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STABILITY OF ABSTRACT DIFFERENTIAL EQUATIONS AT CONSTANTLY ACTING DISTURBANCES

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0. INTRODUCTION

0.1. Speaking about the mathematical description of any object or phenomena, we must remember that, generally, every such description neglects a number of influences, which one does not hold for important or the nature of which is not known. That is why some differences between the behaviour of a model and the real object can arise. From the mathematical point of view we can consider all the neglected influences as constantly acting disturbances, which have to be small in order that the model might represent a good description of reality. But even if disturbances are small enough, the model need not approximate the behaviour of the real object. Mathematically we speak about the stability and instability at constantly acting disturbances.

0.2. In connection with stability at constantly acting disturbances of solutions of ordinary differential equations, several authors (see for example [6]) have proved theorems on linearization. This means, roughly speaking, that they replace the non-linear terms in the equations by their linear approximations and then, assuming that a certain solution of this approximate equation is uniformly asymptotically stable, they prove uniform stability at constantly acting disturbances of the solution of the original equation. There are also results concerning partial differential equations, see for example [7].

0.3. The aim of this paper is to derive results on the linearization principle in the stability at constantly acting disturbances of solutions of abstract differential equations, which could be applied to partial differential equations as well.

0.4. This paper is closely connected with the papers [1–3]. In [1] Lyapunov stability and stability at constantly acting disturbances of solutions of the abstract differential equation

$$(0.4) \qquad \mathscr{L}\, u(t) \equiv u^{(n)}(t) + a_1(A)\, u^{(n-1)}(t) + \ldots + a_n(A)\, u(t) = F(t, u(t))$$

were investigated. The right-hand side F was assumed to be continuous as a function of t in the norm of the space $\mathscr{D}(A^{1/n})$. The paper [2] deals with Lyapunov stability and [3] with stability at constantly acting disturbances of solutions of (0.4) with the right-hand side F continuously differentiable in t in the norm of the space $\mathscr{D}(A^0)$. Stability at constantly acting disturbances of the abstract differential equation

$$u'(t) - A\,u(t) = B(t)\,u(t) + N(t)\,u(t)$$

with linear operators A and B and a nonlinear operator N was studied also in [8]. (The author assumes that the set of all admissible disturbances contains $\mathscr{D}(A)$.) The approach presented in this paper allows us to investigate the cases when the smoothness of the disturbance does not depend on the properties of the right-hand side of (0.4). This is not possible when the theory from [1] and [3] is applied, since there the disturbances are supposed to be from the same functional space as the right-hand side is. Special examples, illustrating the advantage of the present approach, are given in Section 4.

0.5. The paper consists of Introduction and four sections. Section 1 contains the precise formulation of the problem. Two abstract theorems on stability at constantly acting disturbances are given in Sections 2 (Theorem 2.2) and 3 (Theorem 3.2). Section 2 corresponds to the case of exponential stability, when every solution of the equation $\mathscr{L}\,u(t) = 0$ is uniformly exponentially stable. On the other hand, the case of stability, when every solution of the equation $\mathscr{L}\,u(t) = 0$ is uniformly stable, is investigated in Section 3. Theorem 2.2 is an elementary result which asserts, roughly speaking, that if the operator $\mathscr{L}$ is exponentially stable and the right-hand side of the equation is "small" enough, then the solution is stable at constantly acting disturbances. The main result is given in Theorem 3.2 where the operator $\mathscr{L}$ is supposed to be stable only. In that case an arbitrarily small disturbance can cause either the stability or the instability. Therefore we must investigate the influence of the prevailing part of the right-hand side, roughly speaking, its linear part. From this point of view, Theorem 3.2 can be regarded as a linearization theorem. The aim of Section 4 is to illustrate the preceding sections and to show that the assumptions of the theorems from Sections 2, 3 are fulfilled if either the right-hand side F is continuous in the norm of the space $\mathscr{D}(A^{1/n})$ or F is continuously differentiable in t in the norm of the space $\mathscr{D}(A^0)$, and if the disturbances are of the form $R_1 + R_2$, where R_1 is continuous in the norm of the space $\mathscr{D}(A^{1/n})$ and R_2 is continuously differentiable in t in the norm of the space $\mathscr{D}(A^0)$.

Results derived in this paper can be applied to a wide range of partial differential equations, for example to the wave equation $u_{tt} - u_{xx} = F(t, x, u, u_t, u_x)$, to the beam equation $u_{tt} + u_{xxxx} = F(t, x, u, u_t, u_x, u_{xx}, u_{xxx})$, to the Timoshenko type equation and others, with the right-hand side smooth together with its either time or space derivatives, or with the sum of such functions. (It suffices to choose the operator A as a suitable differential operator; see also Remark 4.7.) Of course, these results can be applied to the equations with bounded coefficients (ordinary differential equations) as well.

1. BASIC NOTIONS

1.1. Let A be a linear, selfadjoint, and strictly positive operator in a Hilbert space H with a norm $\|\cdot\|$. (The strict positivity means that $\inf\{s \mid s \in Sp(A)\} > 0$, where $Sp(A)$ is the spectrum of the operator A.)

1.2. We shall define functions of the operator A in the standard way: if $a(s)$ is a continuous, complex-valued function defined on $Sp(A)$, then

$$a(A)\, x = \int_{Sp(A)} a(s)\, \mathrm{d}E(s)\, x$$

for

$$x \in \mathscr{D}(a(A)) := \left\{ x \in H \,\middle|\, \int_{Sp(A)} |a(s)|^2\, \mathrm{d}\|E(s)\, x\|^2 < +\infty \right\},$$

where $E(s)$ is the spectral resolution of identity corresponding to the operator A

1.3. Definition. Let

$$\mathscr{U} := \{u: I \to H \mid I = [\tau, +\infty),\ \tau \in \mathbb{R}^+,\ u^{(i)}(t) \in C(I, \mathscr{D}(A^{(n-i)/n}))$$
$$\text{for} \quad i = 0, \ldots, n-1\}, \quad \text{where} \quad \mathbb{R}^+ = [0, +\infty)\,.$$

1.4. Definition. Let $u \in \mathscr{U}$, $t \in \mathscr{D}(u)$ and let $\|\cdot\|_N$ be a norm on the space $\mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$. Then we set

$$|||u(t)||| = \|(u(t), u'(t), \ldots, u^{(n-1)}(t))\|_N\,.$$

1.5. Definition. Let $a_i(s)$ $(i = 0, \ldots, n)$ be continuous, complex-valued functions defined on $Sp(A)$ such that $a_0(s) \equiv 1$ and there exists a constant C_0^* such that $|a_i(s)\, s^{-i/n}| \leqq C_0^*$ for $s \in Sp(A)$ $(i = 1, \ldots, n)$. Then we define an operator $\mathscr{L}$ by the relation

$$\mathscr{L}\, u(t) \equiv \sum_{i=0}^{n} a_i(A)\, u^{(n-i)}(t)$$

for $u \in \mathscr{D}(\mathscr{L}) := \{u \in \mathscr{U} \mid u^{(n)}(t) \in C(\mathscr{D}(u), \mathscr{D}(A^0))\}$.

1.6. Definition. Let $0 \leqq \alpha \leqq n$, $i \in [\max(1, \alpha), n]$ be integers, $I \subseteq \mathbb{R}^+$ an interval and G a normed space with a norm $\|\cdot\|_G$. Let

$$F\colon \mathscr{D}(F) \subseteq I \times \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{(n-(n-i))/n}) \to H .$$

We shall write $F \in \mathscr{C}^{(\alpha)}(I, G)$ if for every $u \in \mathscr{D}(\mathscr{L})$ such that $\mathscr{D}(u) \subseteq I$ the function

$$\|(F(t, u(t), \ldots, u^{(n-i)}(t)),\ F'(t, u(t), \ldots, u^{(n-i)}(t)), \ldots, F^{(\alpha)}(t, u(t), \ldots, u^{(n-i)}(t)))\|_G$$

is defined and continuous in t for $t \in \mathscr{D}(u)$. Here $F^{(i)}$ is the total derivative of the i-th order of F with respect to t. For simplicity we shall write $F^{(k)}(t, u(t))$ instead of $F^{(k)}(t, u(t), \ldots, u^{(n-i)}(t))$.

1.7. Definition. Let $0 \leqq \alpha \leqq n$, $i \in [\max(1, \alpha), n]$ be integers, $v \in \mathscr{D}(\mathscr{L})$, $r > 0$ and let G be a normed space with a norm $\|\cdot\|_G$. Let

$$R\colon \mathscr{D}(R) \subseteq \mathscr{D}(v) \times \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{(n-(n-i))/n}) \to H .$$

We shall write $R \in \mathscr{B}^{(\alpha)}(v, r, G)$ if for every $u \in \mathscr{D}(\mathscr{L})$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ the function

$$\|(R(t, u(t), \ldots, u^{(n-i)}(t)),\ R'(t, u(t), \ldots, u^{(n-i)}(t)), \ldots, R^{(\alpha)}(t, u(t), \ldots, u^{(n-i)}(t)))\|_G$$

is defined and continuous in t for $t \in \{t \in \mathscr{D}(u) \mid |||u(t) - v(t)||| \leqq r\}$. Here again $R^{(i)}$ is the total derivative of the i-th order of R with respect to t. For simplicity we shall write $R^{(k)}(t, u(t))$ instead of $R^{(k)}(t, u(t), \ldots, u^{(n-i)}(t))$.

1.8. Definition. We say that u *is a solution of the equation*

$$(1.8) \qquad \mathscr{L}\, u(t) = F(t, u(t))\,, \quad F \in \mathscr{C}^{(\alpha)}(I, G)$$

if $u \in \mathscr{D}(\mathscr{L})$, $\mathscr{D}(u) \subseteq I$ and if u fulfils (1.8) for every $t \in \mathscr{D}(u)$.

1.9. Definition. Let v be a solution of (1.8). The equation

$$(1.9) \qquad \mathscr{L}\, u(t) = F(t, u(t)) + \sum_{i=1}^{k} R_i(t, u(t))\,, \quad F \in \mathscr{C}^{(\alpha)}(I, G)\,,$$
$$R_i \in \mathscr{B}^{(\alpha_i)}(v, r_i, G_i)\,, \quad i = 1, \ldots, k\,,$$

will be called *the disturbed equation corresponding to* (1.8), and the functions R_i will be called *constantly acting disturbances* or simply *disturbances*.

We say that u is a solution of the disturbed equation (1.9), if $u \in \mathscr{D}(\mathscr{L})$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $|||u(t) - v(t)||| \leqq \min_{i=1,\ldots,k} r_i$ for every $t \in \mathscr{D}(u)$ and if u fulfils (1.9) for every $t \in \mathscr{D}(u)$.

1.10. Definition. Let $\tau \in \mathbb{R}^+$. Then the symbol $\mathscr{O}_{/[\tau, +\infty)}$ will stay for the function from $\mathscr{D}(\mathscr{L})$ such that $\mathscr{D}(\mathscr{O}_{/[\tau,+\infty)}) = [\tau, +\infty)$ and $\mathscr{O}_{/[\tau,+\infty)} \equiv 0$.

1.11. Definition. Let v be a solution of (1.8). We say that v is *uniformly stable at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|_{G_1}, \ldots, \|\cdot\|_{G_k}$

if for every $\eta \in (0, \min_{i=1,\dots,k} r_i]$ there exist positive constants η_0, η_D so that under the conditions (i), (ii) there exists a solution u of the disturbed equation (1.9) (corresponding to (1.8)) which fulfils the initial conditions $u^{(i)}(\tau) = \varphi_i$ $(i = 0, \dots, n-1)$, and if u is such a solution, then

$$|||u(t) - v(t)||| \leqq \eta \quad \text{for all} \quad t \geqq \tau\,.$$

The conditions (i), (ii) read

(i) $\tau \in \mathscr{D}(v)$ is an arbitrary initial time and $(\varphi_0, \varphi_1, \dots, \varphi_{n-1}) \in \mathscr{D}(A) \times \times \mathscr{D}(A^{(n-1)/n}) \times \dots \times \mathscr{D}(A^{1/n})$ are arbitrary initial conditions fulfilling the relation

$$\|(v(\tau) - \varphi_0,\ v'(\tau) - \varphi_1, \dots, v^{(n-1)}(\tau) - \varphi_{n-1})\|_N \leqq \eta_0\,;$$

(ii) $R_i(t, u(t))$ are arbitrary functions (disturbances) such that

$$\|(R_i(t, u(t)), R_i'(t, u(t)), \dots, R_i^{(\alpha_i)}(t, u(t)))\|_{G_i} \leqq \eta_D \tag{1.11}$$

$(i = 1, \dots, k)$ for $u \in \mathscr{D}(\mathscr{L})$ and such $t \in \mathscr{D}(u)$ that $|||u(t) - v(t)||| \leqq \eta$.

Let us remark that the condition (i) is a restriction to such solutions which initiate sufficiently close to the solution v, while the condition (ii) means that disturbances are small enough.

1.12. Theorem. *Let v be a solution of (1.8). Then v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{G_1}, \dots, \|\cdot\|_{G_k}$ if and only if the solution $\mathcal{O}_{/\mathscr{D}(v)}$ of*

$$\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))\,, \quad F \in \mathscr{C}^{(\alpha)}(I, G) \tag{1.12}$$

is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{G_1}, \dots, \|\cdot\|_{G_k}$.

Proof. Let us write solutions u of (1.9) in the form $u = v_{/\mathscr{D}(u)} + w$. Then the function w fulfils

$$\mathscr{L}\, w(t) = F(t, v(t) + w(t)) - F(t, v(t)) + \sum_{i=1}^{k} \tilde{R}_i(t, w(t))\,,$$

where $\tilde{R}_i(t, w(t)) = R_i(t, v(t) + w(t)) \in \mathscr{B}^{(\alpha_i)}(\mathcal{O}_{/\mathscr{D}(v)}, r_i, G_i)$, $i = 1, \dots, k$. The theorem now follows from the relation $u - v_{/\mathscr{D}(u)} = w$.

1.13. Let v be a solution of (1.8). As we shall not deal with the existence of solutions of the disturbed equation, we shall always suppose that the following condition is fulfilled:

(A) There exists a number $R_0 > 0$ such that if $\tau \in \mathscr{D}(v)$, $(\varphi_0, \dots, \varphi_{n-1}) \in \mathscr{D}(A) \times \dots \times \mathscr{D}(A^{1/n})$ and $\|(\varphi_0, \dots, \varphi_{n-1})\|_N \leqq R_0$, then there exists a solution u of the disturbed equation corresponding to (1.2):

$$(1.13) \qquad \mathscr{L} u(t) = F(t, v(t) + u(t)) - F(t, v(t)) + \sum_{i=1}^{k} R_i(t, u(t)),$$

$$F \in \mathscr{C}^{(\alpha)}(I, G), \; R_i \in \mathscr{B}^{(\alpha_i)}(\mathcal{O}_{/\mathscr{D}(v)}, r_i, G_i) \quad (i = 1, \ldots, k),$$

fulfilling the initial conditions $u^{(i)}(\tau) = \varphi_i$, $i = 0, \ldots, n - 1$.

2. THE CASE OF EXPONENTIAL STABILITY

2.1. In the case of exponential stability we shall suppose that the disturbed euqation is such that its solutions can be estimated as in the following condition (B):

(B) There exist constant $C_1 \geqq 0$, $C_2 \geqq 0$, $R > 0$, $\omega < 0$ and continuous functions $s_i(t) \geqq 0$, $\mathscr{D}(s_i) = \mathbb{R}^+$ $(i = 1, \ldots, k)$ such that if u is a solution of (1.13) and t_0, $t \in \mathscr{D}(u)$ are such that $|||u(\sigma)||| \leqq R$ for all $\sigma \in [t_0, t]$, then the following estimate holds:

$$|||u(t)||| \leqq C_1 |||u(t_0)||| \, e^{\omega(t-t_0)} + C_2 \int_{t_0}^{t} e^{\omega(t-\sigma)} |||u(\sigma)||| \, d\sigma +$$

$$+ \sum_{i=1}^{k} s_i(t - t_0) \max_{\sigma \in [t_0, t]} \|(R_i(\sigma, u(\sigma)), \ldots, R_i^{(\alpha_i)}(\sigma, u(\sigma)))\| G_i .$$

In Section 4 the condition (B) and an analogous condition for the stable case are reformulated in terms of the right-hand sides or their derivatives.

2.2. Theorem. *Let v be a solution of (1.8). Let us assume that the condition (B) is fulfilled and let $\omega + C_2 < 0$. Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{G_1}, \ldots, \|\cdot\|_{G_k}$.*

Proof. By Theorem 1.12 it suffices to prove uniform stability at constantly acting disturbances of the solution $\mathcal{O}_{/\mathscr{D}(v)}$ of (1.12). Let $\eta \in (0, \min_{i=1,\ldots,k} r_i]$ be given. Without loss of generality we may suppose $\eta \leqq \min(R_0, R)$, (R_0 is the number from the condition (A), R is that from (B)). Let us take a number $h > 0$ such that

$$C_1 e^{(\omega + C_2)h} < 1 .$$

Further, let us find numbers $\eta_0 \in (0, \eta/2]$, $\eta_D > 0$ in such a way that

$$(1) \qquad C_1 \eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma) \, e^{-\omega h} \leqq \frac{\eta}{2}$$

and

$$(2) \qquad [C_1 \eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma) \, e^{-\omega h}] \, e^{(\omega + C_2)h} \leqq \eta_0 .$$

Let $\tau \in \mathscr{D}(v)$, $(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$, $\|(\varphi_0, \varphi_1, \ldots \ldots, \varphi_{n-1})\|_N \leqq \eta_0$. Then in virtue of (A) and of the relation $\eta_0 \leqq R_0$ it suffices to prove that if u is a solution of the disturbed equation corresponding to (1.12) and satisfying $u^{(i)}(\tau) = \varphi_i$ $(i = 0, \ldots, n-1)$, then $|||u(t)||| \leqq \eta$ for all $t \geqq \tau$. Let us suppose:

(3) There exists a number $\tilde{h} < h$ such that $|||u(\sigma)||| < \eta$ for $\sigma \in [\tau, \tau + \tilde{h})$, $|||u(\tau + \tilde{h})||| = \eta$.

Then assuming that (1.11) holds we get by (B):

$$|||u(t)||| \leqq (C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma)\, e^{-\omega h})\, e^{\omega(t-\tau)} +$$

$$+ C_2 \int_{t_0}^{t} e^{\omega(t-\sigma)}\, |||u(\sigma)|||\, d\sigma \quad \text{for} \quad t \in [\tau, \tau + \tilde{h}],$$

and thus using Gronwall's lemma we obtain

$$|||u(t)||| \leqq (C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma)\, e^{-\omega h})\, e^{(\omega + C_2)(t-\tau)} \quad \text{for} \quad t \in [\tau, \tau + \tilde{h}]. \tag{4}$$

This with help of the relations $\omega + C_2 < 0$ and (1) implies

$$|||u(t)||| \leqq C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma)\, e^{-\omega h} \leqq \eta/2 < \eta$$

for $t \in [\tau, \tau + \tilde{h}]$, which contradicts (3). So we have proved

$$|||u(t)||| \leqq \eta \quad \text{for} \quad t \in [\tau, \tau + h].$$

Further, by (2), (4) we have

$$|||u(\tau + h)||| \leqq \eta_0. \tag{5}$$

The relation (5) enables us to apply the same procedure m-times. This yields

$$|||u(t)||| \leqq \eta \quad \text{for} \quad t \in [\tau + (m-1)\,h, \tau + mh] \quad (m = 1, 2, \ldots). \tag{6}$$

Now, for an arbitrary $t \in \mathscr{D}(u)$ we can find an integer m and a number $s \in [0, h)$ such that $t = \tau + (m-1)\,h + s$. This with help of (6) proves the theorem.

3. THE CASE OF STABILITY

3.1. As we have mentioned in **0.5**, it will be necessary to consider the influence of the prevailing part of the right-hand side F in this case. That is why we decompose F in the sense of the following condition (C):

(C) $$F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t)),$$

where $F, F_L, F_N \in \mathscr{C}^{(\alpha)}(I, G)$, for $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$. In this section we shall suppose that the condition (C) is fulfilled. Then (1.12) and (1.13) have the form

$$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) \tag{3.1.1}$$

and

$$\mathscr{L}\, u(t) = F_L(t, u(t)) + F_N(t, u(t)) + \sum_{i=1}^{k} R_i(t, u(t)), \tag{3.1.2}$$

respectively. Let us further denote

$$\mathscr{L}\, u(t) = F_L(t, u(t)). \tag{3.1.3}$$

Finally, we shall need an estimate of a solution of (3.1.2) and an estimate for the difference between the solution of the disturbed equation (3.1.2) and the solution of the "linearized" equation (3.1.3). Both these estimates are included in the following condition (D):

(D) There exist nonnegative constants C_i $(i = 1, \ldots, 6)$, $\nu_1 > 0$, $\nu_2 > 0$, $R > 0$ and continuous functions $s_i(t) \geqq 0$, $z_i(t) \geqq 0$, $\mathscr{D}(s_i) = \mathscr{D}(z_i) = \mathbb{R}^+$ $(i = 1, \ldots, k)$ such that if

(i) u_D is a solution of (3.1.2), $t_0 \in \mathscr{D}(u_D)$,

(ii) u_L is a solution of (3.1.3) such that $[t_0, +\infty) \subseteq \mathscr{D}(u_L)$ and $u_L^{(i)}(t_0) = u_D^{(i)}(t_0)$ $(i = 0, \ldots, n-1)$,

(iii) $t \geqq t_0$ is such that $|||u_D(\sigma)||| \leqq R$, $|||u_L(\sigma)||| \leqq R$ for all $\sigma \in [t_0, t]$,

then the following inequalities are satisfied:

$$|||u_D(t)||| \leqq C_1 |||u_D(t_0)||| + C_2 \int_{t_0}^{t} |||u_D(\sigma)|||\, d\sigma + C_3 \int_{t_0}^{t} |||u_D(\sigma)|||^{1+\nu_1}\, d\sigma +$$
$$+ \sum_{i=1}^{k} s_i(t - t_0) \max_{\sigma \in [t_0, t]} \|(R_i(\sigma, u_D(\sigma)), R_i'(\sigma, u_D(\sigma)), \ldots, R_i^{(\alpha_i)}(\sigma, u_D(\sigma)))\|_{G_i},$$

$$|||u_D(t) - u_L(t)||| \leqq C_4 |||u_D(t_0)|||^{1+\nu_2} + C_5 \int_{t_0}^{t} |||u_D(\sigma) - u_L(\sigma)|||\, d\sigma +$$
$$+ C_6 \int_{t_0}^{t} |||u_D(\sigma)|||^{1+\nu_2}\, d\sigma +$$
$$+ \sum_{i=1}^{k} z_i(t - t_0) \max_{\sigma \in [t_0, t]} \|(R_i(\sigma, u_D(\sigma)), R_i'(\sigma, u_D(\sigma)), \ldots, R_i^{(\alpha_i)}(\sigma, u_D(\sigma)))\|_{G_i}.$$

3.2. Theorem. *Let v be a solution of* (1.8). *Let the conditions* (C), (D) *be fulfilled and let $F_L(t, \mathcal{O}_{/[\tau, +\infty)}) = 0$ for every $\tau \in \mathscr{D}(v)$. Finally, let the zero solution $\mathcal{O}_{/\mathscr{D}(v)}$ of* (3.1.3) *be uniformly exponentially stable with respect to the norm $|||\cdot|||$.*

Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{G_1}, \ldots, \|\cdot\|_{G_k}$.

Proof. By Theorem 1.12 it suffices to prove uniform stability at constantly acting disturbances of the solution $\mathcal{O}_{/\mathscr{D}(v)}$ of (3.1.1). Uniform exponential stability of the solution $\mathcal{O}_{/\mathscr{D}(v)}$ of (3.1.3) means:

(1) There exist positive constants C, α, ϱ such that if $(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathscr{D}(A) \times \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$, $\|(\varphi_0, \varphi_1, \ldots, \varphi_{n-1})\|_N \leqq \varrho$ and $\tau \in \mathscr{D}(v)$, then there exists a solution u_L of (3.1.3) fulfilling the initial conditions $u_L^{(i)}(\tau) = \varphi_i$ $(i = 0, \ldots, n-1)$; if u_L is such a solution, then $|||u_L(t)||| \leqq Ce^{-\alpha(t-\tau)}|||u_L(\tau)|||$ for all $t \geqq \tau$.

Let $\eta \in (0, \min_{i=1,\ldots,k} r_i]$ be given. Without loss of generality we may suppose that $\eta \leqq \min(R_0, R, \varrho)$. Let us take numbers $h > 0$ and $R_1 \in (0, \eta]$ such that $Ce^{-\alpha h} < 1$ and

$$Ce^{-\alpha h} + C_4 R_1^{\nu_2} e^{C_5 h} + C_1 C_6 R_1^{\nu_2} h e^{(C_2 + C_5 + C_3 \eta^{\nu_1})h} < 1 .$$

Finally, let us find numbers $\eta_0 \in (0, R_1/2]$, $\eta_D > 0$ such that

$$C\eta_0 \leqq R , \tag{2}$$

$$\left[C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma)\right] e^{(C_2 + C_3\eta^{\nu_1})h} < R_1 , \tag{3}$$

$$\begin{aligned}&\left[Ce^{-\alpha h} + C_4 R_1^{\nu_2} e^{C_5 h} + C_1 C_6 R_1^{\nu_2} h e^{(C_2 + C_5 + C_3\eta^{\nu_1})h}\right] \eta_0 + \\ &+ \left[C_6 R_1^{\nu_2} h \sum_{i=1}^{k} \max_{\sigma \in [0,h]} s_i(\sigma)\, e^{(C_2 + C_5 + C_3\eta^{\nu_1})h} + \sum_{i=1}^{k} \max_{\sigma \in [0,h]} z_i(\sigma)\, e^{C_5 h}\right] \eta_D \leqq \eta_0 .\end{aligned} \tag{4}$$

Now, let $\tau \in \mathscr{D}(v)$, $(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n})$, $\|(\varphi_0, \varphi_1, \ldots, \varphi_{n-1})\|_N \leqq \eta_0$. In virtue of $\eta_0 \leqq R_0$ and (A) it suffices to show that if u_D is a solution of (3.1.2) such that $u_D^{(i)}(\tau) = \varphi_i$ $(i = 0, \ldots, n-1)$ then

$$|||u_D(t)||| \leqq \eta \quad \text{for all} \quad t \geqq \tau .$$

By (1) we can find a solution u_L of (3.1.3) which fulfils the initial conditions $u_L^{(i)}(\tau) = u_D^{(i)}(\tau)$, $i = 0, \ldots, n-1$. It follows from (1) and (2) that

$$|||u_L(t)||| \leqq C|||u_L(\tau)||| \leqq C\eta_0 \leqq R \quad \text{for} \quad t \geqq \tau . \tag{5}$$

Let us suppose:

(6) There exists a number $\tilde{h} < h$ such that $|||u_D(\sigma)||| < R_1$ for $\sigma \in [\tau, \tau + \tilde{h})$, $|||u_D(\tau + \tilde{h})||| = R_1$.

Assuming that (1.11) holds we get from (D) with help of $R_1 \leqq \eta \leqq R$ the inequality

$$|||u_D(t)||| \leqq C_1\eta_0 + (C_2 + C_3\eta^{\nu_1}) \int_\tau^t |||u_D(\sigma)||| \, d\sigma +$$

$$+ \eta_D \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma) \quad \text{for} \quad t \in [\tau, \tau + \tilde{h}] .$$

This inequality with help of Gronwall's lemma and (3) gives

$$|||u_D(t)||| \leqq [C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma)] \, e^{(C_2+C_3\eta^{\nu_1})(t-\tau)} \leqq \tag{7}$$

$$\leqq [C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma)] \, e^{(C_2+C_3\eta^{\nu_1})h} < R_1$$

for $t \in [\tau, \tau + \tilde{h}]$. This contradicts (6). So we have proved

$$|||u_D(t)||| \leqq R_1 \leqq \eta \leqq R \quad \text{for} \quad t \in [\tau, \tau + h] . \tag{8}$$

Using (D) and assuming that (1.11) holds we obtain by combining (5), (7) and (8)

$$|||u_D(t) - u_L(t)||| \leqq C_4\eta_0^{1+\nu_2} + C_5 \int_\tau^t |||u_D(\sigma) - u_L(\sigma)||| \, d\sigma +$$

$$+ C_6 R_1^{\nu_2} h [C_1\eta_0 + \eta_D \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma)] \, e^{(C_2+C_3\eta^{\nu_1})h} +$$

$$+ \eta_D \sum_{i=1}^{k} \max_{\sigma\in[0,h]} z_i(\sigma) \quad \text{for} \quad t \in [\tau, \tau + h] ,$$

and so by Gronwall's lemma

$$|||u_D(t) - u_L(t)||| \leqq \{[C_4 R_1^{\nu_2} + C_1 C_6 R_1^{\nu_2} h e^{(C_2+C_3\eta^{\nu_1})h}] \, \eta_0 + \tag{9}$$

$$+ [C_6 R_1^{\nu_2} h \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma) \, e^{(C_2+C_3\eta^{\nu_1})h} + \sum_{i=1}^{k} \max_{\sigma\in[0,h]} z_i(\sigma)] \, \eta_D\} \, e^{C_5 h} \text{ for } t \in [\tau, \tau + h] .$$

The relations (1), (4), (9) imply

$$|||u_D(\tau + h)||| \leqq |||u_L(\tau + h)||| + |||u_D(\tau + h) - u_L(\tau + h)||| \leqq \tag{10}$$

$$\leqq [C e^{-\alpha h} + C_4 R_1^{\nu_2} e^{C_5 h} + C_1 C_6 R_1^{\nu_2} h e^{(C_2+C_5+C_3\eta^{\nu_1})h}] \, \eta_0 +$$

$$+ [C_6 R_1^{\nu_2} h \sum_{i=1}^{k} \max_{\sigma\in[0,h]} s_i(\sigma) \, e^{(C_2+C_5+C_3\eta^{\nu_1})h} + \sum_{i=1}^{k} \max_{\sigma\in[0,h]} z_i(\sigma) \, e^{C_5 h}] \, \eta_D \leqq \eta_0 .$$

The relations (8), (10) prove the theorem (Cf. the last part of the proof of Theorem 2.2.)

4. EXAMPLES

4.1. In this section we shall give examples illustrating the theory from the preceding sections. We shall use the following norms:

$$\|x\|_{\mathscr{D}(A^{1/n})} = \|A^{1/n}x\| \quad \text{for} \quad x \in \mathscr{D}(A^{1/n}),$$

$$\|x\|_{H\times H} = \max(\|x_1\|, \|x_2\|) \quad \text{for} \quad x = (x_1, x_2) \in H \times H,$$

$$\|(\varphi_0, \varphi_1, \ldots, \varphi_{n-1})\|_N = [\sum_{i=0}^{n-1} \|A^{(n-i)/n}\varphi_i\|^2]^{1/2}$$

$$\text{for} \quad (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathscr{D}(A) \times \mathscr{D}(A^{(n-1)/n}) \times \ldots \times \mathscr{D}(A^{1/n}).$$

(The last identity implies $|||u(t)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n} u^{(i)}(t)\|^2]^{1/2}$.) Further, we shall need the notion "the type of operator $\mathscr{L}$" which was introduced in [1] (Definition 1.2.1).

4.2. Definition. Let $\omega \in R^1$. We say that *the operator $\mathscr{L}$ is of the type ω* if there exists a constant $C(\mathscr{L})$ such that if $m(t; s)$ solves the equation

$$m^{(n)}(t) + a_1(s)\, m^{(n-1)}(t) + \ldots + a_n(s)\, m(t) = 0$$

and fulfils the initial conditions $m^{(i)}(0; s) = \delta_i^{n-1}$ $(i = 0, \ldots, n-1)$ then

$$|m^{(i)}(t; s)\, s^{(n-i-1)/n}| \leqq C(\mathscr{L})\, e^{\omega t}$$

for $i = 0, \ldots, n-1, \quad t \geqq 0, \quad s \in Sp(A)$.

4.3. We shall not present proofs of theorems in this section. They can be found, together with other examples, in [4]. Let us observe only that Theorems 4.4 and 4.5 are consequences of Theorem 2.2, and Theorem 4.6 follows from Theorem 3.2.

Let us now introduce conditions which will be needed in this section:

(E) There exists a constant K^* such that $\|Ax\| \leqq K^*\|a_n(A)\, x\|$ for $x \in \mathscr{D}(A)$.
This condition (together with the inequality from 1.5) is a normalization condition which says, roughly speaking, that the function $a_n(s)$ is "close" to s (i.e., $a_n(A)$ is "approximately equal" to A).

(F) There exist constants $K, R > 0$ such that if u is a solution of (1.13) with $k = 2$, $F \in \mathscr{C}^{(0)}(I, \mathscr{D}(A^{1/n}))$, $R_1 \in \mathscr{B}^{(0)}(v, r, \mathscr{D}(A^{1/n}))$, $R_2 \in \mathscr{B}^{(1)}(v, r, H \times H)$ and if $t_0, t \in \mathscr{D}(u)$ are such that $|||u(\sigma)||| \leqq R$ for all $\sigma \in [t_0, t]$, then

$$\|A^{1/n}[F(t, v(t) + u(t)) - F(t, v(t))]\| \leqq K|||u(t)|||.$$

(G) There exist constants $K_1, K_2, K_3, R > 0$ such that if u is a solution of (1.13) with $k = 2$, $F \in \mathscr{C}^{(1)}(I, H \times H)$, $R_1 \in \mathscr{B}^{(0)}(v, r, \mathscr{D}(A^{1/n}))$, $R_2 \in \mathscr{B}^{(1)}(v, r, H \times H)$ and if $t_0, t \in \mathscr{D}(u)$ are such that $|||u(\sigma)||| \leqq R$ for all $\sigma \in [t_0, t]$ then

$$\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq K_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\|,$$

$$\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq$$

$$\leqq K_2|||u(t)||| + K_3[\|R_1(t, u(t))\|_{\mathscr{D}(A^{1/n})} + \|(R_2(t, u(t)), R_2'(t, u(t)))\|_{H\times H}].$$

Each of these conditions guarantees that the estimate from (B) is fulfilled in the respective cases.

The following condition (H), which substitutes the comparison between the solutions of the "disturbed" and "linearized" equations, implies (D) from Theorem 3.2:

(H) There exist constants $K, K_1, R > 0, \nu > 0$ such that if

(i) u_D is a solution of (3.1.2) with $k = 2$, $F_L, F_N \in \mathscr{C}^{(0)}(I, \mathscr{D}(A^{1/n}))$, $R_1 \in \mathscr{B}^{(0)}(v, r, \mathscr{D}(A^{1/n}))$, $R_2 \in \mathscr{B}^{(1)}(v, r, H \times H)$,

(ii) u_L is a solution of (3.1.3) such that $u_L^{(i)}(t_0) = u_D^{(i)}(t_0)$ for $i = 0, \ldots, n - 1$ and for a certain $t_0 \in \mathscr{D}(u_D) \cap \mathscr{D}(u_L)$,

(iii) $t \in \mathscr{D}(u_D) \cap \mathscr{D}(u_L)$ is such that $|||u_D(\sigma)||| \leqq R$, $|||u_L(\sigma)||| \leqq R$ for all $\sigma \in [t_0, t]$,

then

$$\|A^{1/n}[F_L(t, u_D(t)) - F_L(t, u_L(t))]\| \leqq K|||u_D(t) - u_L(t)|||\,,$$

$$\|A^{1/n}F_N(t, u_D(t))\| \leqq K_1 |||u_D(t)|||^{1+\nu}\,.$$

Finally, the condition (C), in our special situation, will have the form

(C) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$, where $F_L, F_N \in \mathscr{C}^{(0)}(I, \mathscr{D}(A^{1/n}))$, for $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$.

This condition expresses the decomposition of the right-hand side into "linear" and "nonlinear" parts (see Section 0.5).

4.4. Theorem. *Let v be a solution of (1.8), where $F \in \mathscr{C}^{(0)}(I, \mathscr{D}(A^{1/n}))$, let the operator $\mathscr{L}$ be of the type $\omega < 0$ and let the conditions (E), (F) be fulfilled. Finally, let $\omega + nKC(\mathscr{L}) < 0$. Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{\mathscr{D}(A^{1/n})}$, $\|\cdot\|_{H\times H}$.*

4.5. Theorem. *Let v be a solution of (1.8), where $F \in \mathscr{C}^{(1)}(I, H \times H)$, let the operator $\mathscr{L}$ be of the type $\omega < 0$ and let the conditions (E), (G) be satisfied. Further let*

$$\omega + C_5^* K_2 + C_6^* \, C(\mathscr{L}) \, K_1^2 n^{3/2} \delta^{-1/n} < 0\,,$$

where

$$C_5^* = C(\mathscr{L}) \, [K^*(1 + (n - 1) \, C_0^*) + n - 1]\,, \quad C_6^* = K^* \, C(\mathscr{L}) \, (1 + (n - 1) \, C_0^*)\,.$$

Then the solution v is uniformly stable at constantly acting disturbances with respect to the norms $|||\cdot|||$, $\|\cdot\|_{\mathscr{D}(A1/n)}$, $\|\cdot\|_{H\times H}$.

4.6. Theorem. *Let v be a solution of (1.8), where $F \in \mathscr{C}^{(0)}(I, \mathscr{D}(A^{1/n}))$, let the operator $\mathscr{L}$ be of the type 0 and let the conditions (C), (E), (H) be satisfied. Finally,* *let* $F_L(t, \mathcal{O}_{/[\tau,+\infty)}) = 0$ *for every* $\tau \in \mathcal{D}(v)$ *and let the zero solution* $\mathcal{O}_{/\mathcal{D}(v)}$ *of* (3.1.3) *be uniformly exponentially stable with respect to the norm* $|||\cdot|||$.

Then the solution v *is uniformly stable at constantly acting disturbances with respect to the norms* $|||\cdot|||$, $\|\cdot\|_{\mathcal{D}(A^{1/n})}$, $\|\cdot\|_{H\times H}$.

4.7. Remark. We can show that under similar hypotheses as in [1] (Theorem 3.2.1) and in [2] (Theorem 5.2), where $A = (-1)^p \Delta^p$, the conditions (F), (G) and (H), respectively, are satisfied. The proof is quite analogous to the proofs of the theorems mentioned above. The only important difference is that the condition (9) from the proof of Theorem 5.2 from [2] must be replaced by the following one:

There exist constants k_0, k_1, k_2 such that

$$\|u^{(n)}(t)\| \leqq k_0|||u(t)||| + k_1\|R_1(t, u(t))\| + k_2\|R_2(t, u(t))\| \leqq$$
$$\leqq k_0|||u(t)||| + \max(k_1\delta^{-1/n}, k_2)\{\|R_1(t, u(t))\|_{\mathcal{D}(A^{1/n})} +$$
$$+ \|(R_2(t, u(t)), R_2'(t, u(t)))\|_{H\times H}\},$$

where $\delta = \inf\{s \mid s \in Sp(A)\}$.

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