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# ON THE MAXIMUM OF GENERALIZED DARBOUX FUNCTIONS

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*Summary.* The authors show that the proof of a theorem on the maximum of generalized Darboux functions given by Farková contains a gap, and prove the theorem for the special case of the Euclidean space with the collection of all open intervals as a base.

*Keywords:* generalized Darboux functions.

Let $X$ be a topological space with a base $\mathscr{B}$. A real valued function $f$ on $X$ is said to be in $D_0(\mathscr{B})$ if it has the following property:

If $B \in \mathscr{B}$, $x, y \in \bar{B}$, the closure of $B$, and $\eta$ is a real number with $f(x) < \eta < f(y)$, then for an arbitrary $\varepsilon > 0$ there is a point $z \in B$ such that $f(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

The conditions (1*) and (2) below imposed on the base $\mathscr{B}$ are required for some conclusions.

(1*) For arbitrary $x \in X$, $B \in \mathscr{B}$, if $\mathcal{O}$ is an open set and $x \in \mathcal{O} \cap \bar{B}$, then there exists $U \in \mathscr{B}$ such that $U \subset \mathcal{O} \cap B$ and $x \in \bar{U} - U$.

(2) For every $B \in \mathscr{B}$ and every decomposition of $B$, $B = C \cup D$, $C \cap D = \emptyset$, $C \neq \emptyset \neq D$ with the property that $\bar{U} \cap B \subset C$ or $\bar{U} \cap B \subset D$ whenever $U \in \mathscr{B}$ and $U \subset C$ or $U \subset D$, respectively, we have $C' \cap D \neq \emptyset \neq C \cap D'$, where $C'$, $D'$ are the derived sets of $C$, $D$, respectively.

Farková proved some interesting results about the maximum of functions in $D_0(\mathscr{B})$ ([1], pp. 113–114):

**Theorem F1.** *Let $X$ be a topological space with a base $\mathscr{B}$ satisfying (1*) and (2). Let $f, g \in D_0(\mathscr{B})$ be such that every $x \in X$ is a point of the upper semi-continuity of $f$ or $g$. Then $\varphi = \max(f, g) \in D_0(\mathscr{B})$.*

**Theorem F2.** *Let $X$ be a topological space with a base $\mathscr{B}$. Let $f \in D_0(\mathscr{B})$. If $f$ is not upper semi-continuous, then there exists a function $g \in D_0(\mathscr{B})$ such that $\varphi = \max(f, g) \notin D_0(\mathscr{B})$.*

Unfortunately, the function $g$ constructed in the proof of Theorem F2 is not necessarily in $D_0(\mathscr{B})$, as the example below shows. Therefore Theorem F2 is dubious.

We consider the Euclidean plane $E_2$. Let $\mathscr{B}$ be the collection of all open intervals $\{(x, y): a < x < b,\ c < y < d\}$, $a < b$, $c < d$. Define $f$ on $E_2$ as follows:

$$f(x, y) = \frac{y}{x + y} \sin \frac{1}{x + y} \quad \text{if} \quad x \geqq y > 0 ,$$

$$= \frac{x}{x + y} \sin \frac{1}{x + y} \quad \text{if} \quad y > x > 0 ,$$

$$= 0 \quad \text{otherwise} .$$

Clearly $f$ is continuous at every $(x, y) \neq (0, 0)$ and it can be easily shown that $f \in D_0(\mathscr{B})$. $f$ is not upper semi-continuous at $(0, 0)$, since

$$\overline{\lim_{(x,y)\to(0,0)}} f(x, y) = \tfrac{1}{2} > f(0, 0) .$$

The function $g$ constructed in [1] is defined by $g(0, 0) = f(0, 0) = 0$, $g(x, y) = 2K - f(x, y)$ if $(x, y) \neq (0, 0)$, where $K$ is a number with $\frac{1}{4} \geqq K > 0$. It is obviously not in $D_0(\mathscr{B})$.

The purpose of the present paper is to prove the validity of Theorem F2 for the case that $X$ is $E_2$ and $\mathscr{B}$ is the collection of all open intervals in $E_2$. Before we proceed to the main result, we state two theorems given in [2] (p. 418 and p. 422) which will be needed.

**Theorem M1.** *Let $X$ be locally connected topological space, $\mathscr{B}$ a base consisting of open connected sets and satisfying* (1*). *Let $f, g \in D_0(\mathscr{B})$. If each $x$ is a point of continuity of $f$ or $g$, then $f + g \in D_0(\mathscr{B})$.*

**Theorem M2.** *Let $X$ and $\mathscr{B}$ be as in Theorem* M1. *If $g$ is a continuous function on $X$ and $f \in D_0(\mathscr{B})$ such that $f$ is bounded at each $x \in X$ where $g(x) = 0$, then $fg \in D_0(\mathscr{B})$.*

**Theorem 1.** *Let $\mathscr{B}$ be the collection of all open intervals in $E_2$. If $f$ is a function on $E_2$ such that* $\max(f, g) \in D_0(\mathscr{B})$ *for every $g \in D_0(\mathscr{B})$, then $f \in D_0(\mathscr{B})$ and $f$ is upper semi-continuous on $E_2$.*

Proof. Since every constant function is in $D_0(\mathscr{B})$, the hypothesis clearly implies that $f \in D_0(\mathscr{B})$. To show that $f$ is upper semi-continuous, we assume the contrary and construct a function $g \in D_0(\mathscr{B})$ such that $\max(f, g) \notin D_0(\mathscr{B})$.

Suppose $f$ is not upper semi-continuous at $p_0 = (x_0, y_0)$. Then $\overline{\lim}_{p\to p_0} f(p) > f(p_0)$. Let $K$ be a number such that

$$f(p_0) < K < \overline{\lim_{p\to p_0}} f(p) \quad \text{and} \quad 2K < f(p_0) + \overline{\lim_{p\to p_0}} f(p) .$$

Since $f \in D_0(\mathscr{B})$, it can be easily shown that, if $p = (x, y)$,

$$\overline{\lim_{p\to p_0}} f(p) = \overline{\lim_{\substack{p\to p_0 \\ x\neq x_0, y\neq y_0}}} f(p) .$$

Let $p_0(\mathrm{I}) = \{p = (x, y): x > x_0,\ y > y_0\}$, $p_0(\mathrm{II}) = \{p = (x, y): x < x_0,\ y > y_0\}$, $p_0(\mathrm{III}) = \{p = (x, y): x < x_0,\ y < y_0\}$ and $p_0(\mathrm{IV}) = \{p = (x, y): x > x_0,\ y < y_0\}$. Then at least one of

$$\overline{\lim_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}}} f(p) \quad (\Lambda = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$$

is equal to $\overline{\lim}_{p \to p_0} f(p)$.

Let $\hat{f}(p) = \max(f(p), f(p_0))$. Then $\hat{f} \in D_0(\mathscr{B})$,

$$\overline{\lim_{p \to p_0}} \hat{f}(p) = \overline{\lim_{p \to p_0}} f(p) > f(p_0) = \hat{f}(p_0)$$

and

$$\max(\hat{f}, g) = \max(f, \max(f(p_0), g)) \in D_0(\mathscr{B})$$

for every $g \in D_0(\mathscr{B})$. Therefore, every statement above remains valid if $f$ is replaced by $\hat{f}$, and we can assume with no loss of generality that $f$ is bounded below on $E_2$.

Using $f \in D_0(\mathscr{B})$ we can show that, for each $\Lambda = \mathrm{I}, \mathrm{II}, \mathrm{III}$ or $\mathrm{IV}$, there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset p_0(\Lambda)$ such that $p_n \to p_0$ and $f(p_n) \to f(p_0)$. In the case

$$\overline{\lim_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}}} f(p) \leqq 2K - f(p_0)$$

there exists $U_\Lambda \in \mathscr{B}$ such that $U_\Lambda \subset p_0(\Lambda)$, $p_0 \in \overline{U}_\Lambda$ and $f(p) \leqq 2K - f(p_0) + 1$ for every $p \in U_\Lambda$. Thus $f$ is also bounded above on $U_\Lambda$. With no loss of generality, we assume that the above sequence $\{p_n\} \subset U_\Lambda$. Let $X_\Lambda = \mathrm{cl}(p_0(\Lambda)) - \{p_0\}$. Then $\mathscr{B}_\Lambda = \{B \cap X_\Lambda: B \in \mathscr{B},\ B \cap X_\Lambda \neq \emptyset\}$ is a base for the subspace $X_\Lambda$, and the sets $A_{\Lambda 1} = \{p_n: n = 1, 2, \ldots\}$, $A_{\Lambda 2} = X_\Lambda - U_\Lambda$ are two disjoint, closed (relative to $X_\Lambda$) sets on $X_\Lambda$. The function $h_\Lambda$ on $X_\Lambda$ defined for each $p \in X_\Lambda$ by

$$h_\Lambda(p) = \frac{d(p, A_{\Lambda 1})}{d(p, A_{\Lambda 1}) + d(p, A_{\Lambda 2})},$$

where $d$ is the usual distance, is continuous on $X_\Lambda$, $h_\Lambda(A_{\Lambda 1}) = 0$, $h_\Lambda(A_{\Lambda 2}) = 1$ and $h_\Lambda(p) \in (0, 1)$ if $p \in X_\Lambda - A_{\Lambda 1} - A_{\Lambda 2}$. Also, it is easily seen that the restriction $f|X_\Lambda \in D_0(\mathscr{B}_\Lambda)$. Noting that $f$ is bounded on $U_\Lambda$ and $2h_\Lambda(p) - 1 = 0$ only at some points $p \in X_\Lambda - A_{\Lambda 1} - A_{\Lambda 2} \subset U_\Lambda$, we apply Theorems M1 and M2 and conclude that the function $g_\Lambda$ on $X_\Lambda$ defined by

$$g_\Lambda(p) = 2Kh_\Lambda(p) - (2h_\Lambda(p) - 1) f(p) \quad \text{for} \quad p \in X_\Lambda$$

is in $D_0(\mathscr{B}_\Lambda)$.

In the case $\overline{\lim}_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) > 2K - f(p_0)$ we define

$$g_\Lambda(p) = 2K - f(p) \quad \text{for} \quad p \in X_\Lambda,$$

and we also have $g_\Lambda \in D_0(\mathscr{B}_\Lambda)$. In particular, for all $\Lambda = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, the following holds:

($\#$) If $B \in \mathscr{B}$, $B \subset p_0(\Lambda)$, $q_1, q_2 \in \bar{B} - \{p_0\}$ ($\bar{B} - \{p_0\}$ is the closure of $B$ relative to the subspace $X_\Lambda$), $\eta \in R$ such that $g_\Lambda(q_1) < \eta < g_\Lambda(q_2)$, then for given $\varepsilon > 0$, there exists $z \in B$ with $g_\Lambda(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

It should be noted that, for $p \in X_\Lambda \cap X_{\Lambda'}$, $g_\Lambda(p) = g_{\Lambda'}(p)$. Thus we can define $g$ on $E_2$ as follows:

$$\begin{aligned} g(p) &= g_\Lambda(p) \quad \text{if} \quad p \in X_\Lambda \quad (\Lambda = \text{I, II, III, IV}), \\ &= f(p_0) \quad \text{if} \quad p = p_0 . \end{aligned}$$

Now we show that $g \in D_0(\mathscr{B})$. Let $B \in \mathscr{B}$, $q_1, q_2 \in \bar{B}$, $\eta \in R$ such that $g(q_1) < \eta < g(q_2)$, and $\varepsilon > 0$ be given. We want to show there is a $z \in B$ with $g(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

Case 1. $B \subset p_0(\Lambda)$ for some $\Lambda$. If $q_1 \neq p_0 \neq q_2$, then the conclusion follows from ($\#$) above. Hence we assume that either $q_1 = p_0$ or $q_2 = p_0$. Also, for this $\Lambda$, we may have

$$\varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) \leqq 2K - f(p_0) \quad \text{or} \quad \varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) > 2K - f(p_0) .$$

**1.1.** $\varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) \leqq 2K - f(p_0)$ and $q_1 = p_0$ (or $q_2 = p_0$). We recall that the set $A_{\Lambda 1}$ is a sequence $\{p_n\}$ in $p_0(\Lambda)$ such that $p_n \to p_0$ and $f(p_n) \to f(p_0)$. Since $p_0 = q_1$ (or $p_0 = q_2$), $p_0 \in \bar{B}$. Hence we see that there exists $n$ such that $p_n \in B$ and $f(p_n) < \eta$ (or $f(p_n) > \eta$). Also, $p_n \in A_{\Lambda 1}$ implies $h_\Lambda(p_n) = 0$ and $g(p_n) = g_\Lambda(p_n) = f(p_n)$. Consequently, $p_n$ and $q_2$ (or $q_1$ and $p_n$) are points in $\bar{B}$, both different from $p_0$ and satisfying $g(p_n) < \eta < g(q_2)$ (or $g(q_1) < \eta < g(p_n)$). By ($\#$), there exists $z \in B$ with $g(z) = g_\Lambda(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

**1.2.** $\varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) > 2K - f(p_0)$ and $q_1 = p_0$. Since $B \subset p_0(\Lambda)$ and $p_0 \in \bar{B}$, we have

$$\varlimsup_{\substack{p \to p_0 \\ p \in B}} f(p) = \varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) > 2K - f(p_0) = 2K - g(q_1) > 2K - \eta$$

and hence there is a point $p \in B$ with $f(p) > 2K - \eta$. That is, $g(p) = g_\Lambda(p) = 2K - f(p) < \eta$. Now $p, q_2 \in \bar{B}$, $g(p) < \eta < g(q_2)$ and $p \neq p_0 \neq q_2$. We can use ($\#$) again.

**1.3.** $\varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) > 2K - f(p_0)$ and $q_2 = p_0$. By the choice of $K$, $f(p_0) < K$ and hence $f(p_0) < 2K - f(p_0)$. Thus we have $g(q_1) = g_\Lambda(q_1) = 2K - f(q_1)$ and $g(q_2) = g(p_0) = f(p_0) < 2K - f(p_0) = 2K - f(q_2)$. The inequalities $g(q_1) < \eta < g(q_2)$ imply $f(q_2) < 2K - \eta < f(q_1)$. Since $f \in D_0(\mathscr{B})$, there exists $z \in B$ with $f(z) \in (2K - \eta - \varepsilon, 2K - \eta + \varepsilon)$. It follows that $g(z) = g_\Lambda(z) = 2K - f(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

Case 2. $B \not\subset p_0(\Lambda)$ for $\Lambda = $ I, II, III, or IV. Let $B_\Lambda = B \cap p_0(\Lambda)$. Then either $B_\Lambda \neq \emptyset$ for all four $\Lambda$'s or for exactly two $\Lambda$'s (that is, for $\Lambda = $ I, II, or II, III, or III, IV, or IV, I).

**2.1.** $B_\Lambda \neq \emptyset$ for two $\Lambda$'s. For example, $B_{\text{I}} \neq \emptyset \neq B_{\text{II}}$ (the other cases are similar). Then $\bar{B} = \bar{B}_{\text{I}} \cup \bar{B}_{\text{II}}$. If $q_1, q_2$ are both in $\bar{B}_{\text{I}}$ or $\bar{B}_{\text{II}}$, then this is reduced to Case 1. We assume that $q_1 \in \bar{B}_{\text{I}}$, $q_2 \in \bar{B}_{\text{II}}$ and pick any point $q_3 \in B - (B_{\text{I}} \cup B_{\text{II}})$ (thus $q_3 \in \bar{B}_{\text{I}} \cap \bar{B}_{\text{II}}$). There is nothing more to prove if $g(q_3) = \eta$. If $g(q_3) < \eta$, we consider $q_3, q_2 \in \bar{B}_{\text{II}}$. If $g(q_3) > \eta$, we consider $q_1, q_3 \in \bar{B}_{\text{I}}$. In either case, it is solved by Case 1.

**2.2.** $B_\Lambda \neq \emptyset$ for all four $\Lambda$'s. Let $C_1 = B - (\bar{B}_{\text{III}} \cup \bar{B}_{\text{IV}})$ and $C_2 = B - (\bar{B}_{\text{I}} \cup \bar{B}_{\text{II}})$. Then $C_1, C_2 \in \mathscr{B}$, both are of the type in 2.1 above and $\bar{B} = \bar{C}_1 \cup \bar{C}_2$. For this case, the conclusion follows from 2.1 in the same manner as 2.1 follows from Case 1.

We have just showed that $g \in D_0(\mathscr{B})$. It remains to show that $\varphi = \max(f, g) \notin D_0(\mathscr{B})$. Since there exists at least one $\Lambda$ such that

$$\varlimsup_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}} f(p) = \varlimsup_{p \to p_0} f(p) > 2K - f(p_0),$$

we have $g(p) = 2K - f(p)$ for every $p \in X_\Lambda = \text{cl}(p_0(\Lambda)) - \{p_0\}$ for this $\Lambda$. For $B \in \mathscr{B}$ such that $B \subset p_0(\Lambda)$ and $p_0 \in \bar{B}$, $g(p) = 2K - f(p)$ or $f(p) + g(p) = 2K$ for every $p \in B$ and hence $\varphi(p) \geqq K$ for every $p \in B$. But $\varphi(p_0) < K$. Clearly $\varphi \notin D_0(\mathscr{B})$. The proof is completed.

**Theorem 2.** *Let $\mathscr{B}$ be the collection of all open intervals in $E_2$ and $f \in D_0(\mathscr{B})$. Then* $\max(f, g) \in D_0(\mathscr{B})$ *for every $g \in D_0(\mathscr{B})$ if and only if $f$ is upper semi-continuous on $E_2$.*

Proof. In view of Theorem F1 and Theorem 1, all we need to show is that $\mathscr{B}$ satisfies the conditions (1*) and (2). It is trivial that $\mathscr{B}$ satisfies (1*). We now prove that $\mathscr{B}$ also satisfies (2). Let $B \in \mathscr{B}$, $B = C \cup D$, $C \cap D = \emptyset$, $C \neq \emptyset \neq D$ such that for $U \in \mathscr{B}$, $\bar{U} \cap B \subset C$ or $\bar{U} \cap B \subset D$ whenever $U \subset C$ or $U \subset D$, respectively, be given. We want to show that $C' \cap D \neq \emptyset \neq C \cap D'$. Suppose $C' \cap D = \emptyset$. Then $B \cap C' \subset C$, $C$ is closed relative to $B$ and hence $D$ is open. Since $C \neq \emptyset \neq D$, we can pick $p \in C$, $q \in D$ and $B_1 \in \mathscr{B}$ such that $p, q \in B_1$ and $\bar{B}_1 \subset B$. Let $C_1 = B_1 \cap C$, $D_1 = B_1 \cap D$. Then $q$ is a point of the open set $D_1$. We can partially order the collection $\mathscr{I} = \{U \in \mathscr{B}: q \in U \subset D_1\}$ by inclusion. It is clear that every chain is bounded above. By Zorn's lemma, there is a maximal member $U_0$ in $\mathscr{I}$. Now $U_0 \in \mathscr{B}$ and $U_0 \subset D_1 \subset D$. By our assumption, $\bar{U}_0 \cap B \subset D$. That is $\bar{U}_0 \subset D$ since $\bar{U}_0 \subset \bar{B}_1 \subset B$. For the compact interval $\bar{U}_0$ in the open set $D$, we can easily construct a $U \in \mathscr{B}$ such that $\bar{U}_0 \subset U \subset D$. Let $U_1 = B_1 \cap U$. Then $U_1 \in \mathscr{B}$ and $U_0 \subset B_1 \cap \bar{U}_0 \subset U_1 \subset D_1$. Since $C_1 \neq \emptyset$, $B_1 \cap \bar{U}_0$ properly contains $U_0$ and so does $U_1$. This contradicts the maximality of $U_0$. Thus $C' \cap D \neq \emptyset$. Similarly $C \cap D' \neq \emptyset$. Theorem 2 is proved.

**Remark.** The results in this paper can be easily extended to the n-dimensional Euclidean space with the base $\mathscr{B}$ consisting of all open intervals in $E_n$. It is not known whether the same conclusion is true for a general topological space $X$.

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## Souhrn

### O MAXIMU ZOBECNĚNÝCH DARBOUXOVÝCH FUNKCÍ

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Autoři ukazují, že důkaz věty o maximu zobecněných Darbouxových funkcí podaný Farkovou obsahuje mezeru, a dokazují tuto větu pro speciální případ eukleidovského prostoru s bází danou soustavou všech otevřených intervalů.

## Резюме

### О МАКСИМУМЕ ОБОБЩЕННЫХ ФУНКЦИЙ ДАРБУ

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Авторы показывают, что в доказательстве теоремы Фарковой о максимуме обобщенных функций Дарбу имеется пробель, и доказывают эту теорему для специального случая евклидова пространства с базисом состоящим из всех открытых интервалов.

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