Jan Rataj

Some remarks on Fourier seminorms

Časopis pro pěstování matematiky, Vol. 112 (1987), No. 4, 368--372

Persistent URL: http://dml.cz/dmlcz/108551

SOME REMARKS ON FOURIER SEMINORMS

JAN RATAJ, Praha

(Received June 5, 1985)

Summary. The author shows the equivalence between Fourier and Gaussian classes of seminorms on the space of signed measures on $\boldsymbol{R}^k$, which are nonnegative on the complement of the zero point.

Keywords: signed measures, Gaussian and Fourier seminorms, convolution powers.

AMS Subject Classification: 60F (28A).

Classical limit problems for convolution powers and convolution products of measures on finite dimensional spaces can be often successfully studied by using suitable seminorms on the space of signed measures. H. Bergström [1] gave a systematical study of these problems using especially Gaussian seminorms. This note should support the idea that many of the results of H. Bergström can be also formulated in terms of Fourier seminorms which seem to be in some cases a bit simpler than Gaussian seminorms. F. Zítek [2] showed that both the seminorms mentioned are equivalent for signed measures on the real line which are nonnegative on the complement of the zero point. We shall show a similar result for measures on $\boldsymbol{R}^k$.

Let $k \in \boldsymbol{N}$ and denote by $\mathscr{M}_k$ the algebra of all finite Borel signed measures on $\boldsymbol{R}^k$ (with natural additivity and convolution products), and by $\mathscr{M}_k^+$ the cone of nonnegative finite Borel measures.

For $\alpha > 0$ we define

$$F_\alpha(\mu) = \sup_{[t]<\alpha^{-1}} \left| \int_{\boldsymbol{R}^k} \exp\left(i\langle t, x\rangle\right) \mathrm{d}\mu(x) \right|, \quad \mu \in \mathscr{M}_k ,$$

where $[t] = \max\{|t_1|, \ldots, |t_k|\}$, $t = (t_1, \ldots, t_k) \in \boldsymbol{R}^k$, and $\langle \cdot, \cdot \rangle$ is the scalar product on $\boldsymbol{R}^k$. We easily see that for any $\mu, \nu \in \mathscr{M}_k$ and $c \in \boldsymbol{R}$

$$\begin{aligned} &F_\alpha(\mu) \geqq 0 , \quad F_\alpha(0) = 0 , \\ &F_\alpha(\mu + \nu) \leqq F_\alpha(\mu) + F_\alpha(\nu) , \\ &F_\alpha(c\mu) \leqq |c| \, F_\alpha(\mu) . \end{aligned}$$

Thus F_α is a (Fourier) seminorm on $\mathscr{M}_k$ for each $\alpha > 0$. We claim that if $F_\alpha(\mu) = 0$ for all $\alpha > 0$, then $\mu = 0$. Thus (according to the terminology used in [1], see p. 95) $\mathscr{M}_k$ is a seminormed algebra under the system $\{F_\alpha : \alpha > 0\}$ of Fourier seminorms.

Moreover, Fourier seminorms fulfil the inequality

$$F_\alpha(\mu \,.\, \nu) \leqq F_\alpha(\mu)\, F_\alpha(\nu)$$

for each $\mu, \nu \in \mathcal{M}_k$, $\alpha > 0$.

We shall also use other seminorms; for $\alpha > 0$ and $\mu \in \mathcal{M}_k$, define

$$q_\alpha(\mu) = \max \left\{ \int_{\{|x| \geqq \alpha\}} \mathrm{d}|\mu^{(i)}|(x),\ \alpha^{-1} \left| \int_{\{|x| < \alpha\}} x\, \mathrm{d}\mu^{(i)}(x) \right|, \right.$$

$$\left. \alpha^{-2} \int_{\{|x| < \alpha\}} x^2\, \mathrm{d}|\mu^{(i)}|(x),\ |\mu(R^k)|,\ i = 1, \ldots, k \right\}$$

($\mu^{(i)}$ denotes the i-th marginal projection of μ). It can be also easily verified that q_α are seminorms, $\alpha > 0$ (we shall call them majorant seminorms), and that $\mathcal{M}_k$ is a seminormed algebra under the system $\{q_\alpha : \alpha > 0\}$.

Directly from the above definitions we obtain

Proposition 1. *For each $\mu \in \mathcal{M}_k$ and $\alpha > 0$*

i) $F_\alpha(\mu^{(i)}) \leqq F_\alpha(\mu)$, $i = 1, \ldots, k$,

ii) $q_\alpha(\mu) = \max\limits_{1 \leqq i \leqq k} q_\alpha(\mu^{(i)})$.

The next lemma shows that the seminorms q_α are really majorant for F_α:

Lemma 2. *$F_\alpha(\lambda) \leqq (1 + 3k + k^2/2)\, q_\alpha(\lambda)$ whenever $\lambda \in \mathcal{M}_k$ and $\alpha > 0$.*

Proof. Denote the cube $D_\alpha = \{x \in R^k : |x_i| < \alpha,\ i = 1, \ldots, k\}$. The following inequalities hold:

$$\int_{D_\alpha^c} \mathrm{d}|\lambda|(x) \leqq \sum_{i=1}^{k} \int_{\{|x| \geqq \alpha\}} \mathrm{d}|\lambda^{(i)}|(x) \leqq k\, q_\alpha(\lambda)\,, \tag{1}$$

$$\left| \int_{D_\alpha} \langle t, x \rangle\, \mathrm{d}\lambda(x) \right| \leqq \sum_{i=1}^{k} |t_i| \left| \int_{\{|x_i| < \alpha\}} x_i\, \mathrm{d}\lambda^{(i)}(x_i) \right| \leqq [t]\, k\alpha\, q_\alpha(\lambda)\,, \tag{2}$$

$$\begin{aligned} \int_{D_\alpha} \langle t, x \rangle^2\, \mathrm{d}|\lambda|(x) &\leqq \sum_{i=1}^{k} \sum_{j=1}^{k} |t_i t_j| \int_{D_\alpha} |x_i x_j|\, \mathrm{d}|\lambda|(x) \leqq \\ &\leqq \sum_{i=1}^{k} \sum_{j=1}^{k} |t_i t_j| \int_{D_\alpha} (x_i^2 + x_j^2)/2\, \mathrm{d}|\lambda|(x) \leqq [t]^2 k^2 \alpha^2\, q_\alpha(\lambda)\,. \end{aligned} \tag{3}$$

Using (1), (2) and (3) we obtain

$$\begin{aligned} \left| \int_{R^k} \exp(i\langle t, x \rangle)\, \mathrm{d}\lambda(x) \right| &\leqq \left| \int \mathrm{d}\lambda(x) \right| + \\ + \left| \int [\exp(i\langle t, x \rangle) - 1]\, \mathrm{d}\lambda(x) \right| &\leqq |\lambda(R^k)| + \\ + \left| \int_{D_\alpha} [\exp(i\langle t, x \rangle) - 1]\, \mathrm{d}\lambda(x) \right| &+ \int_{D_\alpha^c} 2\, \mathrm{d}|\lambda|(x) \leqq \end{aligned} \tag{4}$$

$$\leqq |\lambda(R^k)| + 2k\, q_\alpha(\lambda) + \left|\int_{D_\alpha} i\langle t, x\rangle \,\mathrm{d}\lambda(x)\right| +$$

$$+ \int_{D_\alpha} [\exp(i\langle t, x\rangle) - 1 - i\langle t, x\rangle] \,\mathrm{d}|\lambda|(x) \leqq |\lambda(R^k)| +$$

$$+ 2k\, q_\alpha(\lambda) + [t]\, k\alpha\, q_\alpha(\lambda) + \int_{D_\alpha} \langle t, x\rangle^2/2 \,\mathrm{d}|\lambda|(x) \leqq$$

$$\leqq q_\alpha(\lambda) + 2k\, q_\alpha(\lambda) + [t]\, k\alpha\, q_\alpha(\lambda) + \tfrac{1}{2}[t]^2\, k^2\alpha^2\, q_\alpha(\lambda)\,.$$

For $[t] \leqq \alpha^{-1}$ we thus have by (4)

$$\left|\int_{R^k} \exp(i\langle t, x\rangle)\,\mathrm{d}\lambda(x)\right| \leqq (1 + 3k + k^2/2)\, q_\alpha(\lambda)\,,$$

which completes the proof.

As to the converse relation, the seminorms F_α are not majorant for q_α in the general case. Thus we shall limit ourselves to the measures of the form $\lambda = \mu - e$, where μ is a nonnegative measure and e is the Dirac measure concentrated at the zero point.

Lemma 3. *Let $\lambda \in \mathscr{M}_1$ have the form $\lambda = \mu - e$, where $\mu \in \mathscr{M}_1^+$. Then $q_\alpha(\lambda) \leqq \leqq 286\, F_\alpha(\lambda)$, $\alpha > 0$.*

Proof – see [2].

Lemma 3 can be easily generalized to the multidimensional case:

Lemma 4. *Let $\lambda \in \mathscr{M}_k$, $k \in N$, $\lambda = \mu - e$, where $\mu \in \mathscr{M}_k^+$, and let $\alpha > 0$. Then $q_\alpha(\lambda) \leqq 286\, F_\alpha(\lambda)$.*

Proof. Using Proposition 1 and Lemma 3 we get

$$q_\alpha(\lambda) = \max_{1 \leqq i \leqq k} q_\alpha(\lambda^{(i)}) \leqq 286 \max_{1 \leqq i \leqq k} F_\alpha(\lambda^{(i)}) \leqq 286\, F_\alpha(\lambda)\,.$$

Now we introduce the Gaussian seminorms on $\mathscr{M}_k$ (see [1], page 112); we define

$$G_\alpha(\lambda) = \sup_x \left|\int \Phi((x - y)/\alpha)\,\mathrm{d}\lambda(y)\right| \quad \text{for } \alpha > 0\,, \quad \lambda \in \mathscr{M}_k$$

(Φ is the distribution function of the Gaussian measure on R^k with the density function $\mathscr{S}(x) = (2\pi)^{-k/2} \exp(-\|x\|^2/2)$). $\mathscr{M}_k$ is a seminormed algebra under the system $\{G_\alpha : \alpha > 0\}$ of Gaussian seminorms. For comparing Gaussian and majorant seminorms we shall use some results from [1].

Lemma 5. a) *For $\lambda \in \mathscr{M}_k$ the margins satisfy $G_\alpha(\lambda^{(i)}) \leqq G_\alpha(\lambda)$, $\alpha > 0$.*

b) *For $\lambda = \mu - e$, $\mu \in \mathscr{M}_1^+$, and for $\alpha > 0$ we have*

(i) $$\int_{\{|y|\geqq\alpha\}} \mathrm{d}\lambda(y) \leqq C_1\, G_\alpha(\lambda)\,,$$

(ii) $$\alpha^{-1}\left|\int_{\{|y|<\alpha\}} y\, \mathrm{d}\lambda(y)\right| \leqq C_2\, G_\alpha(\lambda)\,,$$

(iii) $$\alpha^{-2}\int_{\{|y|<\alpha\}} y^2\, \mathrm{d}\lambda(y) \leqq C_3\, G_\alpha(\lambda)\,,$$

(iv) $$G_\alpha(\lambda) \leqq \int_{\{|y|\geqq\alpha\}} d\lambda(y) + |\lambda(\boldsymbol{R})| +$$
$$+ \alpha^{-1}\|\Phi'\|\left|\int_{\{|y|<\alpha\}} y\, d\lambda(y)\right| + \frac{1}{2}\alpha^{-2}\|\Phi''\|\int_{\{|y|<\alpha\}} y^2\, d\lambda(y)\,.$$

c) *For $\lambda = \mu - e$, $\mu \in \mathscr{M}_k^+$, and $\alpha > 0$ we have*

$$G_\alpha(\lambda) \leqq C_4 k \sum_{i=1}^{k} G_\alpha(\lambda^{(i)})\,.$$

C_1, C_2, C_3, C_4 are absolute constants.

Proof — see [1], pages 112–116.

Lemma 6. *There exist constants $\gamma_1(k)$ and γ_2 such that for any measure $\lambda = \mu - e$, $\mu \in \mathscr{M}_k^+$, and for $\alpha > 0$,*

(i) $$G_\alpha(\lambda) \leqq \gamma_1(k)\, q_\alpha(\lambda)\,,$$

(ii) $$q_\alpha(\lambda) \leqq \gamma_2\, G_\alpha(\lambda)\,.$$

Proof. i) Using Lemma 5 c) and b) (iv) we get

$$G_\alpha(\lambda) \leqq C_4 k^2 \max_{1\leqq i\leqq k} G_\alpha(\lambda^{(i)}) \leqq$$
$$\leqq C_4 k^2 (2 + \|\Phi'\| + \tfrac{1}{2}\|\Phi''\|) \max_{1\leqq i\leqq k} q_\alpha(\lambda^{(i)}) = \gamma_1(k)\, q_\alpha(\lambda)\,,$$

where we put $\gamma_1(k) = C_4 k^2 (2 + \|\Phi'\| + \frac{1}{2}\|\Phi''\|)$.

ii) According to Proposition 1 and Lemma 5 a) it will be sufficient to find a constant γ_2 such that $q_\alpha(\lambda) \leqq \gamma_2\, G_\alpha(\lambda)$ for measures on the real line $\lambda = \mu - e$, $\mu \in \mathscr{M}_1^+$. If x tends to $+\infty$ in the definition of $G_\alpha(\lambda)$ we see that $|\lambda(\boldsymbol{R})| \leqq G_\alpha(\lambda)$. Using this fact together with b) (i), (ii) and (iii) from Lemma 5, we can put $\gamma_2 = \max\{1, C_1, C_2, C_3\}$.

Finally, we see that the Gaussian and Fourier seminorms are equivalent on the space of signed measures on $\boldsymbol{R}^k$ which are nonnegative on the complement of the zero point. This implies, for instance, that the stability condition for convolution products of probability measures (see [1], page 97) is the same for the both seminorms, and that many results could be formulated using both Gaussian and Fourier seminorms.

References

[1] *H. Bergström:* Weak Convergence of Measures. Academic Press, New York, 1982.
[2] *F. Zítek:* Sur la norme de Fourier, II. Časopis pěst. mat. *95* (1970), 62–65.

Souhrn

POZNÁMKY K FOURIEROVSKÝM SEMINORMÁM

JAN RATAJ

V článku je ukázána ekvivalence mezi fourierovskou a gaussovskou třídou seminorem na prostoru znaménkových měr v $\boldsymbol{R}^k$ nezáporných vně nuly.

Резюме

ЗАМЕЧАНИЯ О СЕМИНОРМАХ ФУРЬЕ

JAN RATAJ

В работе доказывается, что на пространстве конечных неотрицательных вне нулевой точки мер в $\boldsymbol{R}^k$ класс семинорм Гаусса и класс семинорм Фурье эквивалентны.

Author's address: Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.