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PERIODIC VIBRATIONS OF AN EXTENSIBLE BEAM

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1. INTRODUCTION

In the last years both free and forced vibrations of an extensible elastic beam have been studied by several authors ([1] – [6]). Under certain conditions forced vibrations of such a beam are described by the equation

$$u_{tt}(t, x) + u_{xxxx}(t, x) + \alpha u_t(t, x) - \beta u_{xx}(t, x) \int_0^\pi u_\xi^2(t, \xi)\, d\xi = f(t, x) .$$

We are interested in the existence of periodic solutions to this equation. In the presence of damping ($\alpha > 0$) this problem is examined in the paper of V. Lovicar [9]. It may be shown (correspondingly to [8]) that there exists a sequence of free vibrations of undamped beam with hinged ends. However, in the case of $f \neq 0$ we are not able to solve this problem for α large. Thus limite ourselves to looking for a solution of the equation

$$(1) \qquad z_{tt}(t, x) + z_{xxxx}(t, x) = g(t, x) + \\ + \varepsilon \left[f(t, x) + z_{xx}(t, x) \int_0^\pi z_\xi^2(t, \xi)\, d\xi + \varepsilon \tilde{F}(z)(t, x) \right]$$

with homogeneous boundary conditions

$$(2) \qquad z(t, 0) = z(t, \pi) = z_{xx}(t, 0) = z_{xx}(t, \pi) = 0$$

and the condition of periodicity

$$(3) \qquad z(t, x) = z(t + \omega, x) .$$

We make use of the results of the paper by N. Krylová, O. Vejvoda [7].

2. NOTATION AND AN AUXILIARY LEMMA

Let H^m be the Hilbert space of real functions $u(x)$ on $[0, \pi]$ which have generalized square integrable derivatives $u^{(j)}(x)$, $j = 0, 1, \ldots, m$ equipped with the norm

$$|u|_{H^m}^2 = \sum_{j=0}^{m} \int_0^\pi [u^{(j)}(x)]^2 \, dx .$$

Denote by ${}^0H^{2m}$ the space of functions from H^{2m} satisfying the conditions $u^{(2j)}(0) = u^{(2j)}(\pi) = 0$, $j = 0, 1, \ldots, m - 1$, with the norm $|u|_{2m} \equiv |u^{(2m)}|_{H^0}$. Denoting

$$u_k = (2/\pi)^{1/2} \int_0^\pi u(x) \sin kx \, dx ,$$

let h^m be the space of real sequences $\{u_k;\ k = 1, 2, \ldots\})$ in the sequel, we write $u \equiv \equiv \{u_k\})$ for which $|u|_m^2 \equiv \sum_{k=1}^{\infty} k^{2m} u_k^2 < +\infty$. The spaces ${}^0H^{2m}$ and h^{2m} are isometric and isomorphic.

The solution of the equation (1) will be sought in the space $\mathscr{U} = \{u \in C(R, {}^0H^4) \cap C^1(R, {}^0H^2) \cap C^2(R, H^0);\ u(t + \omega) = u(t),\ t \in R\}$ with the norm

$$|u|_{\mathscr{U}} \equiv \max_t |u(t)|_4 + \max_t |u_t(t)|_2 + \max_t |u_{tt}(t)|_0 =$$
$$= \max_t \left(\sum_{k=1}^{\infty} [k^4 u_k(t)]^2\right)^{1/2} + \max_t \left[\sum_{k=1}^{\infty} (k^2 u_k'(t))^2\right]^{1/2} + \max_t \left[\sum_{k=1}^{\infty} (u_k''(t))^2\right]^{1/2} .$$

Then $z \in \mathscr{U}$ satisfies the equation (1) in the sence of H^0 for all $t \in R$. The right hand sides of (1) will be elements of the space $\mathscr{G} \equiv \{u \in C(R, {}^0H^2);\ u(t + \omega) = u(t),\ t \in R$ with the norm $|u|_{\mathscr{G}} \equiv \max_t |u(t)|_2 = \max_t \left(\sum_{k=1}^{\infty} k^4 u_k^2(t)\right)^{1/2}\}$.

For a while, let us investigate the limit problem given by (1), (2), (3) with $\varepsilon = 0$ and $g \in \mathscr{G}$. Looking for a solution z in the form $z(t, x) = \sum_{k=1}^{\infty} z_k(t) \sin kx$ we find easily that $z_k(t)$ must satisfy the equation

$$z_k''(t) + k^4 z_k(t) = g_k(t) , \tag{4}$$

for $k = 1, 2, \ldots$. By a well-known theorem from the theory of ordinary differential equations this equation has an ω-periodic solution if and only if g is orthogonal to the every ω-periodic solution to the corresponding homogeneous equation.

If k satisfies the relation $k^2\omega = 2\pi n$ (n integer) then the homogeneous equation (4) has two linearly independent ω-periodic solutions $\cos k^2 t$, $\sin k^2 t$. Denote by S the set of such k. For the other k there exists no ω-periodic solution. Hence, the orthogonality conditions read

$$\int_0^\omega g_k(t) \cos k^2 t \, dt = 0 , \quad \int_0^\omega g_k(t) \sin k^2 t \, dt = 0 , \quad k \in S . \tag{5}$$

Clearly, if $\nu \equiv 2\pi\omega^{-1}$ is rational the set S is infinite. On the other hand, if ν is irrational the set S is empty, but we can not study this case in the sequel, because by the theorem 6.4.1 from [1] the nonlinearity in (1) includes derivatives of too high order. If these conditions are fulfilled the ω-periodic solution of (4) is of the form

$$(6) \qquad z_k(t) = a_k \cos k^2 t + b_k \sin k^2 t + k^{-2} \int_0^t g_k(\tau) \sin k^2(t - \tau) \, d\tau$$

$(k = 1, 2, \ldots)$, where $a_k, b_k, \sum k^8 a_k^2 + \sum k^8 b_k^2 < \infty$ are arbitrary for $k \in S$ and

$$a_k = [2k^2 \sin (k^2 \tfrac{1}{2}\omega)]^{-1} \int_0^\omega g_k(\tau) \cos k^2(\tfrac{1}{2}\omega - \tau) \, d\tau ,$$

$$b_k = -[2k^2 \sin (k^2 \tfrac{1}{2}\omega)]^{-1} \int_0^\omega g_k(\tau) \sin k^2 (\tfrac{1}{2}\omega - \tau) \, d\tau$$

for $k \bar{\in} S$.

Let $g \in \mathscr{G}$, satisfy (5) for $k \in S$ and let $z^0(t, x)$ be the solution to (1), (2), (3) for $\varepsilon = 0$ of the form $z^0(t, x) = \sum_{k=1}^{\infty} z_k^0(t) \sin kx$, where $z_k^0(t)$ is given by (6) with $a_k = b_k = 0$ for $k \in S$. Then the problem (1), (2), (3) may be reduced to that of finding a function u satisfying the equation

$$(1') \qquad u_{tt} + u_{xxxx} = \varepsilon F(u)$$

and the conditions (2), (3), where

$$(7) \qquad F(u)(t, x) \equiv (z^0 + u)_{xx}(t, x) \int_0^\pi (z^0 + u)_\xi^2 (t, \xi) \, d\xi + f(t, x) + \varepsilon \bar{F}(z^0 + u)(t, x) , \quad z = u + z^0 .$$

Hence we have easily

Lemma 1. *Let* $F(u) : \mathscr{U} \to \mathscr{G}$, $F(u)(t, x) = \sum_{k=1}^{\infty} F_k(u)(t) \sin kx$, $u \in \mathscr{U}$, $u(t, x) = \sum_{k=1}^{\infty} u_k(t) \sin kx$. *Then* $u(t, x)$ *is a solution of* (1'), (2), (3) *if and only if there exist* $a, b \in h^4$ *such that*

$$(8) \qquad G(u, a, b, \varepsilon) = 0 ,$$

where

$$G = (G_1, G_2, G_3) ,$$

$$(9) \qquad G_{1k}(u, a, b, \varepsilon)(t) \equiv -u_k(t) + a_k \cos k^2 t + b_k \sin k^2 t + \varepsilon k^{-2} \int_0^t F_k(u)(\tau) \sin k^2(t - \tau) \, d\tau , \quad \text{for} \quad k = 1, 2, \ldots ,$$

(10) $$G_{2k}(u, a, b, \varepsilon) \equiv -a_k + \varepsilon(2k^2 \sin(k^2 \tfrac{1}{2}\omega))^{-1} \int_0^\omega F_k(u)(\tau) \cos k^2(\tfrac{1}{2}\omega - \tau) \, d\tau ,$$

$$G_{3k}(u, a, b, \varepsilon) \equiv b_k + \varepsilon(2k^2 \sin(k^2 \tfrac{1}{2}\omega))^{-1} \int_0^\omega F_k(u)(\tau) \sin k^2(\tfrac{1}{2}\omega - \tau) \, d\tau ,$$

for $k \bar{\in} S$,

(11) $$G_{2k}(u, a. b, \varepsilon) \equiv k^{-2} \int_0^\omega F_k(u)(\tau) \cos(k^2\tau) \, d\tau ,$$

$$G_{3k}(u, a, b, \varepsilon) \equiv k^{-2} \int_0^\omega F_k(u)(\tau) \sin(k^2\tau) \, d\tau ,$$

for $k \in S$.

Note, that

$$u(0, x) = (2/\pi)^{1/2} \sum_{k=1}^{\infty} a_k \sin kx, \; u_t(0, x) = (2/\pi)^{1/2} \sum_{k=1}^{\infty} k^2 b_k \sin kx .$$

These equation will be solved by means of the following implicit function theorem

Theorem 1. *Let the following assumptions be fulfilled:*

(a) *$G(v, \varepsilon)$ is a mapping from Banach space $B_1 \times [-\varepsilon_1, \varepsilon_1]$ into Banach space B_2;*

(b) *the equation $G(v, 0) = 0$ has a solution $v_0 \in B_1$;*

(c) *the mapping $G(v, \varepsilon)$ is continuous in ε and has G-derivative $G'_v(v, \varepsilon)$ continuous in v, ε for $|v - v_0|_{B_1} \leqq K$, $|\varepsilon| \leqq \varepsilon_1$;*

(d) *$[G'_v(v_0, 0]^{-1}$ exists, is bounded and maps B_2 on B_1.*

Then there exists $\varepsilon_0 > 0$ such that the equation $G(v, \varepsilon) = 0$ has a unique solution $v(\varepsilon) \in B_1$ for $\varepsilon \in -[\varepsilon_0, \varepsilon_0]$ which is continuous in ε and such that $v(0) = v_0$.

3. MAIN RESULTS

For the sake of simplicity of calculations we shall find solution to the problem (1), (2), (3) only for g of the form

(12) $$g(t, x) = \cos(\nu k_0 t) \{g_1[1 - (\nu k_0^2)] \sin x + g_3[3^4 - (\nu k_0^2)] \sin 3x\} ,$$

where k_0 is a positive integer such that $\nu k_0 \neq 3$ if $1 \in S$, $\nu k_0 \neq 5$ if 1 or $3 \in S$, $\nu k_0 \neq 4$ if 1 or 2 or $3 \in S$. In that case

(13) $$z^0(t, x) = \cos(\nu k_0 t)(g_1 \sin x + g_3 \sin 3x) .$$

We prove the following

Theorem 2. *Let g be of the form (12), $f \in \mathcal{G}$, $|f|_{\mathcal{G}} + |g|_{\mathcal{G}} > 0$, ω rational. Let $\tilde{F}(u) : \mathcal{U} \to \mathcal{G}$ have a continuous G-derivative in $\mathcal{U}$.*

Then there exists $\varepsilon_0 > 0$, $u^0 \in \mathscr{U}$ *such that the problem* (1), (2), (3) *has a unique solution* $z(\varepsilon) \in \mathscr{U}$ *for* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *which is continuous in* ε *and such that* $z(0) = z^0 + u^0$, $\bar{u}^0$ *is a solution of the equation* $G(u, a, b, 0) = 0$,

$$u^0(t, x) = \sum_{k \in S} [a_k^0 \cos(k^2 t) + b_k^0 \sin(k^2 t)] \sin kx . \tag{14}$$

First, we shall prove two lemmas.

Lemma 2. *Let* $\sigma \geqq 0$, $\sigma_k \geqq 0$, $\sigma_k = 0$ *for* $k \neq 1, 3$, $\sum_{k \in S} k^8(p_k^2 + q_k^2) < +\infty$. *Then the system of algebraic equations*

$$\begin{aligned} a_k[k^2(a_k^2 + b_k^2) + 2(\sigma + \sigma_k)] &= p_k , \\ b_k[k^2(a_k^2 + b_k^2) + 2(\sigma + \sigma_k)] &= q_k \end{aligned} \tag{15}$$

has a unique solution $a_k(\sigma)$, $b_k(\sigma)$, $k \in S$, $\sum_{k \in S} k^8[a_k^2(\sigma) + b_k^2(\sigma)] < +\infty$ *for* $\sigma > 0$, *the function* $A(\sigma) \equiv \sum_{k \in S} k^2[a_k^2(\sigma) + b_k^2(\sigma)]$ *is strictly decreasing on* $(0, +\infty)$, $0 < A(0) < +\infty$ *and* $\lim_{\sigma \to \infty} A(\sigma) = 0$.

Proof. The equations (15) imply

$$a_k = 0 \Leftrightarrow p_k = 0 , \quad b_k = 0 \Leftrightarrow q_k = 0 .$$

Hence we may suppose $p_k^2 + q_k^2 > 0$. Substituting $a_k = p_k y_k$, $b_k = q_k y_k$, $k \in S$ into (15), these equations reduce to the equations

$$y_k^3 + y_k \,.\, [2(\sigma + \sigma_k) k^{-2}(p_k^2 + q_k^2)^{-1}] - k^{-2}(p_k^2 + q_k^2)^{-1} = 0 , \quad k \in S$$

for y_k, which have a unique real root for every $k \in S$, namely

$$y_k(\sigma) = B_k\{[(1 + (4B_k(\sigma + \sigma_k)/3)^3)^{1/2} + 1]^{1/3} -$$
$$- [(1 + (4B_k(\sigma + \sigma_k)/3)^3)^{1/2} - 1]^{1/3}\} \quad \text{where} \quad B_k = [2k^2(p_k^2 + q_k^2)]^{-1/3} .$$

As $y_k(\sigma) \leqq 3[2(\sigma + \sigma_k)]^{-1}$ the following estimate holds

$$a_k^2 + b_k^2 \leqq 9[2(\sigma + \sigma_k)]^{-2} (p_k^2 + q_k^2) \text{ which implies}$$
$$\sum_{k \in S} k^8(a_k^2 + b_k^2) \leqq C\sigma^{-2} \sum_{k \in S} k^8(p_k^2 + q_k^2) .$$

Since $y_k'(\sigma) < 0$ for $\sigma > 0$, $y_k(\sigma)$ is strictly decreasing on $(0, +\infty)$ for $k \in S$ and so is $A(\sigma)$. As $y_k(0) = 2B_k$ for $k \neq 1, 3$ and

$$y_k(0) = B_k\{[(1 + (4B_k\sigma_k/3)^3)^{1/2} + 1]^{1/3} - [(1 + (4B_k\sigma_k/3)^3)^{1/2} - 1]^{1/3}$$

for $k = 1, 3$, we have $0 < A(0) < C[\sum_{k\in S} k^8(p_k^2 + q_k^2)]^{1/3} < +\infty$. Finally, the inequality $A(\sigma) \leqq C\sigma^{-2} \sum k^2(p_k^2 + q_k^2)$ implies $\lim A(\sigma) = 0$ if $\sigma \to \infty$.

Lemma 3. *Let* $\sum k^8(r_k^2 + s_k^2) < +\infty$, $D_k \equiv 2(\sigma + \sigma_k) + k^2(a_k^2 + b_k^2)$, a_k, b_k, σ, σ_k *be from Lemma 2. Then the system of linear equations for* c_k, d_k, $k \in S$

$$(16) \qquad D_k c_k + [2\sum_{j\in S} j^2(a_j c_j + b_j d_j) + k^2(a_k c_k + b_k d_k)]\, a_k = r_k$$

$$D_k d_k + [2\sum_{j\in S} j^2(a_j c_j + b_j d_j) + k^2(a_k c_k + b_k d_k)]\, b_k = s_k\,, \quad k \in S$$

has a unique solution c_k, d_k, $k \in S$ *and the following estimate holds*

$$(17) \qquad \sum_{k\in S} k^8(c_k^2 + d_k^2) \leqq C \sum_{k\in S} k^8(r_k^2 + s_k^2)\,.$$

Proof. If $a_k = b_k = 0$ then

$$c_k^2 + d_k^2 = D_k^{-2}(r_k^2 + s_k^2)$$

Now, let $a_k^2 + b_k^2 > 0$. Multiplying the first equation of (16) by a_k, the second by b_k, multiplying the first equation of (16) by b_k and second by a_k we get an equivalent system to (16)

$$(18) \qquad [D_k + 2k^2(a_k^2 + b_k^2)]\,(a_k c_k + b_k d_k) + 4(a_k^2 + b_k^2)\,\sigma' = r_k a_k + s_k b_k\,,$$

$$D_k(b_k c_k - a_k d_k) = r_k b_k - s_k a_k\,, \quad k \in S$$

where $\sigma' = \sum_{j\in S} j^2(a_j c_j + b_j d_j)$.

Multiplying the first equation by $k^2[D_k + 2k^2(a_k^2 + b_k^2)]^{-1}$ and summing it for $k \in S$ we have

$$\sigma' = \sum_{k\in S} k^2(r_k a_k + s_k b_k)\,[D_k + 2k^2(a_k^2 + b_k^2)]^{-1}\,.$$

$$.\,\{1 + 4\sum_{k\in S} k^2(a_k^2 + b_k^2)\,[D_k + 2k^2(a_k^2 + b_k^2)]^{-1}\}^{-1}$$

which implies the following estimate (using the Hölder inequality)

$$(19) \qquad |\sigma'|^2 \leqq c\sum k^2(r_k^2 + s_k^2)\,.$$

Further, from (18) we get

$$(a_k^2 + b_k^2)\,(c_k^2 + d_k^2) = (r_k b_k - s_k a_k)^2\, D_k^{-2} +$$

$$+ [r_k a_k + s_k b_k - 4(a_k^2 + b_k^2)\,\sigma']^2\,[D_k + 2k^2(a_k^2 + b_k^2)]^{-2}\,,$$

from which follows

$$k^8(c_k^2 + d_k^2) \leqq [r_k^2 + s_k^2 + 16(a_k^2 + b_k^2)\,(\sigma')^2]\, D_k^{-2}\,.$$

This estimate together with (19) imply (17).

Proof of Theorem 2. It suffices to show that the operator G defined by (9), (10), (11) satisfies the assumptions of Theorem 1 with $B_1 = B_2 = \mathscr{U} \times h^4 \times h^4$. The assumptions (a) and (c) are fulfilled in virtue of Lemma 1 and of the assumptions of Theorem 2. To verify the assumption (b) requires to show that the system

$$(20) \qquad -u_k + a_k \cos k^2t + b_k \sin k^2t = 0\,, \quad k = 1, 2, \ldots,$$
$$a_k = 0\,,$$
$$b_k = 0\,, \quad k \in S\,.$$

$$(21) \qquad k^{-2} \int_0^\omega F_k(u, 0)(\tau,) \cos k^2\tau \, d\tau = 0\,,$$
$$k^{-2} \int_0^\omega F_k(u, 0)(\tau,) \sin k^2\tau \, d\tau = 0\,, \quad k \in S$$

has a unique solution $(u^0, a^0, b^0) \in \mathscr{U} \times h^4 \times h^4$, which means, in fact, that the equations (21) have a solutions $a_k^0, b_k^0, k \in S, \sum_{k \in S} k^8[(a_k^0)^2 + (b_k^0)^2] < +\infty$. Inserting (7), (20) into (21) we get after some calculation the equations

$$(22) \quad a_k[g_1^2 + 9g_3^2 + \sum_{j \in S} j^2(a_j^2 + b_j^2) + k^2(a_k^2 + b_k^2) + \sigma_k] = f_k^c\,,$$
$$b_k[g_1^2 + 9g_3^2 + \sum_{j \in S} j^2(a_j^2 + b_j^2) + k^2(a_k^2 + b_k^2) + \sigma_k] = f_k^s\,, \quad k \in S\,,$$

where

$$f_k^c = 2(\pi k^2)^{-1} \int_0^\omega \int_0^\pi f(t, x) \cos k^2t \sin kx \, dx \, dt\,,$$
$$f_k^s = 2(\pi k^2)^{-1} \int_0^\omega \int_0^\pi f(t, x) \sin k^2t \sin kx \, dx \, dt\,,$$
$$\sigma_k = k^2 g_k \quad \text{for} \quad k = 1, 3\,, \quad \sigma_k = 0 \quad \text{for} \quad k \neq 1, 3\,.$$

In the case of more general function $g(t, x)$ the equation (22) will be more complicated.

By Lemma 2 (putting $p_k = f_k^c$, $g_k = f_k^s$, $\sigma = g_1^2 + 9g_3^2 + \sum_{j \in S} j^2(a_j^2 + b_j^2)$) this system has a solution if and only if the equation

$$\sigma = g_1^2 + 9g_3^2 + A(\sigma)$$

has a real solution $\sigma_0 > 0$. However this is an immidiate consequence of Lemma 2. Thus $a_k^0 = a_k(\sigma_0)$, $b_k^0 = b_k(\sigma_0)$, $k \in S$ from Lemma 2 are the solutions to (22). By Lemma 2 $\sum k^8 . [(a_k^0)^2 + (b_k^0)^2]$ is finite for $f \in \mathscr{G}$ and hence $a^0, b^0 = \{a_k^0, b_k^0$, for $k \in S$ $a_k^0 = b_k^0 = 0$, for $k \bar{\in} S\}$ and u^0 are the solutions of (20), (21), $a^0, b^0 \in h^4$ and u^0 is of the form (14).

To prove (d) let us show that the system

$$G'_{(u,a,b)}(u^0, a^0, b^0, 0)(\bar{u}, \bar{a}, \bar{b}) = (\bar{f}, \bar{p}, \bar{q})$$

i.e.

$$-\bar{u}_k(t) + \bar{a}_k \cos k^2 t + \bar{b}_k \sin k^2 t = \bar{f}_k\,, \quad \bar{a}_k = \bar{p}_k\,, \quad \bar{b}_k = \bar{q}_k\,, \quad k \in S\,,$$

$$\int_0^\omega \{\bar{u}_k(t) \sum_{j\in S} j^2(z_j^0(t) + u_j^0(t))^2 + 2(z_k^0(t) + u_k^0(t))\,.$$

$$.\sum_{j\in S} j^2(z_j^0(t) + u_j^0(t))\,\bar{u}_j(t)\} \cos k^2 t \,\mathrm{d}t = \frac{2}{\pi}\,\bar{p}_k\,, \quad k \in S$$

$$\int_0^\omega \{\bar{u}_k(t) \sum_{j\in S} j^2(z_j^0(t) + u_j^0(t))^2 + 2(z_k^0(t) + u_k^0(t)) +$$

$$+ \sum_{j\in S} j^2(z_j^0(t) + u_j^0(t))\,\bar{u}_j(t)\} \sin k^2 t \,\mathrm{d}t = \frac{2}{\pi}\,\bar{q}_k\,, \quad k \in S$$

has a unique solution for every $(\bar{f}, \bar{p}, \bar{q}) \in \mathscr{U} \times h^4 \times h^4$ satisfying

$$|\bar{u}|_4 + |\bar{a}|_{h^4} + |\bar{b}|_{h^4} \leqq C(|\bar{f}|_4 + |\bar{p}|_{h^4} + |\bar{q}|_{h^4})\,. \tag{23}$$

Obviously, it is sufficient to prove this assertion only for the last two equations and $\bar{a}_k$, $\bar{b}_k$, $k \in S$. Integrating we obtain equations (16) with

$$\bar{r}_k = 2\bar{p}_k\,, \quad \bar{s}_k = 2\bar{q}_k\,, \quad c_k = \bar{a}_k\,, \quad d_k = \bar{b}_k\,, \quad \sigma_k = k^2 g_k^2 \quad \text{for} \quad k = 1, 3\,, \quad \sigma_k = 0$$

for $k \neq 1, 3$,

From Lemma 3 it follows the existence and uniqueness of such $\bar{a}_k$, $\bar{b}_k$ and the estimate (23), which completes the proof.

References

[1] *S. Woinowsky-Krieger:* The effect of an axial force on the vibration of hinged bars. Journ. of Appl. Mech., *17* (1950), p. 35—36.

[2] *J. G. Eisley:* Nonlinear vibration of beams and rectangular plates. ZAMP, *15* (1964), p. 167—175.

[3] *J. M. Ball:* Initial-boundary value problems for an extensible beam. Journ. of Math. Anal. Appl. *42* (1973), p. 61—90.

[4] *J. M. Ball:* Stability theory for an extensible beam. Journ. of Diff. Eq. *14* (1973), p. 399—418.

[5] *R. van Dooren:* Two mode sub-harmonic vibrations of order 1/9 of a nonlinear beam forced by a two mode harmonic load. Journ. of Sound and Vibr. *41* (2), (1975), p. 133—142.

[6] *R. van Dooren, R. Bouc:* Two mode sub-harmonic and harmonic vibrations of a nonlinear beam forced by a two mode harmonic load. Int. J. Non-Linear Mech., *10*, (1975) p. 271—280.

[7] *N. Krylová, O. Vejvoda:* A linear and weakly nonlinear equation of a beam. Czech. Math. J. vol. *21 (96)*, (1971), p. 535—566.

[8] *S. Fučík, V. Lovicar:* Periodic solutions of the equation $x''(t) + g(x(t)) = p(t)$. Čas. pro pěst. mat., roč. *100* (1975), 160—175.

[9] *V. Lovicar:* Periodic solutions of nonlinear abstract second order equations with a dissipative term. (to appear).

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