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AN $\mathscr{L}^*$-CONVERGENCE IN DIFFERENTIAL EQUATIONS

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0. In what follows, notations introduced in [1] are used. Especially, $\mathscr{N} = \{1, 2, \ldots\}$; $I = \langle \tau, \tau + \alpha \rangle$, $\alpha > 0$, denotes a fixed compact interval, and G is a region in $\mathscr{R}^n$, for some fixed $n \in \mathscr{N}$. Further, all measurability notions refer to the Lebesgue measure on $\mathscr{R} = \mathscr{R}^1$. A function having a constant value ξ on a domain considered will be denoted by $\hat{\xi}$. The symbol $\mathbf{C}(I; G)$ denotes the set of all continuous mappings from I to G, equipped with the uniform convergence on I; the set of all Lebesgue integrable functions on I will be denoted by $\mathbf{L}(I)$.

The set of all Carathéodory operators T on $\mathbf{C}(I; G)$ (see [1] for this notion) such that the equation

$$x(t) = \xi + \int_\tau^t \mathsf{T}x \tag{0.1}$$

has, for each $\xi \in G$, exactly one solution $\varphi(\,.\,; \xi)$ defined over I, will be denoted by Γ.

For completness, let us state here the Lebesgue-Vitali convergence theorem in the form used in this note: *Let*

1° $f_i \in \mathbf{L}(I)$, $i \in \mathscr{N}$,

2° $f_i \geqq 0$, $i \in \mathscr{N}$,

3° *there exists f such that f_i converge to f asymptotically on I.*

Then $f \in \mathbf{L}(I)$ and $\lim \int_I f_i = \int_T f$ *iff the system $\{\int_\tau^t f_i\}$, $i \in \mathscr{N}$, is equi-AC on I, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that* $\tau \leqq a_1 < b_1 \leqq a_2 < b_2 \leqq \ldots \leqq a_r < b_r \leqq \leqq \tau + \alpha$, $\sum_{j=1}^{r} (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{r} \left|\int_{a_j}^{b_j} f_i\right| < \varepsilon$, *independently of $i \in \mathscr{N}$.*

For the proof, see e.g. [4].

1. This investigation starts from the following theorem on a necessary and sufficient condition for continuous dependence on a parameter, which solves a problem posed in [2].

(1,1) Theorem. *Let $A_i(n \times n)$, $b_i(n \times 1)$, $i \in \mathscr{N}$, be Lebesgue integrable matrices* *on I such that*

1° $A_i \to 0$, $b_i \to 0$ *asymptotically on* I,

2° $A_i \geqq 0$, $b_i \geqq 0$, $i \in \mathcal{N}$.

Let $\xi = [\xi^1, \ldots, \xi^n] \in \mathscr{R}^n$ *be positive, i.e.* $\xi^k > 0$, $k = 1, \ldots, n$, *and let* $\varphi_i(.\,; \xi)$ *be the solution of*

$$x(t) = \xi + \int_\tau^t (Ax + b)$$

defined over I.

Then

3° $\varphi_i(.\,; \xi) \to \xi$ *uniformly on* I

iff

4° $\{\int_\tau^t A_i\}$, $\{\int_\tau^t b_i\}$ *are equi-AC on I.*

Proof. Let $A_i = (a_i^{kl})$, $b_i = (b_i^k)$, $k, l = 1, \ldots, n$. Let 3° be fulfilled. Then, denoting $\varphi_i = [\varphi_i^1, \ldots, \varphi_i^n]$, we have evidently

$$\varphi_i^k(t) = \xi^k + \int_\tau^t \left(\sum_{j=1}^n a_i^{kj}\varphi_i^j + b_i^k\right) \geqq \xi^k + \xi^l \int_\tau^t a_i^{kl} + \int_\tau^t b_i^k \geqq \xi^k\,,$$

hence $\int_I a_i^{kl} \to 0$, $\int_I b_i^k \to 0$. The assertion 4° now follows from the Lebesgue-Vitali theorem.

The step 4° $\Rightarrow$ 3° follows immediately from a more general Theorem 8,4 of [1].

Theorem (1,1) may also be stated as follows.

(2,1) Theorem. *Let* 2° *be fulfilled. Then* 3° *holds iff*

5° $\int_I A_i \to 0$, $\int_I b_i \to 0$.

Proof. From the proof of Theorem (1,1) we see that 2° & 3° $\Rightarrow$ 5°. On the other hand, it is known that 2° & 5° $\Rightarrow$ 1°.

Remark. The positivity of ξ is substantial in the first part of the proof; for, on taking $n = 1$, we have for $\xi \leqq 0$ that $0 = \dot{\xi} = a(t)\,\xi + (-\xi)\,a(t)$ for each $a \in \mathbf{L}(I)$.

2. The above theorem may be given another form, using some more abstract notions.

Recall that a set $\mathscr{E}$ is called an $\mathscr{L}^*$-space (see [3]) iff there are distinguished sequences $\{p_i\} \in \mathscr{E}^{\mathcal{N}}$ called convergent such that the following is fulfilled:

1° if $\{p_i\}$ converges to $p \in \mathscr{E}$, $\lim p_i = p$ in symbol, and if $k_1 < k_2 < \ldots$, then $\lim p_{k_i} = p$,

2° if $p_i = p$, then $\lim p_i = p$,

3° if $\lim p_i \neq p$, then there exist $k_1 < k_2 < \ldots$ such that no subsequence of $\{p_{k_i}\}$ converges to p.

Clearly, each subset of $\mathscr{E}$ is an $\mathscr{L}^*$-space, too.

Let us show a procedure for introducing an $\mathscr{L}^*$-space structure onto a set $\mathscr{E} \neq \emptyset$. Let $\mathscr{C}$ be an $\mathscr{L}^*$-space; for each $p \in \mathscr{E}$, let $\sum_p \neq \emptyset$ be a set of mappings $S : \mathscr{E} \to \mathscr{C}$. When $\{p_i\} \in \mathscr{E}^{\mathscr{N}}$, we write $\lim p_i = p$ iff $\lim S(p_i) = S(p)$, for each $S \in \sum_p$. It is easy to show that 1°, 2°, 3° of the definition are fulfilled.

3. Let $\mathscr{E}$ be an $\mathscr{L}^*$-space, $p \in \mathscr{E}$. We say that $\varrho : \mathscr{E} \to \mathscr{R}$ is an almostmetric at p iff $\lim p_i = p \Leftrightarrow \lim \varrho(p_i) = 0$. We say that $\mathscr{E}$ is almostmetrical iff for each $p \in \mathscr{E}$ there exists an almostmetric at p.

Evidently, when $\mathscr{E}$ is a metrizable topological space with a corresponding metric d, then $x \to d(x, p)$, $x \in \mathscr{E}$, is an almostmetric at p for the induced $\mathscr{L}^*$-structure.

In general, it seems difficult to give necessary and sufficient conditions for an $\mathscr{L}^*$-space to be almostmetrical. In section 5, we give an example pertinent to linear differential equations.

4. Let $\mathsf{T} \in \Gamma$. For each $\xi \in G$, let $S_\xi(\mathsf{T}) = \varphi(.; \xi)$ be the solution of (0.1) over I. To introduce a natural $\mathscr{L}^*$-structure onto Γ, we apply the construction of section 2. Let $\mathsf{T}, \mathsf{T}_i \in \Gamma$, $i \in \mathscr{N}$. We put $\mathscr{C} = \mathbf{C}(I; G)$ and $\sum_{\mathsf{T}} = \{S_\xi;\ \xi \in G\}$; i.e., we write $\lim \mathsf{T}_i = \mathsf{T}$ iff $\lim S_\xi(\mathsf{T}_i) = S_\xi(\mathsf{T})$ uniformly on I, for each $\xi \in G$. It would be of interest to decide whether this structure is almostmetrical. In the next section we define an almostmetric at a point of a subset of Γ.

5. Let $G = \mathscr{R}^n$. Let $\Lambda^+ = \{\mathsf{T} \in \Gamma;\ \mathsf{T}\varphi = [A\varphi + b]$, with A, $b \in \mathbf{L}(I)$ and non-negative a.e. on $I\}$. Using Corollary 8,2 of [1], we see easily that $\mathsf{T} \in \Lambda^+$ iff

1° $\mathsf{U}\varphi = \mathsf{T}\varphi - \mathsf{T}\hat{0}$ is linear,

2° $\varphi \geqq 0 \Rightarrow \mathsf{T}\varphi \geqq 0$.

For each $\mathsf{T} \in \Lambda^+$, put $\varrho(\mathsf{T}) = \sum_{k=1}^{n} \sum_{l=1}^{n} \int_I a^{kl} + \sum_{k=1}^{n} \int_I b^k$.

(5,1) Theorem. ϱ is an almostmetric at 0 of Λ^+.

Proof. This follows from Theorem (2,1) and Theorem 8,4 of [1].

References

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