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GROWTH ESTIMATES FOR NON-OSCILLATORY SOLUTIONS OF A NON-LINEAR DIFFERENTIAL EQUATION

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Introduction. In the real non-linear differential equation

$$y'' + q(t)\,|y|^{\alpha} \operatorname{sgn} y = 0 \quad (' = \mathrm{d}/\mathrm{d}t) \tag{1}$$

let $\alpha > 0$, $\alpha \neq 1$ and assume that q is non-negative and continuous on the half line $\langle 1, \infty)$. A non-trivial solution of (1) is called non-oscillatory if it is eventually of one sign for large t. Our main purpose in this paper is to establish new growth estimates for a non-oscillatory solution of (1) which are similar to previous results of MOORE and NEHARI [11, Theorem IX, p. 50] and BELOHOREC [3, Theorem 4, p. 14]. Our estimates are of general interest since they are phrased in terms of the known necessary conditions for (1) to be non-oscillatory. We also develop new estimates for non-oscillatory solutions of more general differential equations.

We begin with a few related facts. A solution which exists on $\langle 1, \infty)$ is said to be extendable (continuable). As remarked by Nehari [12], the concavity of the solution curve of any non-oscillatory solution implies that the solution is extendable. Further properties of solutions of (1) are given in [1], [2], [8], [9], and the problem of continuing all solutions is discussed in [4], [6], [7], [10].

If $y(t)$ is a positive non-oscillatory solution of (1) on $\langle 1, \infty)$, then the concavity of the solution curve implies that $y(t) \leqq Ct$ where C is a positive constant. However, it is known that some non-oscillatory solutions of (1) are either bounded or grow like a fractional power of t. Bounds for non-oscillatory solutions $y(t)$ which are of the form $y(t) \leqq Ct^{\lambda}$ or $y(t) \geqq Ct^{\lambda}$, $0 < \lambda < 1$, were previously established by Moore and Nehari, Belohorec, respectively. We reproduce their results here for the sake of completeness.

Theorem 1. *If* $\alpha > 1$, $0 < \lambda < 1$,

$$\liminf_{t \to \infty} t^{\lambda(\alpha-1)+1} \int_t^{\infty} q(s)\,\mathrm{d}s > 0 \tag{2}$$

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and $y(t)$ is a non-oscillatory solution of (1), *then*

$$|y(t)| \leqq Ct^{\lambda}$$

where C is a positive constant.

A proof of this result is given in [11, p. 50].

Theorem 2. *Let $0 < \alpha < 1$. If there exists a number β, $0 < \beta \leqq 1$ such that*

$$\liminf_{t \to \infty} t^{\beta} \int_t^{\infty} q(s)\, ds > 0 \tag{3}$$

and $y(t)$ is a non-oscillatory solution of (1), *then*

$$|y(t)| > C(t - 1)^{(1-\beta)/(1-\alpha)}$$

where C is a positive constant.

A proof of this result is given in [3, p. 14].

Main Results. The main results of the paper are given in Theorems 3 and 4. These results are but a special case of Theorem 5.

Theorem 3. *In* (1) *let $\alpha > 1$ and assume that $y(t)$ is a non-oscillatory solution on $\langle 1, \infty)$. Then for $t \geqq 1$,*

$$\int_t^{\infty} s\, q(s)\, ds\, |y(t)|^{\alpha - 1} \leqq C\,, \tag{4}$$

where C is a positive constant.

Theorem 4. *In* (1) *let $0 < \alpha < 1$ and assume that $y(t)$ is a non-oscillatory solution on $\langle 1, \infty)$. Then for $t \geqq 1$,*

$$t^{1-\alpha} \int_t^{\infty} s^{\alpha} q(s)\, ds\, |y(t)|^{\alpha - 1} \leqq C \tag{5}$$

where C is a positive constant.

Remark. The estimates given here provide no information if the improper integrals vanish for any $t_0 \geqq 1$. In this case $q(t) \equiv 0$ for $t \geqq t_0$ and (1) reduces to a trivial case.

These estimates are of interest since they are phrased in terms of the known necessary conditions for (1) to be non-oscillatory.

By considering the equation

$$y'' + \frac{\lambda(1-\lambda)}{t^{\lambda(\alpha-1)+2}} |y|^\alpha \operatorname{sgn} y = 0\,, \quad \alpha > 0\,, \quad 0 < \lambda < 1\,,$$

and comparing the non-oscillatory solution $y(t) = t^\lambda$ with the estimates given in Theorems 3 and 4, we see that the results are in general sharp with respect to the exponent λ. However, the estimates given in Theorems 1 and 2 are also sharp for the same reasons. Thus the main interest in the estimates given in Theorems 3 and 4 is due to their form. In order to compare our results with those given in Theorems 1 and 2, we will state two simple corollaries.

Corollary 1. *If for some* λ, $0 < \lambda < 1$,

$$\liminf_{t\to\infty} t^{\lambda(\alpha-1)} \int_t^\infty s\, q(s)\, \mathrm{d}s > 0\,, \tag{6}$$

then every non-oscillatory solution of (1) *with* $\alpha > 1$ *satisfies*

$$|y(t)| \leqq Ct^\lambda$$

where C *is a positive constant.*

The proof of the corollary follows from (4). We note that (2) implies (6) and hence Corollary 1 is a slight improvement of the original result of Moore and Nehari.

Corollary 2. *If for some* λ, $0 < \lambda < 1$,

$$\liminf_{t\to\infty} t^{(1-\lambda)(1-\alpha)} \int_t^\infty s^\alpha\, q(s)\, \mathrm{d}s > 0 \tag{7}$$

then every non-oscillatory solution of (1) *with* $0 < \alpha < 1$ *satisfies*

$$|y(t)| \geqq Ct^\lambda$$

where C *is a positive constant.*

The proof of the corollary follows from (5). We note that (3) implies (7) and hence Corollary 2 is also a slight improvement of the original estimate of Belohorec.

Upper and Lower Estimates. In [5] we studied the oscillatory behavior of solutions of the equation

$$y'' + (q_1(t)\,|y|^\alpha + q_2(t)\,|y|^\sigma) \operatorname{sgn} y = 0 \tag{8}$$

where q_1, q_2 are non-negative and continuous on the half line $\langle 1, \infty)$ and $0 < \sigma < 1 < \alpha$. Upper *and* lower estimates for a non-oscillatory solution of (8) are given in the following

Theorem 5. *In* (8) *let* $0 < \sigma < 1 < \alpha$ *and assume that* q_1, q_2 *are non-negative and continuous on the half line* $\langle 1, \infty)$. *If* $y(t)$ *is a non-oscillatory solution on* $\langle 1, \infty)$, *then*

$$\int_t^\infty s\, q_1(s)\, \mathrm{d}s\, |y(t)|^{\alpha-1} \leqq C\,, \tag{9}$$

$$t^{1-\sigma} \int_t^\infty s^\sigma\, q_2(s)\, \mathrm{d}s\, |y(t)|^{\sigma-1} \leqq C \tag{10}$$

where C *is a positive constant.*

Theorem 3(4) follows from Theorem 5 by taking $q_2 \equiv 0 (q_1 \equiv 0)$.

The estimates given in this theorem are in general sharp since $y(t) = t^\lambda$ $(0 < \lambda < 1)$ is a non-oscillatory solution of

$$y'' + \frac{\lambda(1-\lambda)}{2}\, t^{\lambda-2}\, (t^{-\alpha\lambda}\, |y|^\alpha + t^{-\sigma\lambda}\, |y|^\sigma) \operatorname{sgn} y = 0\,.$$

Before giving the proof of Theorem 5 we want to point out a few well known properties of a non-oscillatory solution $y(t)$ of (8). These properties will be used in the sequel without further comment. Since $-y(t)$ is also a solution of (8) we can assume that $y(t)$ is positive for large t. We can also assume, without loss of generality, that $y(t)$ is positive on $\langle 1, \infty)$. If $y(t)$ is a positive non-oscillatory solution of (8) then $y'(t)$ is positive since $-y''(t)$ is non-negative. If $y(t)$ is a positive non-oscillatory solution of (8) on $\langle 1, \infty)$, then

$$t\, y'(t)\, y(t)^{-1} \leqq k\,, \quad k = (1 + y'(1)\, y(1)^{-1})\,. \tag{11}$$

This is true since

$$(t\, y'(t) - y(t))' = -t(q_1(t)\, y(t)^\alpha + q_2(t)\, y(t)^\sigma) \leqq 0\,.$$

Proof of Theorem 5. Let $y(t)$ be a positive non-oscillatory solution of (8) on $\langle 1, \infty)$. It was shown in [5, p. 388] that $\int^\infty \{s q_1 + s^\sigma q_2\} < \infty$ was a necessary condition for (8) to be non-oscillatory. From (8),

$$y''(t) + q_1(t)\, y(t)^\alpha \leqq 0$$

on $\langle 1, \infty)$. Multiply by $t\, y(t)^{-\alpha}$ and integrate by parts to obtain

$$s\, y(s)^{-\alpha}\, y'(s) \Big|_t^x + \int_t^x s\, q_1(s)\, \mathrm{d}s \leqq$$

$$\leqq -\alpha \int_t^x s\, y'(s)^2\, y(s)^{-\alpha-1}\, \mathrm{d}s + (-\alpha+1)^{-1}\, y(s)^{-\alpha+1} \Big|_t^x\,, \quad t \leqq x\,.$$

Since $\alpha > 1$ and $y'(s) > 0$ we can let x approach infinity to obtain

$$\int_t^\infty s\, q_1(s)\, ds \leqq t\, y(t)^{-\alpha}\, y'(t) + (\alpha - 1)^{-1}\, y(t)^{-\alpha+1} .$$

Since $t\, y'(t)\, y(t)^{-1}$ is bounded (see (11)), the previous estimate leads to the final estimate

$$\int_t^\infty s\, q_1(s)\, ds \leqq C\, y(t)^{-\alpha+1} , \tag{4}$$

where C is a constant.

From (8),

$$y''(t) + q_2(t)\, y(t)^\sigma \leqq 0$$

on $\langle 1, \infty)$. Multiply by $y'(t)^{-\sigma}$, integrate and use (11) to obtain

$$\frac{y'(x)^{-\sigma+1}}{1-\sigma} + k^{-\sigma} \int_t^x s^\sigma\, q_2(s)\, ds \leqq \frac{y'(t)^{1-\sigma}}{1-\sigma}, \quad t \leqq x .$$

Since $0 < \sigma < 1$ and $y'(x)$ is positive we can let x approach infinity in the previous inequality to obtain

$$\int_t^\infty s^\sigma\, q_2(s)\, ds \leqq C\, y'(t)^{1-\sigma}$$

or

$$C \int_1^t \left(\int_s^\infty x^\sigma\, q_2(x)\, dx \right)^{1/(1-\sigma)} ds + y(1) \leqq y(t) \tag{12}$$

where C is a positive constant which is not necessarily the same in each estimate. It remains to estimate the integral in (12). However, an integration by parts shows that

$$\int_1^t \left(\int_s^\infty x^\sigma\, q_2(x)\, dx \right)^{1/1-\sigma)} ds = s \int_s^\infty x^\sigma\, q_2(x)\, dx^{1/(1-\sigma)} \Big|_1^t +$$
$$+ (1-\sigma)^{-1} \int_1^t \left(\int_s^\infty x^\sigma\, q_2(x)\, dx \right)^{\sigma/(1-\sigma)} s^{\sigma+1}\, q_2(s)\, ds .$$

Consequently (12) can be written as

$$Ct \left(\int_t^\infty s^\sigma\, q_2(s)\, ds \right)^{1/(1-\sigma)} \leqq y(t) + \int_1^\infty s^\sigma\, q_2(s)\, ds^{1/(1-\sigma)} \leqq C_1\, y(t)$$

since $0 < \sigma < 1$ and $y(t) \geqq y(1)$. The estimate given in (5) is now clear and the proof of Theorem 5 is complete.

Generalizations and a conjecture. Growth estimates for non-oscillatory solutions of the non-linear self-adjoint equation

$$(r(t)\, y')' + \{q_1(t)\, |y|^\alpha + q_2(t)\, |y|^\sigma\} \operatorname{sgn} y = 0$$

can also be given. Here $0 < \sigma < 1 < \alpha$, q_1, q_2 are non-negative and continuous on the half line $\langle 1, \infty)$ and r is positive and continuous on the half line $\langle 1, \infty)$. Standard transformations [13, p. 227] reduce this equation to the type considered in (8) and corresponding estimates can be given. We omit the details.

All of the previous work presented in this paper dealt with solutions of a differential equation on the half line $\langle 1, \infty)$. The properties of some solutions of

$$(1)' \qquad y'' + q(t)\,|y|^{\alpha} \operatorname{sgn} y = 0\,, \quad \alpha > 1$$

on the line $(-\infty, \infty)$ are largely determined once we have information concerning solutions on the half line $\langle 0, \infty)$. The transformation $\tau = -t$ in $(1)'$ can be used in conjunction with Theorem 3 to give estimates for a non-oscillatory solution on $(-\infty, 0\rangle$ since the Atkinson condition

$$(13) \qquad \int_{-\infty}^{\infty} |s|\, q(s)\, \mathrm{d}s = \lim_{m,n\to\infty} \left(\int_{-m}^{0} + \int_{0}^{n}\right) |s|\, q(s)\, \mathrm{d}s = \infty$$

is clearly a sufficient condition for all non-trivial solutions to be oscillatory (e.g., given $N > 0$, there exist positive *and* negative values of t such that $y(t) = 0$, $|t| \geqq N$). However, it is somewhat surprising that it is not clear whether (13) is a necessary condition for oscillation if we, by definition, assume that a non-trivial solution $y(t)$ is non-oscillatory if $|y(t)| \neq 0$ when $|t|$ is large. The argument given by Atkinson [1] only shows that $(1)'$ has a solution on $(-\infty, \infty)$ which satisfies $y(t) > 0$ if t is large. Since q is positive, the concavity of the solution curve of any non-oscillatory solution $y(t)$ implies that the solution must have at least one zero on $(-\infty, \infty)$. Thus the non-oscillatory solution constructed by Atkinson for a half line has at least one zero on $(-\infty, \infty)$, but its behavior for negative t is not known. A necessary and sufficient condition for all non-trivial solutions of (1) to be oscillatory on $(-\infty, \infty)$ is presently lacking. We conjecture that (13) is also a necessary condition for all solutions of $(1)'$ to be oscillatory on $(-\infty, \infty)$. The question is also open when $0 < \alpha < 1$.

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