REFERÁTY

CONFERENCE ON CONVERGENCE PROBLEMS IN PROBABILITY THEORY

The Conference, organized by the Mathematical Institute of the Czechoslovak Academy of Sciences, was held in Liblice, from 31 May to 3 June, 1967. Its main aim was to give the participants an occasion for presenting new results and exchanging views on problems in the following three domains: a) Convergence as a fundamental notion of Probability Theory; b) Classical limit theorems; c) Convergence in stochastic processes.

There were 43 mathematicians attending at the Conference, 18 of them were from abroad; they gave 24 lectures. As there will be no special proceedings of the conference, we bring here short summaries of these papers and/or indication of where the full texts can eventually be found.

H. BERGSTRÖM (Göteborg): *Limit theorems for convolution powers of random functions.*

Let us consider the L^p-space ($p \geqq 1$) on a probability space. A random function $\xi(t)$, $-\infty < t < +\infty$, is said to belong to L^p if $\xi(t) \in L^p$ for all t. Let M^p be the class of all functions $\xi(t) \in L^p$ that are almost surely non-negative and non-decreasing and satisfy $\xi(t) = \frac{1}{2}[\xi(t + 0) + \xi(t - 0)]$ for all t, the limits being taken with respect to the L^p-norm. If $\xi \in M^p$, $\eta \in M^q$, where $p \geqq 1$, $q \geqq 1$ and $(1/p) + (1/q) = (1/r)$, $r \geqq 1$, there exists a generalized convolution $\xi * \eta$ belonging to M^r. Hence $\xi^{*n} \in M^p$, if $\xi \in M^{np}$. Sufficient conditions for the L^p-complete convergence of a sequence $\{\xi_n^{*k_n}\}$ to a random function were given, $\{k_n\}$ being a sequence of positive integers, $k_n \to \infty$ for $n \to \infty$.

For more details see Report No M 117, The Florida State University, Dept. of Statistics, Tallahassee, 1966.

H. HEYER (Erlangen): *Convergence problems for convolutions of probability measures on locally compact groups.*

Generalizing two well-known classical limit theorems for real valued and circular random variables, the author proved the following theorems. (G is a locally compact group with denumerable basis.)

Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent G-valued random variables with distribution laws μ_k. If G has no non-trivial compact subgroup, then the stochastic *convergence, the almost sure convergence and the convergence in law of the sequence* $\{\prod_{k=1}^{n} X_k\}_{n=1}^{\infty}$ *are equivalent.*

Let $\{X_k\}_{k=1}^{\infty}$ *have the same properties. If G is compact, there exists a sequence* $\{A_k\}_{k=1}^{\infty}$ *in G such that* $\{\mu_1 * \mu_2 * \ldots * \mu_n * \varepsilon_{A_n}\}_{n=1}^{\infty}$ *converges vaguely.*

The "only if" parts can also be proved. For more details see a paper by the author (Mathematica Scandinav., *19* (1966), 211–216) and another by I. Csiszár (Zeitschrift für Wahrscheinlichkeitstheorie, *5* (1966), 279–295).

A. A. Borovkov (Novosibirsk): *On large deviations in functional spaces.*

Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent, equally distributed random variables, $\mathbf{E}\xi_n = 0$, $\mathbf{E}(e^{\lambda\xi_n}) = \varphi(\lambda)$, $s_n = \sum_{k=1}^{n} \xi_k$. In many problems (arising in connection with large deviations) the function $m(\alpha) = \inf_{\lambda} e^{-\lambda\alpha}\varphi(\lambda)$ proves to be very useful. Let $\Lambda(\alpha) = \lg m(\alpha)$; it is a convex function.

Let $S_n(t)$, $0 \leqq t \leqq 1$, be the continuous broken line passing through the points $[k/n, s_k/x]$, $k = 0, 1, \ldots, n$, where $n \to \infty$ and $n^{-1/2}x \to \infty$. We can now ask for asymptotic properties of the probabilities $\mathbf{P}\{S_n(t) \in G\}$ where G are open sets in $C(0, 1)$; such problems are in no way trivial. The following concrete result was presented:

Let G be an open set in $C(0, 1)$, $\bar{G}$ *its closure,* $|\varphi(\lambda)| < \infty$ *for all* λ, *let* $x \sim \beta n$. *Let* $W(g) = \int_0^1 \Lambda((\mathrm{d}g(t)/\mathrm{d}t)\,\beta)\,\mathrm{d}t$ *for* $g \in C(0, 1)$, $W(G) = \sup W(g)$ *over the set of piece-wise convex* $g \in G$. *Then*

$$W(G) = \lim_{n\to\infty} n^{-1} \lg \mathbf{P}\{S_n(t) \in G\}\,.$$

For more details see the author's papers in Сибирский матем. журнал *5* (1964), 253–289 and 750–767 and in Теория вэроятностей *12* (1967), 634–654.

V. M. Zolotarev (Moskva): *A generalization of the theorem of Lindeberg-Feller.*

The well known theorem of Lindeberg-Feller, concerning the convergence in law of a sequence of series of independent random variables to a normal random variable was generalized in a direction suggested by P. Lévy. Let $\zeta_n = \xi_{n1} + \xi_{n2} + \ldots$ for $n = 1, 2, \ldots$, where ξ_{nk}, $k = 1, 2, \ldots$ are, for each n, independent random variables with distribution functions F_{nk}; F_n is the distribution function of ζ_n. Let $\int x\,\mathrm{d}F_{nk}(x) = 0$, $\sigma_{nk}^2 = \int x^2\,\mathrm{d}F_{nk}(x) < \infty$, $\sum_k \sigma_{nk}^2 = 1$. Let Φ be the normal $N(0, 1)$ distribution function and $\Phi_{nk}(x) = \Phi(x/\sigma_{nk})$. *In order to have* $F_n \to \Phi$ *for* $n \to \infty$ *it is necessary and sufficient that*

1) $\alpha_n = \sup_k L(F_{nk}, \Phi_{nk}) \to 0$, *L being the Lévy metric;*

2) *for any* $\delta > 0$ *we have* $\sum_{k\in A} \int_{|x|\geqq\delta} x^2\,\mathrm{d}F_{nk}(x) \to 0$, *where* $A = \{k\colon \sigma_{nk}^2 < \alpha_n^r\}$ (*r being any number in* (0, 1)).

M. DĄBEK (Lublin): *On the comparative speed of convergence of sums of certain random variables.*

No summary available.

H. DINGES (Frankfurt a. M.): *Conical measures in ergodic theory.*

The ergodic theorems of the Birkhoff type concern the almost sure convergence of $(x_1 + x_2 + \ldots + x_n)/(p_1 + p_2 + \ldots + p_n)$, where x_i and p_i are generated by a contraction of the space L_1. In this paper, the author defined a distribution of $\{x_i, p_i\}$; it is a conical measure in R^∞. The space of all conical measures can be ordered. There exists a theorem by Blackwell-Sherman-Stein-Strassen with the aid of which it should be possible — so the author hopes — to prove the convergence (the ergodic theorem) under the assumption that the $\{x_i, p_i\}$ are monotonous in law according to this order.

U. KRENGEL (Erlangen): *On the theorem of McMillan.*

A generalization to σ-finite measure spaces of the theorem of McMillan was given (both a Cesàro-convergence theorem and a ratio-limit theorem), under some assumptions which are always true in the finite case, but seem to be difficult to check in infinite measure spaces. They can be verified, however, for the Markov shift of the coin-tossing random walk.

V. STATULEVIČIUS (Vilnius): *Some investigation on limit theorems in Probability Theory.*

Let $\{\xi_k\}_{k=1}^\infty$ be a sequence of independent random variables with distribution functions F_k, let $S_n = \sum_{k=1}^n \xi_k$, let B_n^2 be the variance of S_n and $Z_n = (S_n - \mathbf{E}S_n) B_n^{-1}$. We further assume that $\mathbf{E}|\xi_k - \mathbf{E}\xi_k|^3 < \infty$ for all k and that F_k can be decomposed in $F_k = a_k G_k + b_k H_k$, where G_k is the absolutely continuous part, $G_k' = g_k$. Now, if $\overline{\lim_{n\to\infty}} B_n^{-2} \sum_{k=1}^n |\xi_k - \mathbf{E}\xi_k|^3 < \infty$, $g_k(x) \leqq C_k < \infty$, $\sum_{k=1}^\infty a_k C_k^{-2} = \infty$, then

$$\sup_A \left| \mathbf{P}\{Z_n \in A\} - (2\pi)^{-1/2} \int_A \exp(-\tfrac{1}{2}x^2)\, dx \right| \to 0 \tag{1}$$

for $n \to \infty$.

Let $\{\xi_n\}$ be a Markov chain with $\alpha^{(n)} = 1 - \max_{1\leqq k\leqq n} \sup_{x,y,A} |P_k(x, A) - P_k(y, A)|$. If the ξ_k are uniformly bounded with probability 1, $\alpha^{(n)} > 0$, and $\alpha^{(n)3} \sum_{k=1}^n a_k C_k^{-2} \to \infty$ for $n \to \infty$, then also (1) holds true.

For other results see Zeitschrift für Wahrscheinlichkeitstheorie, *6* (1966), 133—144; Теория вероятностей, *10* (1965), 645—659; Литовский матем. сборник, *6* (1966), 569—585.

C. LENÁRT (Košice): *On the stochastic solution of a lattice point problem.*

Let $S = S(1)$ be a closed hypersurface in E_r, let V be the volume it envelops. For real x, $0 \leqq x < \infty$, let $S(x)$ be the hypersurfaces corresponding homothetically to S, the ratio being $\sqrt{(x)} : 1$.

Let Q_i, $i = 1, 2, \ldots$ be the unit cubes of the integer-lattice in E_r, ordered in a simple sequence; for $i = 1, 2, \ldots$ let Z_i be a random variable uniformly distributed in Q_i. Let $A(x)$ be the number of those Z_i that take values in the interior of $S(x)$, let $R(x) = A(x) - x^{r/2} V$. Then under quite general assumptions one can prove the following theorem:

Let $x_n = Cn^\alpha$, $C > 0$, $\alpha > 0$, $n = 1, 2, \ldots$, *then, with probability one*, $R(x_n) = O(x_n^{(r-1)/4} \log^{1/2} x_n)$ *and* $R(x_n) = \Omega(x_n^{(r-1)/4} \log^{1/2} x_n)$.

See also a forthcoming paper in Matematický časopis SAV.

J. Lamperti (Cambridge): *Semi-stable Markov processes.*

A strong Markov process in R^n with right-continuous paths is said to be semi-stable if its transition function obeys an identity of the form

$$P_{at}(x, E) = P_t(xa^{-\alpha}, a^{-\alpha}E), \quad \alpha > 0\,,\ \text{const.}\,, \tag{1}$$

for all $a > 0$. Such processes arise naturally in connection with certain common types of limit theorems; a partial explanation can be found in Trans. Amer. Math. Soc., *104* (1962), 62–78. For the case where the state space is the half-line $\langle 0, \infty)$ a method of investigating the structure of semi-stable Markov processes will be described.

Define the additive functional

$$\varphi_\tau(\omega) = \int_0^\tau x_s(\omega)^{-1/\alpha}\, ds\,, \quad x_0(\omega) = x_0 > 0\,, \tag{2}$$

and let $T(\tau)$ denote the inverse function to φ_τ. Let $y_t = x_{T(t)}$ provided $T(t)$ is defined, the $\{y_t\}$ then form a right-continuous strong Markov process, possibly with a finite life time.

Let $z_t = \lg y_t$, *then* $\{z_t\}$ *is a (possibly terminating) process with stationary independent increments.*

The proof is by semi-group methods, and the paths of $\{z_t\}$ reflect the behaviour of $\{x_t\}$ until its first passage to state 0. For example, if $\{x_t\}$ has a.s. continuous paths, so does $\{z_t\}$ and the latter does not terminate. But then $\{z_t\}$ is a Wiener process with constant drift and examining the effect of the above transformations it is quite easy to obtain:

The infinitesimal generator of a semi-stable Markov process with continuous paths is given in $(0, \infty)$ *by the differential operator*

$$A f(x) = dx^{2-1/\alpha} f''(x) + (c + d)\, x^{1-1/\alpha} f'(x)\,; \quad d > 0\,,\ c \text{ const.} \tag{3}$$

It thus appears that all semi-stable processes $\{x_t\}$ can be studied in terms of additive ones. The method can be extended to the case where the state space is the whole line $(-\infty, +\infty)$, but apparently not to higher dimensions.

R. Z. CHASMINSKIJ (Moskva): *Some limit theorems for random processes defined by differential equations.*

Let $(X^{(\varepsilon)}(t), Y^{(\varepsilon)}(t))$ be a family of l-dimensional Markov processes defined by the stochastic differential equations

$$(1) \qquad \mathrm{d}X_i^{(\varepsilon)}(t) = A_i(X^{(\varepsilon)}, Y^{(\varepsilon)})\,\mathrm{d}t + \sum_{r=1}^{l} \sigma_i^{(r)}\,\mathrm{d}\xi_r(t)\,,$$

$$\mathrm{d}Y_j^{(\varepsilon)}(t) = \varepsilon^{-1}B_j(X^{(\varepsilon)}, Y^{(\varepsilon)})\,\mathrm{d}t + \varepsilon^{-1/2}\sum_{r=1}^{l} \varphi_j^{(r)}(X^{(\varepsilon)}, Y^{(\varepsilon)})\,\mathrm{d}\xi_r(t)$$

$i = 1, 2, \ldots, l_1$; $j = 1, 2, \ldots, l_2$; $l_1 + l_2 = l$; the $\xi_r(t)$, $r = 1, 2, \ldots, l$ being independent Wiener processes with $\mathbf{E}\xi_r(t) = 0$, $\mathbf{D}^2\xi_r(t) = t$ and the initial conditions $X^{(\varepsilon)}(0) = x_0$, $Y^{(\varepsilon)}(0) = y_0$.

Conditions were given under which the process $X^{(\varepsilon)}(t)$ converges (for $\varepsilon \to 0$) weakly to a Markov process $X^{(0)}(t)$ which is the solution of

$$\mathrm{d}X^{(0)}(t) = \bar{A}(X^{(0)})\,\mathrm{d}t + \sum_{r=1}^{l} \bar{\sigma}^{(r)}(X^{(0)})\,\mathrm{d}\xi_r(t)\,,$$

where $\bar{\sigma}(x) = \|\sigma_i^{(r)}(x)\|$ is the square root of a symmetric matrix.

P. MANDL (Praha): *Some convergence problems in controlled Markov processes.*

Let $\mathbf{X} = \{X_n, n = 0, 1, \ldots\}$ be a discrete homogeneous Markov chain with state space I and transition probability matrix $\|p_{ij}\|$. Let the states of $\mathbf{X}$ form one recurrent class. Denote $\{\pi_i, i \in I\}$ the invariant measure of $\mathbf{X}$. In many problems the cost or reward from the chain has the form $\vartheta = \sum_i \pi_i\, c(i) / \sum_i \pi_i\, d(i)$ where $c(i), d(i) \geqq 0$ are given functions on I, $\sum_i \pi_i |c(i)| < \infty$, $0 < \sum \pi_i\, d(i) < \infty$.

The following characterization of ϑ derived for finite chains by R. Bellman (Journ. Math. and Mech., *6* (1957), 679–684) is important in control theory: *ϑ is the unique number for which the system of equations*

$$(1) \qquad w_j = c(j) - \vartheta\, d(j) + \sum_k p_{jk} w_k\,, \quad j \in I\,,$$

posseses a solution $\{w_j, j \in I\}$.

In the lecture, (1) was generalized to chains with countable state space using a suitable definition of the quantities w_j. Then its validity for general Markov chains was discussed and illustrated by a result from the control theory for chains which are not observable directly. See also Revue Roumaine de Math. pures et appliquées, *11* (1966), 533–539.

K. KRICKEBERG (Heidelberg): *Notions of convergence in Probability Theory, that can be described by extreme limits.*

In the first part of the lecture, the author presented the results of his paper in Bulletin de la Société Mathématique de Grèce, 1964. In the second part, he discussed the stable convergence in the sense of A. Rényi.

J. NOVÁK (Praha): *On convergence algebras.*

A convergence algebra $(L, \mathscr{L}, \lambda, +, .)$ is a convergence space $(L, \mathscr{L}, \lambda)$ with a convergence $\mathscr{L}$ and a convergence topology λ and a ring $(L, +, .)$ with the unit $e \in L$ and such that both operations $x - y$ and $x . y$ are sequentially continuous on $L \times L$. It was shown that it is possible to define a probability function on convergence algebras which fail to be Boolean algebras:

Let X be a non empty point set and $\mathbf{A}$ a set algebra on X. Denote $\dot{\mathbf{A}}$ the collection of all pairs $[E_1, E_2]$ where E_i are disjoint subsets belonging to $\mathbf{A}$, $i = 1, 2$. Define

$$[E_1, E_2] + [F_1, F_2] = [(E_1 - \bigcup F_i) \cup (F_1 - \bigcup E_i) \cup E_2 \cap F_2 ,$$
$$(E_2 - \bigcup F_i) \cup (F_2 - \bigcup E_i) \cup E_1 \cap F_1] ,$$
$$[E_1, E_2] . [F_1, F_2] = [E_1 \cap F_1 \cup E_2 \cap F_2, E_1 \cap F_2 \cup E_2 \cap F_1] ,$$
$$\lim [E_1^n, E_2^n] = [\text{Lim } E_1^n, \text{Lim } E_2^n] .$$

Now, let P be any probability measure on $\mathbf{A}$. Put $p([E_1, E_2]) = P(E_1) + P(E_2)$. It can be proved that the function p has the basic properties of probability measure: it is non negative, $p([X, \emptyset]) = 1$, it is additive and continuous on $\dot{\mathbf{A}}$ and it can be continuously extended to the least λ-closed (in 2^X) unitary ring containing $\dot{\mathbf{A}}$ as a subring.

I. CSISZÁR (Budapest): *On topological properties of f-divergences.*

See the full text in Studia Sci. Math. Hung., *2* (1967), 329–339.

M. DRIML (Praha): *Convergence in law for abstract-valued random variables.*

See the paper Les fonctions de répartition sur les espaces semi-ordonnés in Bulletin des Sci. Math., 2e Sér., *90* (1966), 129–140.

E. CSÁKI (Budapest): *An iterated logarithm law for semimartingales and its application to the empirical distribution function.*

See a forthcoming paper in Studia Sci. Math. Hung.

J. HÁJEK (Praha): *Asymptotic normality under dependence.*

For the full text see a forthcoming paper in the Annals of Math. Statistics; an abstract has already been published in Proceedings of the National Academy of Sciences (U.S.A.), *57* (1967), 19–20.

V. Dupač, J. Hájek (Praha): *Asymptotic normality of the Wilcoxon statistic under divergent alternatives.*

See a forthcoming paper in Applicationes Mathematicae (Zastosowania Matematyki), Hugo Steinhaus special volume.

J. Jurečková (Praha): *Asymptotic linearity of some rank statistics in the translation parameter.*

Let $S_{0N} = \sum_{i=1}^{N} (c_i - \bar{c})\, a_N(R_{Ni})$, $S_{\Delta N} = \sum_{i=1}^{N} (c_i - \bar{c})\, a_N(R_{Ni}^{\Delta})$, where $c_1, \ldots, c_N$ are some regression constants, a_N are some scores in a rank statistic, $R_{N1}, \ldots, R_{NN}$ is the vector of ranks of the random sample $X_1, \ldots, X_N$, and $R_{N1}^{\Delta}, R_{N2}^{\Delta}, \ldots, R_{NN}^{\Delta}$ is the vector of ranks of a shifted sample $X_1 + \Delta d_1, \ldots, X_N + \Delta d_N$. Under certain assumptions, usual in the theory of rank tests, it is shown that for arbitrary $\varepsilon > 0$, $K > 0$

$$\lim_{N\to\infty} \max_{|\Delta| \leqq K} \mathbf{P}\{S_{\Delta N} - S_{0N} - b_N \Delta| \geqq \varepsilon \sqrt{\operatorname{var} S_{0N}}\} = 0$$

where b_N is a certain constant related to the efficiency of the S_{0N}-test.

S. Zubrzycki (Wrocław): *A problem on sampling in the plane.*

The aim of the talk was to recall a conjecture put forward in T. Dalenius, J. Hájek, S. Zubrzycki: On plane sampling and related geometrical problems (Proceedings of the Fourth Berkeley Symposium, 1961, 125–150) – Problem 4.2, p. 141 – on the occasion of some partial results by B. Kopociński (Zastosowania Matematyki, *8* (1965), 1–12 and *9* (1966), 63–73) which gave a new support to it. The conjecture is that among sampling patterns in the plane of a given density the pattern obtained from a net of equilateral triangles yields the smallest limiting variance of estimation of the mean of a stationary isotropic stochastic process with an exponential correlation function. B. Kopociński has proved that this pattern cannot be improved by small deformations.

F. Zítek (Praha): *On the Fourier norm.*

The Fourier norm (introduced for functions with bounded variation in an earlier paper – Časopis pro pěst. matem. *91* (1966), 453–462) was compared with the major norm of H. Bergström (see his book Limit Theorems for Convolutions, Stockholm and New York, 1963). Some connections with the theory of random functions of intervals were considered. The main results will appear in a paper in Časopis pro pěst. matem. in 1968.

Z. Šidák (Praha): *Some remarks on the ratio limit theorem in general Markov chains.*

Let $p(x, A)$ be the transition probability in a Markov chain with discrete time and general state space; let the chain be irreducible. Let μ be a subinvariant measure. It was proved that in case of a recurrent chain one has

$$\lim_{n \to \infty} [\sum_{m=1}^{n} p^{(m)}(x, A) / \sum_{m=1}^{n} p^{(m)}(y, B)] = \mu(A)/\mu(B)$$

for all x, y. If the chain is transient, one can prove an analogous, but weaker theorem, for μ-almost all x and μ-almost all y.

A problem formulated by the author (see Transactions of the Fourth Prague Conference, Praha 1967, 547–571) was solved: in a transient chain, it is possible that $\sum_{n=1}^{\infty} p^{(n)}(x, A) = \infty$ for some x even if $\mu(A) < \infty$.

M. JIŘINA (Praha): *A simplified proof of Sevastyanov theorem on branching processes.*

No summary available.

František Zítek, Praha