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SOME REMARKS ON MENGER'S THEOREM

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One of the most well-knowu theorems of the graph theory is Menger's theorem concerning maximal vertex-disjoint systems of arcs (simple paths, see [3]) joining two given sets of vertices. Here we shall prove a generalization of this theorem.

First we shall give some definitions. Let G be an undirected graph. (For this purpose we may assume that G is without loops and multiple edges, because by adding them nothing would change.) Let V be its vertex set. Now if $M \subset V$ and P is an arc in G joining the vertices u, v, we say that the arc P meets the set M n times, where n is a positive integer, if and only if there exist n pairwise different elements $a_1, \ldots, a_n$ of M and $n - 1$ pairwise different elements $b_1, \ldots, b_{n-1}$ of $V \dot{-} M$ such that on the arc P

(1) a_1 lies between u and b_1;

(2) a_i lies between b_{i-1} and b_i for $i = 2, \ldots, n - 1$;

(3) a_n lies between b_{n-1} and v;

(4) b_i lies between a_i and a_{i+1} for $i = 1, \ldots, n - 1$.

Evidently the arc P cannot meet the set M more than $|M|$ times or more than $|V \dot{-} M|$ times.

Let a decomposition $\mathscr{V}$ of the vertex set V be given, $\mathscr{V} = \{V_1, \ldots, V_k\}$, $V_i \cap V_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} V_i = V$. We say that a system $\mathscr{P}$ of arcs is a $\mathscr{V}$-system, if and only if any arc of the system $\mathscr{P}$ meets any set of $\mathscr{V}$ at most once and any set of $\mathscr{V}$ is met at most by one arc of $\mathscr{P}$.

Let U_1, U_2 be two subsets of V such that each of them is a union of some classes of $\mathscr{V}$. The vertex connectivity degree between U_1 and U_2 with respect to $\mathscr{V}$ is the minimal number of classes of $\mathscr{V}$ which must be deleted from G in order that all arcs joining vertices of U_1 with vertices of U_2 and meeting any class of $\mathscr{V}$ at most once might be destroyed.

Theorem 1. *Let G be an undirected graph with the vertex set V, let $\mathscr{V}$ be a decomposition of the set V into pairwise disjoint subsets. Let U_1, U_2 be two subsets of V such that each of them is a union of some classes of $\mathscr{V}$. Let ω be the vertex connectivity degree between U_1 and U_2 with respect to $\mathscr{V}$. Then the maximal number of arcs in a $\mathscr{V}$-system of arcs joining vertices of U_1 with vertices of U_2 is ω.*

Proof. In the case that each class of $\mathscr{V}$ consists only of one vertex we obtain Menger's theorem. If not, we shall prove this theorem with help of Menger's theorem [2]. Let A be a class of $\mathscr{V}$, let $G(A)$ be the subgraph of G generated by the set A. We shall identify any two vertices which belong to the same connected component of $G(A)$. We do this for all classes of $\mathscr{V}$; if some loops and multiple edges occur in this process, we may omit loops and substitute single edges for multiple ones. The resulting graph will be denoted by G_0, its vertex set by V_0. As we have identified only the pairs of vertices belonging both to the same class of $\mathscr{V}$, we have obtained a decomposition $\mathscr{V}_0$ of V_0 such that two vertices of V_0 belong to the same class of $\mathscr{V}_0$ if and only if the corresponding vertices of V belong to the same class of $\mathscr{V}$. Now let $\mathscr{R}(\mathscr{V}_0)$ be the system of sets of representants of $\mathscr{V}_0$, i.e. the system of all subsets of $\mathscr{V}_0$, each of which has exactly one element in common with any class of $\mathscr{V}_0$. If $M \in \mathscr{R}(\mathscr{V}_0)$, let $G_0(M)$ be the subgraph of G_0 generated by the set M. Each system of vertex-disjoint arcs in $G_0(M)$ is evidently a $\mathscr{V}_0$-system in G_0 and corresponds to a $\mathscr{V}$-system in G. Namely, if some arc meets a set only once, all its vertices belonging to this set form a subarc, thus a connected subgraph, therefore in V_0 one vertex corresponds to all vertices of this subarc. On the other hand, if we have a $\mathscr{V}$-system $\mathscr{P}$ of arcs in G, we may choose $M \in \mathscr{R}(\mathscr{V}_0)$ so that M contains all vertices corresponding to the mentioned subarcs and the arc system corresponding to $\mathscr{P}$ in G_0 is contained in $G_0(M)$. Now let $\omega(M)$ be the minimal number of vertices of $G_0(M)$ which must be deleted in order that all arcs joining vertices of U_1 with vertices of U_2 might be destroyed. (We speak about the sets U_1 and U_2 also in G_0; they are the same sets as in G, only some pairs of elements which were different in G may be identical in G_0.) According to Menger's theorem the maximal number of vertex-disjoint arcs joining vertices of U_1 with vertices of U_2 in $G_0(M)$ is $\omega(M)$. Now let $\omega = \max_{M \in \mathscr{R}(\mathscr{V}_0)} \omega(M)$. In the graph $G_0(M_0)$ where M_0 is such a set of $\mathscr{R}(\mathscr{V}_0)$ that $\omega(M_0) = \omega$, there exists a system of ω vertex-disjoint arcs joining vertices of U_1 with vertices of U_2. To this system a $\mathscr{V}$-system of arcs in G joining vertices of U_1 with vertices of U_2 corresponds. As a system of vertex-disjoint arcs in some $G_0(M)$ corresponds to each $\mathscr{V}$-system in G, we cannot have a $\mathscr{V}$-system of arcs in G joining vertices of U_1 with vertices of U_2 with more than ω arcs, q.e.d.

Now if we have two vertices a, b in G, we shall define a $\mathscr{V}$-system of arcs joining a and b. The $\mathscr{V}$-system of arcs joining a and b is a system $\mathscr{P}$ of vertex-disjoint (up to the vertices a and b) arcs joining a and b such that any arc of $\mathscr{P}$ meets any class of $\mathscr{V}$ at most once and any class of $\mathscr{V}$ except for the classes containing a and b is met at most by one arc of $\mathscr{P}$.

The vertex connectivity degree between the vertices a and b with respect to $\mathscr{V}$ is defined analogously to that between the sets U_1 and U_2. (Only classes containing neither a nor b may be deleted.)

Theorem 2. *Let G be an undirected graph with the vertex set V, let $\mathscr{V}$ be a decomposition of the set V into pairwise disjoint subsets. Let a, b be two vertices of G not belonging to the same class of the decomposition $\mathscr{V}$ and such that any arc joining a and b meets at least one class of $\mathscr{V}$ containing neither a nor b. Let ω be the vertex connectivity degree between a and b with respect to $\mathscr{V}$. Then the maximal number of arcs in a $\mathscr{V}$-system of arcs joining a and b is ω.*

Proof is similar to that of Theorem 1. Let $V(a)$ or $V(b)$ be the class of $\mathscr{V}$ containing a or b respectively. Let $G(a)$ or $G(b)$ be the subgraph of G generated by $V(a)$ or $V(b)$ respectively. At first we delete all vertices of G which belong to connected components of $G(a)$ not containing a and to connected components of $G(b)$ not containing b. (If some arc joining a and b in G contains a vertex of a connected component of $G(a)$ not containing a, it meets $V(a)$ at least twice, because the subarc between a and this vertex must contain a vertex not belonging to $V(a)$. Analogously for $G(b)$.) Then we construct the graph G_0, the set system $\mathscr{R}(\mathscr{V}_0)$ and the graphs $G_0(M)$ for $M \in \mathscr{R}(\mathscr{V}_0)$ as in the proof of Theorem 1. The further procedure is quite analogous to that in the proof of Theorem 1.

R. Halin [1] has proved a theorem analogous to Menger's theorem and concerning one-way infinite arcs. From this theorem we can derive the following theorem by the same way as we have derived preceding theorems from Menger's theorem. Before formulating this theorem, we shall define a $\mathscr{V}$-system of one-way infinite arcs outgoing from a vertex a. It is a system $\mathscr{P}$ of one-way infinite arcs outgoing from a such that any two arcs of $\mathscr{P}$ have only the vertex a in common, any arc of $\mathscr{P}$ meets any class of $\mathscr{V}$ at most once and any class of $\mathscr{V}$, except for the class containing a is met at most by one arc of $\mathscr{P}$.

Theorem 3. *Let G be an undirected locally finite graph with the vertex set V, let $\mathscr{V}$ be a decomposition of V into pairwise disjoint finite subsets. Let a be its vertex. Let ω be the minimal number of classes of $\mathscr{V}$ which must be deleted in order that all one-way infinite arcs outgoing from a and meeting any class of $\mathscr{V}$ at most once might be destroyed. Then the maximal number of arcs in a $\mathscr{V}$-system of one-way infinite arcs outgoing from a is ω.*

We adopt the assumption that all sets of the decomposition $\mathscr{V}$ are finite in order that, after identifying vertices of connected components of graphs generated by these sets, the graph might remain locally finite.

Now we shall define a simple $\mathscr{V}$-system of arcs joining U_1 and U_2 as a $\mathscr{V}$-system of arcs joining a vertex of U_1 with a vertex of U_2 such that any arc of this system has at most one vertex in common with any class of $\mathscr{V}$. The simple vertex connectivity degree between U_1 and U_2 with respect to $\mathscr{V}$ will be the minimal number of classes of $\mathscr{V}$ which must be deleted in order that all arcs joining a vertex of U_1 with a vertex of U_2 such that any of them has at most one vertex in common with any class of $\mathscr{V}$ might be destroyed. Analogously for the vertices a and b.

The following theorems can be proved by a similar way as preceding ones.

Theorem 4. *Let G be an undirected graph with the vertex set V, let $\mathscr{V}$ be a decomposition of the set V into pairwise disjoint subsets. Let U_1, U_2 be two subsets of V such that each of them is a union of some classes of $\mathscr{V}$. Let ω be the simple vertex connectivity degree between the sets U_1 and U_2 with respect to $\mathscr{V}$. Then the maximal number of arcs in a simple $\mathscr{V}$-system of arcs joining vertices of U_1 with vertices of U_2 is ω.*

Theorem 5. *Let G be an undirected graph with the vertex set V, let $\mathscr{V}$ be a decomposition of the set V into pairwise disjoint subsets. Let a, b be two vertices of G not belonging to the same class of the decomposition $\mathscr{V}$ and such that any arc joining a and b meets at least one class of $\mathscr{V}$ containing neither a nor b. Let ω be the simple vertex connectivity degree between a and b with respect to $\mathscr{V}$. Then the maximal number of arcs in a simple $\mathscr{V}$-system of arcs joining a and b is ω.*

The simple $\mathscr{V}$-system of one-way infinite arcs outgoing from a vertex a is defined analogously to the preceding concepts.

Theorem 6. *Let G be an undirected locally finite graph with the vertex set V, let $\mathscr{V}$ be a decomposition of V into pairwise disjoint subsets. Let a be a vertex of G such that the subgraph of G generated by the class of $\mathscr{V}$ containing a does not contain any one-way infinite arc outgoing from a. Let ω be the minimal number of classes of $\mathscr{V}$ which must be deleted from G in order that all one-way infinite arcs outgoing from a, each of which has at most one vertex in common with any class of $\mathscr{V}$, might be destroyed. Then the maximal number of arcs in a simple $\mathscr{V}$-system of one-way infinite arcs outgoing from a is ω.*

The proofs of these theorems are a little simpler than those of Theorems 1, 2, 3. We do not construct the graph G_0 and in the further procedure we have to do with the original graph G instead of G_0.

We remark that the vertex connectivity degree between two vertices with respect to $\mathscr{V}$ exists only if the assumptions of Theorem 2 are satisfied. The same holds also for the simple vertex connectivity degree between two vertices with respect to $\mathscr{V}$.

Now we shall prove an analogue of Theorem 1 dealing with edges instead of vertices. At first we shall define an $\mathscr{E}$-system of arcs, which is an analogue of the $\mathscr{V}$-system.

Let G be an undirected graph (we admit multiple edges, but we may assume that G is without loops). Let E be its edge set. If $M \subset E$ and P is an arc in G joining the vertices u and v, we say that the arc P meets the set M n times, where n is a positive integer, if and only if there exist n pairwise different elements $e_1, \ldots, e_n$ of M and $n - 1$ pairwise different elements $h_1, \ldots, h_{n-1}$ of $E - M$ such that on the arc P

(1) e_1 lies between u and h_1;
(2) e_i lies between h_{i-1} and h_i for $i = 2, \ldots, n - 1$;
(3) e_n lies between h_{n-1} and v;
(4) h_i lies between e_i and e_{i+1} for $i = 1, \ldots, n - 1$.

Let a decomposition $\mathscr{E}$ of the edge set E of the graph G be given, $\mathscr{E} = \{E_1, .., E_k\}$, $E_i \cap E_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} E_i = E$. We say that a system $\mathscr{P}$ of arcs is an $\mathscr{E}$-system, if and only if any arc of the system $\mathscr{P}$ meets any set of $\mathscr{E}$ at most once and any set of $\mathscr{E}$ is met at most by one arc of $\mathscr{P}$.

Let a, b be two vertices of G. The edge connectivity degree between the vertices a and b with respect to $\mathscr{E}$ is the minimal number of classes of $\mathscr{E}$ which must be deleted in order that all arcs joining the vertex a with the vertex b and meeting any class of $\mathscr{E}$ at most once might be destroyed.

Theorem 7. *Let G be an undirected graph with the edge set E, let $\mathscr{E}$ be a decomposition of the set E into pairwise disjoint subsets. Let a, b be two vertices of G. Let ω be the edge connectivity degree between a and b with respect to $\mathscr{E}$. Then the maximal number of arcs in an $\mathscr{E}$-system of arcs joining a and b is ω.*

Proof. In the case that each class of $\mathscr{E}$ consists only of one edge we obtain a well-known analogue of Menger's theorem. If not, we shall do the proof with its help. Let A be a class of $\mathscr{E}$, let $G(A)$ be the subgraph of G consisting of all edges of A and vertices incident with them. We shall make the transitive closure of $G(A)$, i.e. if some pair of vertices of a connected component of $G(A)$ is not joined by an edge of A, we shall adjoin such an edge (even if it is joined by an edge not belonging to A). The transitive closure of $G(A)$ will be denoted by $G^*(A)$, the graph obtained from G by constructing $G^*(A)$ for each $A \in \mathscr{E}$ will be denoted by G^*, its edge set by E^*, the decomposition of E^* into the edge sets of $G^*(A)$ for $A \in \mathscr{E}$ by $\mathscr{E}^*$. Now let $\mathscr{R}(\mathscr{E}^*)$ be the system of sets of representants of $\mathscr{E}^*$, i.e. the system of all subsets of E, each of which has exactly one element in common with any class of $\mathscr{E}^*$. If $M \in \mathscr{R}(\mathscr{E}^*)$, let $G^*(M)$ be the subgraph of G^* formed by all vertices of G^* and all edges of M. Each system of edge-disjoint arcs in $G^*(M)$ joining a and b corresponds to an $\mathscr{E}^*$-system in G^* and to an $\mathscr{E}$-system in G. Namely, if e is such en edge that $\{e\} = M \cap G^*(A)$ for some $A \in \mathscr{E}$ and an arc P joining a and b in $G^*(M)$ contains e, then we may substitute e by an arc joining the end vertices of e in $G(A)$. If we do this for each e, we obtain an arc joining a and b in G and meeting any class of $\mathscr{E}$ at most once (because $G^*(M)$ contains exactly one edge of each $G^*(A)$ for $A \in \mathscr{E}$). On the other hand, if we have an $\mathscr{E}$-system $\mathscr{P}$ of arcs in G, we may choose $M \in \mathscr{R}(\mathscr{E}^*)$ so that M contains all edges corresponding to the mentioned subarcs and the arc system corresponding to $\mathscr{P}$ in G^* is contained in $G^*(M)$. Now let $\sigma(M)$ be the minimal number of edges of $G^*(M)$ which must be deleted in order that all arcs joining a and b might be destroyed. According to the edge analogue of Menger's theorem (see for example [3]) the maximal number of edge-disjoint arcs joining a and b in $G^*(M)$ is $\sigma(M)$. Now let $\sigma = \max_{M \in \mathscr{R}(\mathscr{E}^*)} \sigma(M)$. In the graph $G^*(M_0)$, where M_0 is such a set of $\mathscr{R}(\mathscr{E}^*)$ that $\sigma(M_0) = \sigma$, there exists a system of σ edge-disjoint arcs joining a and b in $G^*(M_0)$. To this system, an $\mathscr{E}$-system of arcs in G joining a and b corresponds. As a system of edge-disjoint arcs in some $G^*(M)$ corresponds to any $\mathscr{E}$-system in G, we cannot have an $\mathscr{E}$-system of arcs in G joining a and b with more than σ arcs, q.e.d.

Now we shall define a simple $\mathscr{E}$-system of arcs joining a and b as an $\mathscr{E}$-system of arcs joining a with b such that any arc of this system has at most one edge in common with any class of $\mathscr{E}$. The simple edge connectivity degree between a and b with respect to $\mathscr{E}$ will be the minimal number of classes of $\mathscr{E}$ which must be deleted in order that all arcs joining the vertex a with the vertex b such that any of them has at most one edge in common with any class of $\mathscr{E}$ might be destroyed.

Theorem 8. *Let G be an undirected graph with the edge set E, let $\mathscr{E}$ be a decomposition of the set E into pairwise disjoint subsets. Let a, b be two vertices of G. Let σ be the simple edge connectivity degree between the vertices a, b with respect to $\mathscr{E}$. Then the maximal number of arcs in a simple $\mathscr{E}$-system of arcs joining a and b is σ.*

This theorem can be proved analogously as Theorem 7. We do not construct the graph G^* and in the further procedure we have to do with the original graph G instead of G^*.

Remark. In the case of infinite graphs, in all considerations of this paper we must assume the validity of the Axiom of Choice.

References

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