Časopis pro pěstování matematiky

Jindřich Kerndl
Affine deformation of surfaces

Časopis pro pěstování matematiky, Vol. 96 (1971), No. 2, 126--132

Persistent URL: http://dml.cz/dmlcz/108515

AFFINE DEFORMATION OF SURFACES

JINDŘICH KERNDL, Brno

(Received October 21, 1969)

In the paper the so called conjugate correspondences between surfaces immersed in $(n + 1)$-dimensional affine unimodular space A_{n+1} are studied. Conditions for an affine deformation of the second order are derived supposing $n > 3$. The case of a surface immersed in a 4-dimensional affine space A_4 is not sufficiently general and it was studied in a special paper. (See [2].)

I would like to express my gratitude to Professor K. SVOBODA for some suggestions leading to the simplification of arguments.

I.

Let (A) be a surface immersed in $(n + 1)$-dimensional affine space A_{n+1}, $n > 3$ generated by the point $A = A(u, v)$, $(u, v) \in C^2$, $(C$ = complex numbers). To each point of the surface we associate a frame consisting of the point A and linearly independent vectors $I_1, I_2, \ldots, I_{n+1}$ such that

$$(1.1) \qquad [I_1 I_2 \ldots I_{n+1}] = 1 .$$

The fundamental system of difierential equation is

$$(1.2) \qquad \mathrm{d}A = \sum_{k=1}^{n+1} \omega_k I_k , \quad \mathrm{d}I_j = \sum_{k=1}^{n+1} \omega_{jk} I_k , \quad j = 1, 2, \ldots, n + 1 ,$$

ω_k, ω_{jk} being linear differential forms in parameters determining the specialization of the moving frame.

Differentiating (1.1) and using (1.2), we obtain

$$(1.3) \qquad \sum_{k=1}^{n+1} \omega_{kk} = 0 .$$

Further, the forms ω fulfil the structure equations of the affine space

$$(1.4) \qquad \mathrm{d}\omega_j = \sum_{k=1}^{n+1} \omega_k \wedge \omega_{kj} , \quad \mathrm{d}\omega_{ij} = \sum_{k=1}^{n+1} \omega_{ik} \wedge \omega_{kj} ; \quad i, j = 1, 2, \ldots, n + 1 .$$

Moreover, we shall suppose that the surface (A) is not developable and we shall consider (A) being a surface immersed in the projective space P_{n+1} obtained from the space A_{n+1} by its projective extension. Then each vector of A_{n+1} is an improper point of P_{n+1}. These points generate the n-dimensional improper projective space P_n. We can speak about the points I_1, I_2 etc. when thinking of the improper points of the space P_n determined by the mentioned vectors. According to our suppositions, improper straight lines of tangent planes of the surface (A) generate the line congruence L in the space P_n. We shall suppose that the congruence L is non-parabolic with the character three (see [1], p. 12).

We shall suppose the frame to be specialized so that the following equations hold

$$\omega_j = 0, \qquad j = 3, 4, \ldots, n+1; \quad \omega_1 \wedge \omega_2 \neq 0; \tag{1.5}$$

$$\begin{aligned} &\omega_{13} = \omega_1, \quad \omega_{24} = \omega_2, \\ &\omega_{14} = 0, \quad \omega_{23} = 0, \\ &\omega_{12} = \alpha_1\omega_2, \quad \omega_{21} = \alpha_2\omega_1, \quad \alpha_1\alpha_2 \neq 0, \\ &\omega_{1j} = 0, \quad \omega_{2j} = 0; \qquad j = 5, 6, \ldots, n+1. \end{aligned} \tag{1.6}$$

By exterior differentiation of equations (1.6) we obtain

$$\begin{aligned} &\omega_1 \wedge (2\omega_{11} - \omega_{33}) - \alpha_2\omega_1 \wedge \omega_2 = 0, \\ &\omega_2 \wedge (2\omega_{22} - \omega_{44}) + \alpha_1\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge \omega_{34} = 0, \\ &\omega_2 \wedge \omega_{43} = 0, \\ &\omega_1 \wedge \omega_{32} + \omega_2 \wedge (\mathrm{d}\alpha_1 - \alpha_1\omega_{11}) - \alpha_1^2\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge (\mathrm{d}\alpha_2 - \alpha_2\omega_{22}) + \omega_2 \wedge \omega_{41} + \alpha_2^2\omega_1 \wedge \omega_2 = 0, \\ &\omega_1 \wedge \omega_{3j} = 0, \\ &\omega_2 \wedge \omega_{4j} = 0; \quad j = 5, 6, \ldots, n+1. \end{aligned} \tag{1.7}$$

As usual, let us denote by δ the differentiation so that $\delta u = \delta v = 0$ and let us write $\omega_{ij}(\delta) = e_{ij}$. Then we have from (1.7)

$$\delta\alpha_1 = \alpha_1 e_{11}, \qquad \delta\alpha_2 = \alpha_2 e_{22}.$$

Moreover, it holds

$$\delta\omega_1 = -e_{11}\omega_1, \quad \delta\omega_2 = -e_{22}\omega_2$$

and we are able to verify

Lemma 1. *The form* $\varphi = \alpha_1\alpha_2\omega_1\omega_2$ *is invariant* (i.e. $\delta\varphi = 0$). *Equation* $\varphi = 0$ *is the equation of the conjugate net of the surface* (A).

II.

Let us consider a surface (B) immersed in an affine space A'_{n+1}, $n > 3$ and generated by a point $B = B(u', v')$. Let us consider the same suppositions on (B) as those on (A). Let the frame of (B) consist of the point B and of the vectors $J_1, J_2, \ldots, J_{n+1}$ such that

$$[J_1 J_2 \ldots J_{n+1}] = 1 \,. \tag{1.1'}$$

Suppose this frame to be specialized in the same way as that associated with (A). We denote all expressions connected with (B) by a dash. So the surface (B) is determined by the system of equations (1.3'), (1.5'), (1.6') together with the exterior quadratic relations (1.7'). There is no need of writing these equations here.

Now, let us consider a correspondence $C : (A) \to (B)$ such that the point $B = CA$ of the surface (B) corresponds to the point A of the surface (A). Let C be regular. Then it is given by

$$\begin{aligned} \omega'_1 &= \lambda_{11}\omega_1 + \lambda_{12}\omega_2 \,, \\ \omega'_2 &= \lambda_{21}\omega_1 + \lambda_{22}\omega_2 \,, \end{aligned} \quad \begin{vmatrix} \lambda_{11}, \lambda_{12} \\ \lambda_{21}, \lambda_{22} \end{vmatrix} \neq 0 \,. \tag{2.1}$$

We shall use the following specification

$$\tau_{ij} = \omega'_{ij} - \omega_{ij} \,, \quad t_{ij} = e'_{ij} - e_{ij} \,.$$

The correspondence $C : (A) \to (B)$ is called *conjugate* in the case it is given by the relations

$$\omega'_1 = \lambda_1\omega_1 \,, \quad \omega'_2 = \lambda_2\omega_2 \,, \quad \lambda_1\lambda_2 \neq 0 \,. \tag{2.2}$$

Let us remark that the geometrical characterization of the conjugate correspondences follows from Lemma 1.

By exterior differentiation of (2.2), we get

$$\begin{aligned} &\omega_1 \wedge (\mathrm{d}\lambda_1 + \lambda_1\tau_{11}) - \lambda_1(\lambda_2\alpha'_2 - \alpha_2)\,\omega_1 \wedge \omega_2 = 0 \,, \\ &\omega_2 \wedge (\mathrm{d}\lambda_2 + \lambda_2\tau_{22}) + \lambda_2(\lambda_1\alpha'_1 - \alpha_1)\,\omega_1 \wedge \omega_2 = 0 \,. \end{aligned} \tag{2.3}$$

Hence, we have

$$\delta\lambda_1 = -\lambda_1 t_{11} \,, \quad \delta\lambda_2 = -\lambda_2 t_{22} \,.$$

Now, it is easy to see that the choice $\lambda_1 = \lambda_2 = 1$ corresponds to the specialization of the frames by the relations $t_{11} = t_{22} = 0$. The conjugate correspondence is then given by the equations

$$\omega'_1 = \omega_1 \,, \quad \omega'_2 = \omega_2 \,. \tag{2.4}$$

Corresponding exterior quadratic relations are

$$(2.5) \qquad \begin{aligned} \omega_1 \wedge \tau_{11} - (\alpha_2' - \alpha_2)\,\omega_1 \wedge \omega_2 = 0\,, \\ \omega_2 \wedge \tau_{22} + (\alpha_1' - \alpha_1)\,\omega_1 \wedge \omega_2 = 0\,. \end{aligned}$$

Using Cartan's lemma, we obtain

$$(2.6) \qquad \begin{aligned} \tau_{11} = f_1\omega_1 + (\alpha_2' - \alpha_2)\,\omega_2\,, \\ \tau_{22} = (\alpha_1' - \alpha_1)\,\omega_1 + f_2\omega_2\,. \end{aligned}$$

By exterior differentiation of (2.6), we get (when denoting $\bar{\alpha}_1 = \alpha_1' - \alpha_1$, $\bar{\alpha}_2 = \alpha_2' - - \alpha_2$)

$$(2.7) \qquad \begin{aligned} &\omega_1 \wedge (df_1 - f_1\omega_{11} + \tau_{31}) + \omega_2 \wedge (\mathrm{d}\bar{\alpha}_2 - \bar{\alpha}_2\omega_{22}) + \\ &\quad + (\alpha_1\alpha_2 - \alpha_1'\alpha_2' + \alpha_2 f_1 - \alpha_1\bar{\alpha}_2)\,\omega_1 \wedge \omega_2 = 0\,, \\ &\omega_1 \wedge (\mathrm{d}\bar{\alpha}_1 - \bar{\alpha}_1\omega_{11}) + \omega_2 \wedge (\mathrm{d}f_2 - f_2\omega_2 + \tau_{42}) + \\ &\quad + (\alpha_1'\alpha_2' - \alpha_1\alpha_2 + \bar{\alpha}_1\alpha_2 - \alpha_1 f_2)\,\omega_1 \wedge \omega_2 = 0\,. \end{aligned}$$

Finally, let us remark that the tangent plane of the surface (A) at any point A is determined by $[AI_1I_2]$. The above mentioned line congruence L is generated by $[I_1I_2]$. Similarly, L' is the marking of the congruence of the improper lines $[J_1J_2]$. Suppose $C : (A) \to (B)$ to be the correspondence. Now, the correspondence $\gamma : L \to L'$ is determined in a natural way so that the improper lines of the tangent planes at the points A, $B = CA$ of the surfaces (A), (B) correspond to each other. Particularly, $C : (A) \to (B)$ being conjugate then $\gamma : L \to L'$ is developable.

III.

In this section, we shall deal with the affine deformation of surfaces.

Let (A) be a surface immersed in an affine space A_{n+1}, $n > 3$. Suppose the frames of the surface (A) to be specialized so that equations (1.3), (1.5), (1.6) hold. Let us make the analogous suppositions concerning the surface (B) immersed in an affine space A'_{n+1}, $n > 3$. Finally, let us consider the correspondence $C : (A) \to (B)$ given by relations (2.1).

The correspondence $C : (A) \to (B)$ is called *an affine deformation of order k* if for each point A of the surface (A) there exists an affinity $T : A_{n+1} \to A'_{n+1}$ such that the surfaces (TA), (B) have an analytic contact of order k at the point $B = CA$. We shall say that T realizes the affine deformation C.

Conditions for the correspondence C to be an affine deformation of the first order consist in the existence of the affinity T so that it holds

$$(3.1) \qquad TA = B\,, \quad T\mathrm{d}A = \mathrm{d}B\,.$$

Let the affinity T be given by

(3.2) $$TA = B\,, \quad TI_j = \sum_{k=1}^{n+1} a_{jk} J_k\,; \quad j = 1, 2, \ldots, n+1\,.$$

Further, we shall always assume the determinant of the matrix $M = \|a_{jk}\|$ being equal to one, i.e.

(3.3) $$\det M = 1\,.$$

Now, we have

$$\mathrm{d}A = \omega_1 I_1 + \omega_2 I_2\,, \quad \mathrm{d}B = \omega_1' J_1 + \omega_2' J_2\,.$$

Making use of the affinity (3.2) and equations (2.1), we get from conditions (3.1)

Lemma 2. *Any correspondence $C : (A) \to (B)$ given by relations (2.1) is an affine deformation of the first order. The affinity T realizing this deformation is of the form*

(3.4) $$TA = B$$

$$M = \begin{Vmatrix} \lambda_{11} & \lambda_{21} & 0 & 0 & \ldots & 0 \\ \lambda_{12} & \lambda_{22} & 0 & 0 & \ldots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots & a_{3,n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n+1,1} & \ldots & \ldots & \ldots & \ldots & a_{n+1,n+1} \end{Vmatrix}.$$

Now, let $C : (A) \to (B)$ be an affine deformation of the second order. Then for each point A of the surface (A) there exists the affinity $T : A_{n+1} \to A'_{n+1}$ so that it holds

(3.5) $$TA = B\,, \quad T\mathrm{d}A = \mathrm{d}B\,, \quad T\mathrm{d}^2A = \mathrm{d}^2B\,.$$

We can suppose that the affinity T is of the form (3.4). Further, it holds

$$\mathrm{d}^2A = \sum_{k=1}^{4} \Phi_k I_k\,,$$

where we denote

(3.6) $$\begin{aligned} \Phi_1 &= \mathrm{d}\omega_1 + \omega_1\omega_{11} + \alpha_2\omega_1\omega_2\,, \\ \Phi_2 &= \mathrm{d}\omega_2 + \omega_2\omega_{22} + \alpha_1\omega_1\omega_2\,, \\ \Phi_3 &= \omega_1^2\,, \quad \Phi_4 = \omega_2^2\,. \end{aligned}$$

Making use of the affinity (3.4), we compute

(3.7) $$\begin{aligned} T\mathrm{d}^2A = {} & (\lambda_{11}\Phi_1 + \lambda_{12}\Phi_2 + a_{31}\Phi_3 + a_{41}\Phi_4)\,J_1 + \\ & + (\lambda_{21}\Phi_1 + \lambda_{22}\Phi_2 + a_{32}\Phi_3 + a_{42}\Phi_4)\,J_2 + \\ & + \sum_{k=3}^{n+1} (a_{3k}\Phi_3 + a_{4k}\Phi_4)\,J_k\,. \end{aligned}$$

Furthermore, we have

$$(3.8) \qquad d^2B = \sum_{k=1}^{4} \Phi'_k J_k$$

where we denote

$$(3.9) \qquad \begin{aligned} \Phi'_1 &= d\omega'_1 + \omega'_1\omega'_{11} + \alpha'_2\omega'_1\omega'_2 \,, \\ \Phi'_2 &= d\omega'_2 + \omega'_2\omega'_{22} + \alpha'_1\omega'_1\omega'_2 \,, \\ \Phi'_3 &= \omega'^2_1 \,, \quad \Phi'_4 = \omega'^2_2 \,. \end{aligned}$$

With regard to the last equation (3.5) and using (3.7), (3.8), we get by comparing the coefficients of linearly independent vectors $J_1, J_2, \ldots, J_{n+1}$

$$(3.10) \qquad \begin{aligned} &\lambda_{11}\Phi_1 + \lambda_{12}\Phi_2 + a_{31}\Phi_3 + a_{41}\Phi_4 = \Phi'_1 \,, \\ &\lambda_{21}\Phi_1 + \lambda_{22}\Phi_2 + a_{32}\Phi_3 + a_{42}\Phi_4 = \Phi'_2 \,, \\ &a_{33}\Phi_3 + a_{43}\Phi_4 = \Phi'_3 \,, \\ &a_{34}\Phi_3 + a_{44}\Phi_4 = \Phi'_4 \,, \\ &a_{3j}\Phi_3 + a_{4j}\Phi_4 = 0 \,; \quad j = 5, 6, \ldots, n + 1 \,. \end{aligned}$$

Using (2.1), (3.6), (3.9), we obtain in the first place

$$(3.11) \qquad a_{3j} = a_{4j} = 0 \,, \quad j = 5, 6, \ldots, n + 1 \,,$$

then

$$(3.12) \qquad \begin{aligned} &a_{33} = \lambda^2_{11} \,, \quad a_{43} = \lambda^2_{12} \,, \quad \lambda_{11}\lambda_{12} = 0 \,, \\ &a_{34} = \lambda^2_{21} \,, \quad a_{44} = \lambda^2_{22} \,, \quad \lambda_{22}\lambda_{21} = 0 \,. \end{aligned}$$

We can assume $\lambda_{11} \neq 0$. Then (3.12_3) yields $\lambda_{12} = 0$. Now, we have $\lambda_{22} \neq 0$ (see (2.1)) and (3.12_6) yields $\lambda_{21} = 0$. Therefore, it is necessary for C to be a conjugate correspondence. We may suppose that it is of the form (2.4). Moreover, equations (2.6) hold.

Now, equations $(3.10_{1,2})$ yield

$$(3.13) \qquad a_{31} = f_1 \,, \quad a_{41} = a_{32} = 0 \,, \quad a_{42} = f_2$$

and also

$$(3.14) \qquad \alpha'_1 = \alpha_1 \,, \quad \alpha'_2 = \alpha_2 \,.$$

These conditions are sufficient, too.

Theorem 1. *Let (A) be a surface in an affine space A_{n+1}, $n > 3$. Let (B) be a surface in an affine space A'_{n+1}, $n > 3$. The correspondence $C : (A) \to (B)$ is an affine deformation of the second order if and only if it is conjugate and equations (3.14) hold.*

As regards the affinity T realizing an affine deformation of the second order, we find out

Lemma 3. *Let $C : (A) \to (B)$ be an affine deformation of the second order. The affinity T realizing this deformation is of the form*

$$TA = B \tag{3.15}$$

$$M = \left\| \begin{matrix} 1 & 0 & 0\,0\,0 & \dots & 0 \\ 0 & 1 & 0\,0\,0 & \dots & 0 \\ f_1 & 0 & 1\,0\,0 & \dots & 0 \\ 0 & f_2 & 0\,1\,0 & \dots & 0 \\ a_{51} & \dots & \dots & \dots & a_{5,n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n+1,1} & \dots & \dots & \dots & a_{n+1,n+1} \end{matrix} \right\| .$$

Let us notice the following fact. Supposing $C : (A) \to (B)$ to be an affine deformation of the second order then $\varphi = \varphi'$ holds where $\varphi = \alpha_1\alpha_2\omega_1\omega_2$ is the point form of the congruence L([1], Prop. 1.). Taking into account [1], Prop. 2., we get

Lemma 4. *Let the correspondence $C : (A) \to (B)$ be an affine deformation of the second order. Then the correspondence $\gamma : L \to L'$ is a point deformation.*

Finally, let us suppose that the surface (A) immersed in A_{n+1} is given. Let us consider the pairs $[C, (B)]$ where (B) is a surface in A'_{n+1} and $C : (A) \to (B)$ is an affine deformation of the second order. These pairs are determined by the system (1.3′), (1.5′), (1.6′), (2.4), (2.6) together with the exterior quadratic relations (1.7′), (2.7). Of course, conditions (3.14) are to be considered. The system is involutive.

Theorem 2. *Let (A) be a given surface immersed in an affine space A_{n+1}, $n > 3$. Then the pairs $[C, (B)]$, (B) being a surface in A'_{n+1}, $n > 3$ and $C : (A) \to (B)$ being an affine deformation of the second order, exist and depend on $2(n + 1)$ functions of one argument.*

References

[1] *Švec A.:* Projective Differential Geometry of Line Congruences, Prague 1965.

[2] *Kerndl J.:* Deformation of surfaces immersed in unimodular affine space of four dimensions, Čas. pěst. mat. *94* (1969), Praha.

Author's address: Brno, Leninova 75 (VAAZ).