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Distances between directed graphs

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DISTANCES BETWEEN DIRECTED GRAPHS

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Summary. The concepts of the edge distance and of the edge rotation distance introduced by other authors for undirected graphs are adapted for directed graphs.

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In this paper we shall modify the concepts of the edge distance (introduced by V. Kvasnička, V. Baláž, M. Sekanina and J. Koča [2]) and of the edge rotation distance (introduced by G. Chartrand, F. Saba and H.-B. Zou [1]) for directed graphs. We shall consider finite directed graphs without loops in which there exist no two edges with the common initial vertex and with the common terminal vertex.

Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of directed graphs, let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. Let G_{12} be a graph which is isomorphic simultaneously to a subgraph of G_1 and to a subgraph of G_2 and has the maximum number of edges from all graphs with this property. Let V_1 (or V_2) be the vertex set of G_1 (or G_2, respectively). Let E_1 (or E_2, or E_{12}) be the edge set of G_1 (or G_2, or G_{12}, respectively). Then the edge distance of G_1 and G_2 is

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = |E_1| + |E_2| - 2|E_{12}| + ||V_1| - |V_2|| .$$

In the sequel we shall use such pairs $\mathfrak{G}_1, \mathfrak{G}_2$ that $|V_1| = |V_2| = n$, $|E_1| = |E_2| = m$. Then we have

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2m - 2|E_{12}| . \tag{1}$$

In this paper the edge distance will be used only as an auxiliary concept. The main concept will be the edge rotation distance.

As we work with directed graphs, we may introduce two types of the edge rotation. Let x, y, z be three distinct vertices of a digraph G such that the edge xy is in the edge set $E(G)$ of G, while the edge xz is not. (We omit arrows over the symbols of edges for typographical reasons.) A rotation of the edge xy around its initial vertex (shortly I-rotation) is a transformation of G by deleting the edge xy and adding the edge xz. If we suppose that the edge zy is not in $E(G)$ (the edge xz may be), then a rotation of xy around its terminal vertex (shortly T-rotation) is a transformation of G by deleting xy and adding zy. An edge rotation is either an I-rotation, or a T-rotation.

Let $\mathfrak{G}_1$, $\mathfrak{G}_2$ be two isomorphism classes of directed graphs such that any graph $G_1 \in \mathfrak{G}_1$ has the same number n of vertices and the same number m of edges as any graph $G_2 \in \mathfrak{G}_2$. We define the edge rotation distance $d_r(\mathfrak{G}_1, \mathfrak{G}_2)$ as the minimum number of edge rotations which are necessary for transforming a graph from $\mathfrak{G}_1$ into a graph from $\mathfrak{G}_2$. Instead of speaking about the distance between isomorphism classes of graphs we shall sometimes speak about the distance between graphs. The distance between the graphs G_1, G_2 is the distance between isomorphism classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ such that $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$.

Now we define the degree vectors of a digraph. Let G be a digraph with n vertices. For $i = 0, 1, \ldots, n-1$ let $v_i^+(G)$ (or $v_i^-(G)$) be the number of vertices of G of the outdegree (or indegree, respectively) equal to i. Now we have two n-dimensional vectors

$$\mathbf{v}^+(G) = (v_0^+(G), v_1^+(G), \ldots, v_{n-1}^+(G)),$$
$$\mathbf{v}^-(G) = (v_0^-(G), v_1^-(G), \ldots, v_{n-1}^-(G)).$$

The vector $\mathbf{v}^+(G)$ (or $\mathbf{v}^-(G)$) is called the outdegree (or indegree, respectively) vector of G. Both these vectors are called degree vectors of G.

Theorem 1. *Let G_1, G_2 be two digraphs with the same number n of vertices and the same number m of edges. Then the following two assertions are equivalent:*

(i) $\mathbf{v}^+(G_1) = \mathbf{v}^+(G_2)$.

(ii) *The graph G_1 can be transformed into a graph isomorphic to G_2 by a sequence of I-rotations.*

Proof. (i) $\Rightarrow$ (ii). If $\mathbf{v}^+(G_1) = \mathbf{v}^+(G_2)$, we may find a one-to-one correspondence between the vertex sets of G_1 and of G_2 such that the corresponding vertices have equal outdegrees G_1 and in G_2, respectively. Now we identify each vertex of G_1 with the corresponding vertex of G_2; then G_1 and G_2 may be considered as graphs with the common vertex set. Let u be a vertex of this set; then the set of edges outgoing from u, belonging to G_1 and not belonging to G_2, has the same cardinality as the set of edges outgoing from u, belonging to G_2 and not belonging to G_1. Thus each edge from the first set can be transferred by an I-rotation into an edge of the second set. If we do this for all vertices u, the graph G_1 is transformed into G_2.

(ii) $\Rightarrow$ (i). After performing an I-rotation, evidently the outdegree vector of the graph remains unchanged; this implies the assertion.

Theorem 1'. *Let G_1, G_2 be two digraphs with the same number n of vertices and the same number m of edges. Then the following two assertions are equivalent:*

(i) $\mathbf{v}^-(G_1) = \mathbf{v}^-(G_2)$.

(ii) *The graph G_1 can be transformed into G_2 by a sequence of T-rotations.*

Proof is dual to the proof of Theorem 1.

If $\mathbf{v}^+(G_1) = \mathbf{v}^+(G_2)$ (or $\mathbf{v}^-(G_1) = \mathbf{v}^-(G_2)$) for graphs G_1, G_2, we say that G_1, G_2 are outdegree equivalent (or indegree equivalent, respectively). Two isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of digraphs are called outdegree equivalent (or indegree equivalent, respectively), if so are the graphs $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$.

Now we may define the distances d_{Ir}, d_{Tr}. If $\mathfrak{G}_1, \mathfrak{G}_2$ are outdegree (or indegree) equivalent isomorphism classes of digraphs, then their I-rotation distance $d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$ (or their T-rotation distance $d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2)$) is the minimum number of I-rotations (or T-rotations, respectively) which are necessary for transforming a graph from $\mathfrak{G}_1$ into a graph from $\mathfrak{G}_2$.

The fact that d_r, d_{Ir}, d_{Tr} are metrics can be easily proved analogously as in [1] for undirected graphs.

The following two assertions are evident, because any I-rotation and any T-rotation is an edge rotation.

Proposition 1. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two outdegree equivalent isomorphism classes of directed graphs. Then*

$$d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) \geqq d_r(\mathfrak{G}_1, \mathfrak{G}_2).$$

Proposition 1'. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two indegree equivalent isomorphism classes of directed graphs. Then*

$$d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2) \geqq d_r(\mathfrak{G}_1, \mathfrak{G}_2).$$

It might seem that d_{Ir} and d_{Tr}, if they are defined, are always equal to d_r. But the following theorem shows that it is not so.

Theorem 2. *There exist outdegree equivalent isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of directed graphs such that*

$$d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) > d_r(\mathfrak{G}_1, \mathfrak{G}_2).$$

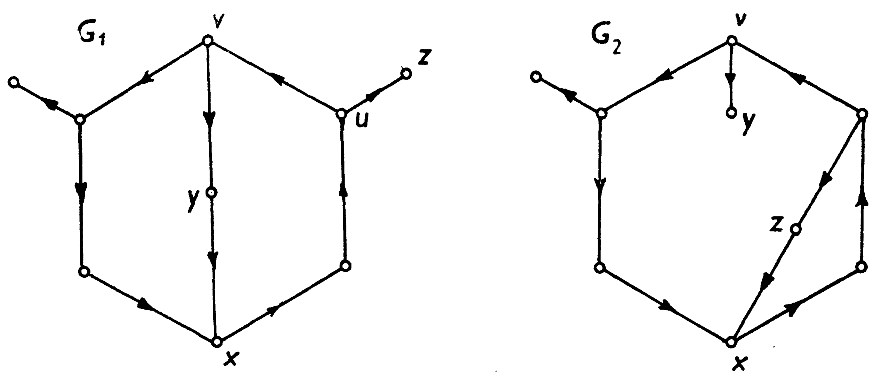

Fig. 1

Proof. In Fig. 1 we see a graph $G_1 \in \mathfrak{G}_1$ and a graph $G_2 \in \mathfrak{G}_2$; we consider them as graphs with the common vertex set. Evidently

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) = 1 ,$$

because G_2 is obtained from G_1 by a T-rotation of the edge yx to the position zx. For the graphs G_1, G_2 we have

$$\mathbf{v}^+(G_1) = \mathbf{v}^+(G_2) = (2, 4, 3, 0, 0, 0, 0, 0, 0)$$

and hence $d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$ is defined. It is equal to 2, because G_2 can be obtained from G_1 by the I-rotation of uz into the position uy and by the I-rotation of vy into the position vz, while by one I-rotation it is evidently not possible.

Theorem 2'. *There exist indegree equivalent isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of directed graphs such that*

$$d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2) > d_r(\mathfrak{G}_1, \mathfrak{G}_2) .$$

Proof. These classes are those containing the graphs obtained from the graphs, in Fig. 1 by reversing the orientations of all edges.

Consider again two outdegree equivalent isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ and graphs $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$ such that G_2 is obtained from G_1 by $d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$ I-rotations. Thus G_1 and G_2 have the common vertex set V. Let E_1 (or E_2) be the edge set of G_1 (or G_2, respectively). Denote $F_0 = E_1 \cap E_2$, $F_1 = E_1 - E_2$, $F_2 = E_2 - E_1$. If an edge e_2 is obtained from an edge e_1 by I-rotations, then evidently it can be obtained from e_1 by one I-rotation. Thus $|F_1| = |F_2| = d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$, because each edge from F_2 is obtained by an I-rotation of one edge from F_1 and the edges from F_0 may remain unchanged. The graph G_0 with the vertex set V and with the edge set F_0 has $m - d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$ edges, where m is the number of edges of G_1 and of G_2. The graph G_0 has the property that it can be embedded into a graph $G_1 \in \mathfrak{G}_1$ and simultaneously into a graph $G_2 \in \mathfrak{G}_2$ in such a way that in both the cases each vertex of G_0 is mapped onto vertices of equal outdegrees in G_1 and in G_2. The class of such graphs will be denoted by $OC(\mathfrak{G}_1, \mathfrak{G}_2)$. Analogously we define the class $IC(\mathfrak{G}_1, \mathfrak{G}_2)$ by replacing the word "outdegrees" by "indegrees".

Theorem 3. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two outdegree equivalent isomorphism classes of directed graphs with m edges. Then the maximum number of edges of a graph from $OC(\mathfrak{G}_1, \mathfrak{G}_2)$ is equal to $m - d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$.*

Proof. From the above considerations we have seen that $G_0 \in OC(\mathfrak{G}_1, \mathfrak{G}_2)$ and has $m - d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$ edges. On the other hand, suppose that there exists $F^1 \in \in OC(\mathfrak{G}_1, \mathfrak{G}_2)$ with m' edges. We embed F' into $G_1 \in \mathfrak{G}_1$ and into $G_2 \in \mathfrak{G}_2$ and then we choose a one-to-one correspondence between the vertex sets of G_1 and G_2, such that if a vertex x_1 of G_1 and a vertex x_2 of G_2 are images of the same vertex of F' in both the embeddings, then they correspond to each other. If we identify each pair of the corresponding vertices, the graphs G_1, G_2 become graphs with the common vertex set and with the common subgraph F'. There are $m - m'$ edges of G_1 not belonging to F' and also $m - m'$ edges of G_2 not belonging to F'. As $\mathbf{v}^+(G_1) = \mathbf{v}^+(G_2)$, for each vertex x the number of edges outgoing from x which belong to G_1 and not to F' is equal to the number of edges adjacent to x which belong to G_2 and not to F'. Thus each edge belonging to G_1 and not to F' either belongs also to G_2, or can be transferred by an I-rotation into an edge of G_2 not belonging to F'. As there are $m - m'$ edges belonging to G_1 and not to F', there exist I-rotations which transform G_1 into G_2 and whose number is less than or equal to $m - m'$. Thus $d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) \leqq m - m'$, which implies

$$m' \leqq m - d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) .$$

Theorem 3'. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two indegree equivalent isomorphism classes of directed graphs with m vertices. Then the maximum number of edges of a graph from $IC(\mathfrak{G}_1, \mathfrak{G}_2)$ is equal to $m - d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2)$.*

Proof is dual to that of Theorem 3.

We shall compare this with the definition of d_e. By $C(\mathfrak{G}_1, \mathfrak{G}_2)$ we denote the class of all graphs which are isomorphic simultaneously to a subgraph of a graph from $\mathfrak{G}_1$ and to a subgraph of a graph from $\mathfrak{G}_2$. Obviously $OC(\mathfrak{G}_1, \mathfrak{G}_2) \subseteq C(\mathfrak{G}_1, \mathfrak{G}_2)$, $IC(\mathfrak{G}_1, \mathfrak{G}_2) \subseteq C(\mathfrak{G}_1, \mathfrak{G}_2)$. Thus the maximum number m_c of edges of a graph from $C(\mathfrak{G}_1, \mathfrak{G}_2)$ is greater than or equal to the maximum number m_{OC} of edges of a graph from $OC(\mathfrak{G}_1, \mathfrak{G}_2)$ and also greater than or equal to the maximum number m_{IC} of edges a graph from $IC(\mathfrak{G}_1, \mathfrak{G}_2)$.

We have another theorem.

Theorem 4. *For two outdegree equivalent isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of directed graphs we have*

$$d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) \geqq \tfrac{1}{2} d_e(\mathfrak{G}_1, \mathfrak{G}_2) .$$

Proof. We have

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2m - 2m_C$$

by the definition. Further, Theorem 3 implies that

$$d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) = m - m_{OC} .$$

From these two equalities we obtain the assertion.

Theorem 4'. *For two indegree equivalent isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of directed graphs we have*

$$d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2) \geqq \tfrac{1}{2} d_e(\mathfrak{G}_1, \mathfrak{G}_2) .$$

Proof is dual to the proof of Theorem 4.

Now we shall consider $d_r(\mathfrak{G}_1, \mathfrak{G}_2)$. We shall again consider two isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of digraphs with the same number n of vertices and the same number m

of edges. Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$ and let C_2 be obtained from G_1 by $d_r(\mathfrak{G}_1, \mathfrak{G}_2)$ edge rotations. Thus G_1, G_2 have a common vertex set V. Similarly as above we denote the edge sets of G_1, G_2 by E_1, E_2, respectively, and further, $F_0 = E_1 \cap E_2$, $F_1 = E_1 - E_2$, $F_2 = E_2 - E_1$. Evidently there exists a mapping φ of F_1 onto F_2 such that when transforming G_1 into G_2 by the minimum number of edge rotations the edge $e \in F_1$ is transferred by edge rotations to $\varphi(e) \in F_2$. The mapping φ is one-to-one. Now we prove a lemma.

Lemma. *Let x_1, y_1, x_2, y_2 be vertices of a digraph G (having at least three vertices) such that the edge x_1y_1 belongs to G and x_2y_2 does not. If $x_2 \neq y_1$ or $y_2 \neq x_1$, then the edge x_1y_1 can be transferred into x_2y_2 by one or two edge rotations. If $x_2 = y_1$ and $y_2 = x_1$, then this can be done by three edge rotations.*

Proof. If $x_1 = x_2$, the edge x_1y_1 can be transferred into x_2y_2 by one I-rotation around x_1. If $y_1 = y_2$, this can be done by one T-rotation around y_1. Let $x_1 \neq x_2$, $y_1 \neq y_2$. Suppose $x_2 \neq y_1$. If x_2y_1 does not belong to G_2, we perform the T-rotation of x_1y_1 into the position x_2x_1 and then the I-rotation of x_2y_1 into the position x_2y_2. If x_2y_1 belongs to G_2, we perform first the I-rotation of x_2y_1 into x_2y_2 and then the T-rotation of x_1y_1 into x_2y_1. If $x_2 = y_1$ but $x_1 \neq y_2$, we may proceed dually. Now the case $x_2 = y_1$, $y_2 = x_1$ remains; then x_2y_2 is oriented inversely to x_1y_1. Choose a vertex z different from x_1, y_1. If neither zy_1 nor zx_1 is in G, then we perform a T-rotation R_1 of x_1y_1 into zy_1, an I-rotation R_2 of zy_1 into zx_1 and a T-rotation R_3 of zx_1 into $y_1x_1 = x_2y_2$. If zy_1 is in G and zx_1 is not, we perform first R_2, then R_3 and R_1. If both zy_1, zx_2 are in G, we perform the rotations in the order R_3, R_2, R_1. If zx_1 is in G and zy_1 is not, we perform them in the order R_1, R_3, R_2. On the other hand, we see that two edge rotations are not sufficient, because we do not admit loops.

Theorem 5. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be isomorphism classes of directed graphs with the same number n of vertices and the same number m of edges. If $d_e(\mathfrak{G}_1, \mathfrak{G}_2) > 2$, then*

$$\tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2) \leqq d_r(\mathfrak{G}_1, \mathfrak{G}_2) \leqq d_e(\mathfrak{G}_1, \mathfrak{G}_2)\,.$$

If $d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2$, then

$$1 \leqq d_r(\mathfrak{G}_1, \mathfrak{G}_2) \leqq 3\,.$$

Remark. As we see from (1), the value of $d_e(\mathfrak{G}_1, \mathfrak{G}_2)$ under the described conditions is always even.

Proof. The formula (1) can be rewritten with the new notation:

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2m - 2m_C\,.$$

Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. Let $G_0 \in C(\mathfrak{G}_1, \mathfrak{G}_2)$ and let G_0 have m_C edges. We perform the same procedure with G_1, G_2, G_0 as with G_1, G_2, F' in the proof of Theorem 3. Then G_1, G_2 have the common vertex set V and a common subgraph G_0. Use again the notation E_1, E_2, F_0, F_1, F_2 as above. We have $|F_1| = |F_2| = m - m_C = \frac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2)$. Choose a one-to-one mapping φ of F_1 onto F_2. If $|F_1| \geqq 2$, this mapping can be chosen is such a way that no edge is mapped onto the edge obtained from it by reversing the orientation. Now each edge $e \in F_1$ will be transformed into its image $\varphi(e) \in F_2$ by edge rotations. If $\varphi(e)$ is not obtained from e by reversing the orientation for any e, then, according to Lemma, the number of the necessary edge rotations is greater than or equal to $|F_1|$ and less than or equal to $2|F_1|$. Thus

$$\tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2) \leqq d_r(\mathfrak{G}_1, \mathfrak{G}_2) .$$

If $|F_1| = 1$, then the element of F_1 can be transferred (again according to Lemma) into the element of F_2 by one, two or three edge rotations, and thus

$$1 \leqq d_r(\mathfrak{G}_1, \mathfrak{G}_2) \leqq 3 .$$

Theorem 6. *Let α, β be positive integers such that $\alpha \leqq \beta \leqq 2\alpha$. Then there exist isomorphism classes $\mathfrak{G}_1, \mathfrak{G}_2$ of digraphs with equal numbers of vertices and equal numbers of edges such that*

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2\alpha ,$$
$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) = \beta .$$

Proof. First construct the graph G_0. Its vertex set is $\{u_1, \ldots, u_{2\alpha}, v_1, \ldots, v_{2\alpha}\}$and its edge set consists of the edges u_iu_j, v_iv_j for $1 \leqq i < j \leqq 2\alpha$. (Thus G_0 consists of two connected components which are acyclic tournaments.) The graph G_1 contains all vertices and edges of G_0 and, moreover, the edges $u_{2i}v_{2i}$ for $i = 1, \ldots, \alpha$. The graphs G_2 contains also all vertices and edges of G_0 and, moreover, the edges $u_{2i-1}v_{2i-1}$ for $i = 1, \ldots, \beta - \alpha$ and $u_{2i}v_{2i-1}$ for $i = \beta - \alpha + 1, \ldots, \alpha$. The reader may verify himself that G_0 is a graph from $C(\mathfrak{G}_1, \mathfrak{G}_2)$ (where $\mathfrak{G}_1, \mathfrak{G}_2$ are the isomorphism classes containing G_1, G_2, respectively) with the maximum number of edges; there are α edges belonging to G_1 and not to G_0, hence

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = \alpha .$$

Any edge $u_{2i}v_{2i}$ for $i = 1, \ldots, \beta - \alpha$ can be transferred by two edge rotations $u_{2i-1}v_{2i-1}$ and any edge $u_{2i}v_{2i}$ for $i = \beta - \alpha + 1, \ldots, \alpha$ can be transferred by one I-rotation into $u_{2i}v_{2i-1}$; the total number of these edge rotations is β and this is evidently the minimum number of edge rotations necessary to transform G_1 into G_2.

At the end we shall consider outdegree regular graphs and indegree regular ones. A directed graph is called outdegree regular (or indegree regular), if all of its vertices have equal outdegrees (or indegrees, respectively).

Theorem 7. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of outdegree regular directed graphs with the same number n of vertices and the same number m of edges. Then*

$\mathfrak{G}_1, \mathfrak{G}_2$ *are outdegree equivalent and*

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) = d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) = \tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2).$$

Proof. Let $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. As G_1, G_2 have the same number n of vertices and the same number m of edges and are outdegree regular, all vertices of G_1 and all vertices of G_2 have the same outdegree m/n. From the definitions it is evident that $OC(\mathfrak{G}_1, \mathfrak{G}_2) = C(\mathfrak{G}_1, \mathfrak{G}_2)$ and thus $m_{OC} = m_C$. As

$$m_{OC} = m - d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2),$$

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2m - 2m_C,$$

we have

$$d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2) = \tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2).$$

As, according to Proposition 1,

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) \leqq d_{Ir}(\mathfrak{G}_1, \mathfrak{G}_2)$$

and according to Theorem 5

$$\tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2) \leqq d_r(\mathfrak{G}_1, \mathfrak{G}_2),$$

we have

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) = \tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2).$$

Theorem 7′. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of indegree regular directed graphs with the same number n of vertices and the same number m of edges. Then $\mathfrak{G}_1, \mathfrak{G}_2$ are indegree equivalent and*

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) = d_{Tr}(\mathfrak{G}_1, \mathfrak{G}_2) = \tfrac{1}{2}d_e(\mathfrak{G}_1, \mathfrak{G}_2).$$

Proof is dual to the proof of Theorem 7.

Theorem 8. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of directed graphs with the same number n of vertices and the same number $m \neq 0$ of edges. Then*

$$d_e(\mathfrak{G}_1, \mathfrak{G}_2) \leqq 2m - 2$$

and this bound cannot be improved.

Proof. If $m \neq 0$, then the graph consisting of two vertices and an edge joining them belongs to $C(\mathfrak{G}_1, \mathfrak{G}_2)$ and thus $m_C \geqq 1$ and $d_e(\mathfrak{G}_1, \mathfrak{G}_2) \leqq 2m - 2$. An example of classes $\mathfrak{G}_1, \mathfrak{G}_2$ for which $d_e(\mathfrak{G}_1, \mathfrak{G}_2) = 2m - 2$ are the classes containing the graph G_1 which is a directed path of the length m and the graph G_2 which is a star with m edges directed from the center.

Note that for the classes $\mathfrak{G}_1, \mathfrak{G}_2$ from the proof of Theorem 8 we have $d_r(\mathfrak{G}_1, \mathfrak{G}_2) = m - 1$, because they are classes of indegree regular graphs. This leads us to a conjecture.

Conjecture. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes of directed graphs with the same number n of vertices and the same number m of edges. Then*

$$d_r(\mathfrak{G}_1, \mathfrak{G}_2) \leqq m - 1 .$$

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Souhrn

VZDÁLENOSTI MEZI ORIENTOVANÝMI GRAFY

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Pojmy hranové vzdálenosti a hranové rotační vzdálenosti zavedené dříve jinými autory pro neorientované grafy jsou modifikovány pro případ orientovaných grafů.

Резюме

РАССТОЯНИЯ МЕЖДУ ОРИЕНТИРОВАННЫМИ ГРАФАМИ

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Понятия реберного расстояния и реберно-вращательного расстояния, введеные другими авторами для неориентированных графов, здесь применены к ориентированным графам.

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