infty} \nu^l v_l\,, \tag{46}$$

then $z = \nu(v_0 + U + V)$ and the determination of u_l, v_l from equations (45) is equivalent to the solution of the system for U, V

$$\Phi_1 \equiv U - \frac{2K}{\nu} S(\nu(v_0 + U + V)) = 0 \tag{47}$$

$$\Phi_2 \equiv (1 + e\bar{G})\, V - e\left(\frac{1}{\nu^2} S(\nu(v_0 + U + V)) - S_2\right) = 0$$

in the form (46). As for $\nu = U = V = 0$, there is $\Delta = 1$, the assumption (31) is fulfilled and from system (47) it is possible to determine U and V as unambiguous and continuous functions of ν. From analogical considerations as in the section b1) there follows that the series (33) converges absolutely and uniformly according to s and ν in $\langle a, b\rangle$ and in a neighbourhood of the point $\nu = 0$ to functions $(y_j(s) - y_0(s))$ $(j = \overline{1, 2^p})$ which are continuous in $\langle a, b\rangle$.

Hence, the assertions of the theorem are proved.

Reference

[1] *W. Schmeidler:* Algebraische Integralgleichungen I., Math. Nachrichten, *8* (1952), 31—40.

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