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A REMARK TO ONE RESULT OF M. E. MULDOON

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1. DEFINITIONS AND NOTATION

A function $\varphi(x)$ is said to be n-times monotonic (or monotonic of order n) on an interval I if

$$(1.1)\qquad (-1)^i \varphi^{(i)}(x) \geqq 0\,, \quad i = 0, 1, 2, \ldots, n\,, \quad x \in I\,.$$

For such a function we write $\varphi(x) \in M_n(I)$ or $\varphi(x) \in M_n(a, b)$ provided I is an open interval (a, b). If the strict inequality holds in (1.1) we write $\varphi(x) \in M_n^*(I)$ or $\varphi(x) \in \in M_n^*(a, b)$. We say that $\varphi(x)$ is completely monotonic on I if (1.1) holds for $n = \infty$.

A sequence $\{\mu_k\}_{k=1}^{\infty}$ denoted simply by $\{\mu_k\}$ is said to be n-times monotonic if

$$(1.2)\qquad (-1)^i \Delta^i \mu_k \geqq 0\,, \quad i = 0, 1, 2, \ldots, n\,; \quad k = 0, 1, 2, \ldots\,.$$

Here $\Delta\mu_k = \mu_{k+1} - \mu_k$; $\Delta^2\mu_k = \Delta(\Delta\mu_k)$, etc. For such a sequence we write $\{\mu_k\} \in M_n$. If the strict inequality holds in (1.2) we write $\{\mu_k\} \in M_n^*$. $\{\mu_k\}$ is called completely monotonic if (1.2) holds for $n = \infty$.

As usual, $\varphi(x) \in C_n(I)$ means that $\varphi(x)$ has (on I) continuous derivatives including the n-th order.

$D_x\, \varphi(x)$ denotes the first derivative $d\varphi(x)/dx$ and

$D_x^n\, \varphi(x)$ denotes the n-th derivative $d^n\varphi(x)/dx$.

As usual we write $[a, b)$ to denote the interval $\{x \mid a \leqq x < b\}$.

2. PRELIMINARY REMARKS

Consider an equation

$$(2.1)\qquad [g(x)\, y'(x)]' + f(x)\, y = 0\,, \quad g(x) > 0$$

with $f(x)$ and $g(x)$ continuous for $a < x < \infty$. The change of variable

$$(2.2)\qquad \xi = \int_a^x \frac{du}{g(u)\, \psi^2(u)} \quad \psi(x) > 0\,, \quad \psi(x) \in C_2(a, \infty)\,,$$

where the integral is assumed to be convergent for $x \in (a, \infty)$ and divergent for $x = \infty$, transforms (2.1) into

$$\frac{d^2\eta}{d\xi^2} + \varphi(\xi)\,\eta = 0\,, \quad \xi \in (0, \infty) \tag{2.3}$$

where

$$\eta(\xi) = \frac{y(x)}{\psi(x)} \quad \text{and} \quad \varphi(\xi) = [(g\psi')' + f\psi]\,\psi^3 g \quad (\text{see [5] p. 597})\,.$$

In our further investigation we shall need [1], Theorem 2.1:

Let $y(x)$, $z(x)$ be solutions of (2.1) on (a, ∞), where

$$0 < \lim_{x\to\infty} \{[(g\psi')' + f\psi]\,\psi^3 g\} \leqq \infty \tag{2.4}$$

for some function $\psi(x)$, $\psi(x) \in C_2(a, \infty)$, and suppose that $z(x)$ has consecutive zeros at $x_1, x_2, \ldots$ on $[a, \infty)$. Suppose also that $g(x)\,\psi^2(x)$, $D_x[\varphi(\xi)]$ and $W(x)$ are positive and belong to the class $M_n(a, \infty)$ for some $n \geqq 0$. Then, for fixed $\lambda > -1$,

$$\left\{\int_{x_k}^{x_{k+1}} W(x) \frac{1}{g(x)\,\psi^2(x)} \left|\frac{y(x)}{\psi(x)}\right|^\lambda dx\right\} \in M_n^*\,. \tag{2.5}$$

Let $y(x)$, $z(x)$ be solutions of (2.1). Let $f(x) > 0$. Then gy', gz' (see [2] p. 354) are solutions of

$$\left(\frac{1}{f}u'\right)' + \frac{1}{g}u = 0\,. \tag{2.6}$$

3. REMARK TO [3], THEOREM 6.1

In this section we are going to prove that [3], Theorem 6.1 is a corollary of [1], Theorem 2.1 applied to the equation (2.6).

Consider the differential equation

$$y'' + f(x)\,y = 0\,, \quad f(x) > 0 \tag{2.1'}$$

which is a differential equation (2.1) with $g(x) \equiv 1$. Let $y(x)$, $z(x)$ be solution of (2.1'). Then $y'(x)$, $z'(x)$ are solutions of

$$\left(\frac{1}{f}u'\right)' + u = 0\,. \tag{2.6'}$$

If [1] Theorem 2.1 is applied to (2.6'), we obtain

Corollary 3.1. *Let $y(x)$, $z(x)$ be solutions of (2.1') on (a, ∞), where*

(2.4') $$0 < \lim_{x\to\infty}\left\{\left[\left(\frac{\psi'}{f}\right)' + \psi\right]\frac{\psi^3}{f}\right\} \leqq \infty$$

for some function $\psi(x) > 0$, $\psi(x) \in C_2(a, \infty)$ *and suppose that* $z'(x)$ *has consecutive zeros at* $x'_1, x'_2, \ldots$ *on* $[a, \infty)$. *Suppose also that*

$$\frac{\psi^2(x)}{f(x)}, \quad D_x\left\{\left[\left(\frac{\psi'}{f}\right)' + \psi\right]\frac{\psi^3}{f}\right\}$$

and $W(x)$ *are positive and belong to* $M_n(a, \infty)$ *for some* $n \geqq 0$. *Then, for fixed* $\lambda > -1$,

(2.5') $$\left\{\int_{x'_k}^{x'_{k+1}} W(x)\frac{f}{\psi^2}\left|\frac{y'(x)}{\psi(x)}\right|^\lambda dx\right\} \in M_n^*, \quad k = 0, 1, 2, \ldots.$$

Now let us choose $\psi(x) = [f(x)]^c$, $c \in (-\infty, \infty)$ in Corollary 3.1. Then we have

(3.1) $$\frac{\psi^2}{f} = [f(x)]^{2c-1}$$

and

(3.2) $$\varphi(\xi) = c(c-2)[f(x)]^{4c-4}f'^2(x) + c[f(x)]^{4c-3}f''(x) + [f(x)]^{4c-1}.$$

If $c = \frac{1}{2}$, then

$$\varphi(\xi) = f(x) + \frac{1}{2}\frac{f''(x)}{f(x)} - \frac{3}{4}\frac{f'^2(x)}{f^2(x)}.$$

This transformation was considered by J. Vosmanský in [4].

Now let $4c - 1 = 0$ in (3.2). Hence $c = \frac{1}{4}$. Then we get

$$\varphi(\xi) = 1 + \frac{1}{4}\frac{f''(x)}{[f(x)]^2} - \frac{7}{16}\frac{f'^2(x)}{[f(x)]^3} = 1 - \frac{1}{3}[f(x)]^{-1/4}D_x^2\{[f(x)]^{-3/4}\},$$

which is the mapping considered in [3], Theorem 6.1. The assumptions of [3], Theorem 6.1 follow from the known properties of monotonic functions.

Finally, let $4c - 1 = 1/m$, where $m \in [1, \infty)$. Then $c = (m+1)/4m$ and $2c - 1 = (-2m+2)/4m$ and we get

(3.3) $$\varphi(\xi) = \frac{m+1}{4m}\left(\frac{-7m+1}{4m}\right)[f(x)]^{(-3m+1)/m}f'^2(x) + \frac{m+1}{4m}[f(x)]^{(-2m+1)/m}f''(x) + [f(x)]^{1/m}.$$

Let $f(x) = x^m$. Then (2.4') holds and

$$\varphi(\xi) = x - \frac{3}{16}\frac{(m+1)^2}{x^{m+1}}.$$

Hence $D_x[\varphi(\xi)] \in M_\infty(0, \infty)$ and (3.1) holds for $m \geqq 1$. We have obtained the following result:

Corollary 3.2. *Let $y(x)$, $z(x)$ be solutions of*

$$y'' + x^m y = 0\,, \quad x \in (0, \infty) \tag{3.4}$$

where $m \in [1, \infty)$. Suppose that $z'(x)$ has consecutive zeros at $x'_0, x'_1, \ldots$ on $[0, \infty)$. Let $W(x)$ be positive and belong to $M_\infty(0, \infty)$. Then, for fixed $\lambda > -1$,

$$\left\{\int_{x'_k}^{x'_{k+1}} W(x)\, x^{(m-1)/2} \left|\frac{y'(x)}{x^{(m+1)/4}}\right|^\lambda \mathrm{d}x\right\} \in M_\infty^*\,, \quad k = 0, 1, 2, \ldots\,. \tag{3.5}$$

Remark 3.1. Choose $W(x) = x^{[(m+1)\lambda - 2m - 2]/4}$ in (3.5). If $(m + 1)\lambda - 2m - 2 \leqq 0$ then we can write (3.5) in the form

$$\left\{\int_{x'_k}^{x'_{k+1}} |y'(x)|^\lambda \,\mathrm{d}x\right\} \in M_\infty^*\,, \quad k = 0, 1, 2, \ldots\,. \tag{3.5'}$$

If $\lambda = 0$ we get

$$\{\Delta x'_k\} \in M_\infty^*\,, \quad k = 0, 1, 2, \ldots\,.$$

Remark 3.2. Corollary 3.2 can be applied to the generalized Airy equation

$$y'' + \beta^2\gamma^2 x^{2\beta-2} y = 0\,, \quad 0 < x < \infty$$

for $\beta \geqq \frac{3}{2}$ which is an extension of $1 \leqq \beta \leqq \frac{3}{2}$ considered e.g. in [2].

Remark 3.3. Passing to the limit for $m \to \infty$ in (3.3) we obtain the mapping considered in [3], Theorem 6.1.

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