Časopis pro pěstování matematiky, roč. 109 (1984), Praha

GENERALIZED HARDY'S INEQUALITY

PETR GURKA, Praha

(Received September 1, 1983)

0. INTRODUCTION

In the present paper we deal with the inequality

$$\left(\int_0^a |u(x)|^q s(x)\,\mathrm{d}x\right)^{1/q} \leqq C^{1/q}\left(\int_0^a |u'(x)|^p r(x)\,\mathrm{d}x\right)^{1/p}, \tag{0.1}$$

where $0 < a \leqq +\infty$, $1 < p \leqq q < +\infty$; $r(x) > 0$, $s(x) \geqq 0$ are measurable functions on $]0, a[$. We find necessary and sufficient conditions on r, s for the inequality (0.1) to hold for every absolutely continuous function $u(x)$ on $]0, a[$ such that $u(0) = 0$, with a constant $C > 0$ independent of $u(x)$.

Some necessary and sufficient conditions have been given in literature. In papers by Bradley [2], Kokilašvili [3] and Maz'ja [6] we find the condition

$$\sup_{0<x<a} \int_x^a s(t)\,\mathrm{d}t \left[\int_0^x r^{1-p'}(t)\,\mathrm{d}t\right]^{q/p'} < +\infty\,. \tag{0.2}$$

These papers represent generalizations of a paper by Muckenhoupt [2], where the inequality (0.1) is studied and the condition (0.2) is derived for $p = q$. The case $p = q$ was also studied by other authors and besides (0.2) there exist two other types of conditions (for $p = q$) different from (0.2). The approaches are different as well. Beesack in [1] or Tomaselli in [8] connect the inequality with a condition of the type that a certain nonlinear differential equation has a solution with certain special properties. A paper by Kufner and Triebel [5] contains explicit formulas expressing r in terms of s or vice versa and a proof that for such functions r, s the inequality (0.1) holds.

Here we shall give generalizations of the latter two approaches for $1 < p \leqq q < +\infty$. In the whole paper we suppose $0 < a \leqq +\infty$ and $p' = p/(p-1)$.

1. GENERALIZED HARDY'S INEQUALITY AND A DIFFERENTIAL EQUATION

Throughout this section let us suppose that $g(x)$ is a nonnegative measurable function on $]0, a[$.

Lemma 1.1. *If the differential equation*

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x}((y'(x))^{q/p'}) + g(x)\, y^{q/p'}(x) = 0 \quad (\lambda > 0) \tag{1.1}$$

has a solution y (with a locally absolutely continuous first derivative) such that

$$y(x) > 0\,, \quad y'(x) > 0 \quad (0 < x < a)\,, \tag{1.2}$$

then the following inequality

$$\left(\int_0^a |u(x)|^q\, g(x)\,\mathrm{d}x\right)^{1/q} \leqq \lambda^{1/q}\left(\int_0^a |u'(x)|^p\,\mathrm{d}x\right)^{1/p} \tag{1.3}$$

holds for every function $u(x)$ such that

$$\begin{cases} u(x) \text{ is absolutely continuous on }]0, a[\,, \\ u(0) = \lim\limits_{t\to 0} u(t) = 0\,. \end{cases} \tag{1.4}$$

Proof. Let $u(x)$ be an absolutely continuous function on $]0, a[$ such that $u(0) = 0$. The case $\int_0^a |u'(x)|^p\,\mathrm{d}x = +\infty$ is trivial. So we can suppose that $\int_0^a |u'(x)|^p\,\mathrm{d}x < +\infty$. Let $y(x)$ be a solution of the problem (1.1)–(1.2); by the Hölder inequality we have

$$|u(x)|^q\, g(x) \leqq \left(\int_0^x |u'(t)|\,\mathrm{d}t\right)^q g(x) = g(x)\left(\int_0^x |u'(t)|\,(y'(t))^{1/p'}\,(y'(t))^{-1/p'}\,\mathrm{d}t\right)^q \leqq$$

$$\leqq g(x)\left(\int_0^x y'(t)\,\mathrm{d}t\right)^{q/p'}\left(\int_0^x |u'(t)|^p\,(y'(t))^{-p/p'}\,\mathrm{d}t\right)^{q/p} \leqq$$

$$\leqq g(x)\, y^{q/p'}(x)\left(\int_0^x |u'(t)|^p\,(y'(t))^{-p/p'}\,\mathrm{d}t\right)^{q/p} =$$

$$= -\lambda \frac{\mathrm{d}}{\mathrm{d}x}((y'(x))^{q/p})\left(\int_0^x |u'(t)|^p\,(y'(t))^{-p/p'}\,\mathrm{d}t\right)^{q/p} =$$

$$= \left(\int_0^x \left[-\lambda \frac{\mathrm{d}}{\mathrm{d}x}((y'(x))^{q/p'})\right]^{p/q} |u'(t)|^p\,(y'(t))^{-p/p'}\,\mathrm{d}t\right)^{q/p}.$$

Hence, using the Minkowski integral inequality we have

$$\left\{\int_0^a |u(x)|^q\, g(x)\,\mathrm{d}x\right\}^{p/q} \leqq$$

$$\leqq \left\{ \int_0^a \left(\int_0^x \left[-\lambda \frac{\mathrm{d}}{\mathrm{d}x} ((y'(x))^{q/p'}) \right]^{p/q} |u'(t)|^p (y'(t))^{-p/p'} \,\mathrm{d}t \right)^{q/p} \mathrm{d}x \right\}^{p/q} \leqq$$

$$= \int_0^a \left(\int_t^a -\lambda \frac{\mathrm{d}}{\mathrm{d}x} ((y'(x))^{q/p'}) (|u'(t)|^p (y'(t))^{-p/p'})^{q/p} \,\mathrm{d}x \right)^{p/q} \mathrm{d}t =$$

$$= \int_0^a |u'(t)|^p (y'(t))^{-p/p'} \left(\int_t^a -\lambda \frac{\mathrm{d}}{\mathrm{d}x} ((y'(x))^{q/p'}) \,\mathrm{d}x \right)^{p/q} \mathrm{d}t \leqq$$

$$\leqq \int_0^a |u'(t)|^p (y'(t))^{-p/p'} (\lambda (y'(t))^{q/p'})^{p/q} \,\mathrm{d}t = \lambda^{p/q} \int_0^a |u'(t)|^p \,\mathrm{d}t = \lambda^{p/q} \int_0^a |u'(x)|^p \,\mathrm{d}x .$$

Raising to the $1/p$-th power $(1/p > 0)$ we obtain the inequality (1.3).

Let us put

$$\tilde{K} = \frac{p'}{q} \inf \sup_{0<x<a} \frac{1}{f(x) - x} \int_0^x g(t) (f(t))^{(q/p'+1)} \,\mathrm{d}t , \tag{1.5}$$

where the infimum is taken over the class of measurable functions such that

$$f(x) > x \quad \text{for all} \quad x \in]0, a[. \tag{1.6}$$

The following lemma gives the relation betwen the number $\tilde{K}$ and the problem (1.1)–(1.2).

Lemma 1.2. *Let λ be a number from Lemma* 1.1 *and let $\tilde{K}$ be given by* (1.5). *Then*

(i) *if the problem* (1.1)–(1.2) *has a solution with a locally absolutely continuous first derivative, then $\lambda \geqq \tilde{K}$;*

(ii) *if $\tilde{K} < +\infty$, then the problem* (1.1)–(1.2) *has a solution for every $\lambda > \tilde{K}$.*

Proof. Let $y(x)$ be a solution of (1.1)–(1.2). Let us take $w = y/y'$. We immediately see that w is a positive solution of the equation

$$w' = \frac{p'}{q\lambda} g w^{(q/p'+1)} + 1 . \tag{1.7}$$

On the other hand we have

$$w(x) \geqq \int_0^x w'(t) \,\mathrm{d}t .$$

Hence (1.7) implies

$$w(x) \geqq \frac{p'}{q\lambda} \int_0^x g(t) (w(t))^{(q/p'+1)} \,\mathrm{d}t + x . \tag{1.8}$$

From (1.8) we obtain

$$w(x) > x \quad (0 < x < a) , \tag{1.9}$$

$$\lambda \geqq \frac{p'}{q} \frac{1}{w(x) - x} \int_0^x g(t)\,(w(t))^{(q/p'+1)}\,dt\,. \tag{1.10}$$

From (1.9), (1.10), (1.5) and (1.6) it follows that $\lambda \geqq \tilde{K}$ and the proof of (i) is complete.

To establish (ii) let us fix $\lambda > \tilde{K}$. By the definition of $\tilde{K}$ there exists a measurable function $f(x)$ such that

$$f(x) \geqq \frac{p'}{q\lambda} \int_0^x g(t)\,(f(t))^{(q/p'+1)}\,dt + x\,. \tag{1.11}$$

Let us define a sequence $w_n(x)$ by setting

$$\begin{aligned} w_0(x) &= f(x)\,, \\ w_{n+1}(x) &= \frac{p'}{q\lambda} \int_0^x g(t)\,(w_n(t))^{(q/p'+1)}\,dt + x\,, \quad (n = 0, 1, \ldots)\,. \end{aligned} \tag{1.12}$$

Then (1.11) becomes $w_0(x) \geqq w_1(x)$; moreover,

$$w_{n+1}(x) - w_n(x) = \frac{p'}{q\lambda} \int_0^x g(t)\,[w_n^{(q/p'+1)}(t) - w_{n-1}^{(q/p'+1)}(t)]\,dt\,,$$

hence the sequence $w_n(x)$ is decreasing. Since $w_n(x) \geqq 0$, the sequence (1.12) converges. We denote its limit by $w(x)$:

$$w(x) = \lim_{n\to\infty} w_n(x)\,. \tag{1.13}$$

By the Levi monotone convergence theorem it follows that w is a nonnegative solution of the integral equation

$$w(x) = \frac{p'}{q\lambda} \int_0^x g(t)\,(w(t))^{(q/p'+1)}\,dt + x\,. \tag{1.14}$$

Hence, w is absolutely continuous and satisfies the differential equation

$$w'(x) = \frac{p'}{q\lambda}\, g(x)\,(w(x))^{(q/p'+1)} + 1\,. \tag{1.15}$$

At this point it is easy to see that the function

$$y(x) = \exp\left(\int_c^x w^{-1}(t)\,dt\right) \quad (c \text{ fixed in }]0, a[) \tag{1.16}$$

satisfies the problem (1.1)–(1.2).

Theorem 1.1. *Let $\tilde{C}$ be the smallest constant such that the inequality*

$$(1.17)\qquad \left(\int_0^a |u(x)|^q g(x)\,\mathrm{d}x\right)^{1/q} \leqq \tilde{C}^{1/q}\left(\int_0^a |u'(x)|^p\,\mathrm{d}x\right)^{1/p}$$

holds for every function $u(x)$ which satisfies (1.4). *Then $\tilde{C} \leqq \tilde{K}$, where $\tilde{K}$ is given by* (1.5).

Remark 1.1. It is easy to see that

$$(1.18)\qquad \tilde{C} = \sup \int_0^a g(x)\left(\int_0^x v(t)\,\mathrm{d}t\right)^q \mathrm{d}x\,,$$

where the supremum is taken over over the class of measurable functions $v(x)$ such that

$$(1.19)\qquad v(x) \geqq 0\,, \quad \int_0^a v^p(x)\,\mathrm{d}x = 1\,.$$

Proof of Theorem 1.1. The assertion is an easy consequence of Lemmas 1.1 and 1.2. If $\tilde{C} > \tilde{K}$, then there exists $\lambda_0 > 0$ such that $\tilde{C} > \lambda_0 > \tilde{K}$; moreover $\tilde{K} < +\infty$, so by Lemma 1.2 – (ii) the problem (1.1)–(1.2) with λ_0 has a solution. By Lemma 1.1 we have the inequality (1.3) with the constant λ_0 for every function $u(x)$ which satisfies the condition (1.4). This implies that $\tilde{C}$ is not the smallest constant in the inequality (1.17). This contradiction proves the theorem.

Notation 1.1. Let $r(x) > 0$, $s(x) \geqq 0$ be two measurable functions defined on $]0, a[$. We shall suppose that

$$(1.20)\qquad \int_0^\xi r^{1-p'}(x)\,\mathrm{d}x < +\infty \quad \text{for all} \quad \xi \in]0, a[\,.$$

Let C be the smallest constant such that the inequality

$$(1.21)\qquad \left(\int_0^a |u(x)|^q s(x)\,\mathrm{d}x\right)^{1/q} \leqq C^{1/q}\left(\int_0^a |u'(x)|^p r(x)\,\mathrm{d}x\right)^{1/p}$$

holds for every function $u(x)$ satisfying (1.4). As in Remark 1.1 it is easy to see that

$$(1.22)\qquad C = \sup \int_0^a s(x)\left(\int_0^x v(t)\,\mathrm{d}t\right)^q \mathrm{d}x\,,$$

where the supremum is taken over the class of all measurable functions $v(x)$ such that

$$(1.23)\qquad v(x) \geqq 0\,, \quad \int_0^a v^p(x)\,r(x)\,\mathrm{d}x = 1\,.$$

Theorem 1.2. *Let C be given by* (1.22)–(1.23) *and let*

$$(1.24)\qquad K = \frac{p'}{q} \inf \sup_{0<x<a} \frac{1}{f(x)} \int_0^x s(t)\left[f(t) + \int_0^t r^{1-p'}(y)\,\mathrm{d}y\right]^{(q/p'+1)} \mathrm{d}t\,,$$

where the infimum is taken over the class of all measurable functions $f(x)$ satisfying the condition

$$f(x) > 0 \quad \textit{for all} \quad x \in]0, a[\, . \tag{1.25}$$

Then $C \leqq K$.

Proof. Let us define the function $z(x)$ by the formula

$$z(x) = \int_0^x r^{1-p'}(y) \, \mathrm{d}y \, ; \tag{1.26}$$

this function is strictly increasing and absolutely continuous and maps the interval $]0, a[$ onto the interval $]0, A[$, where

$$A = \lim_{x \to a} \int_0^x r^{1-p'}(y) \, \mathrm{d}y \, . \tag{1.27}$$

After the change of variables $x \to z$ in the integrals, the inequality (1.21) assumes the form (1.17) (there is A instead of a, z instead of x and the function U instead of the function u), the condition (1.4) for a function $u(x)$ transforms to the same condition for $U(z)$. The function g from (1.17) is given by the formula

$$g(z) = r^{p'-1}(x) \, s(x) \, . \tag{1.28}$$

The change of variables $x \to z$ in (1.22)–(1.23) and (1.24) enables us to see that C, K transform to $\tilde{C}$ and $\tilde{K}$, respectively. Theorem 1.2 is an easy consequence of Theorem 1.1. •

Corollary 1.1. *By Theorem* 1.2 *and the definition of* K *we obtain the estimate*

$$C \leqq \frac{p'}{q} \sup_{0<x<a} \frac{1}{f(x)} \int_0^x s(t) \left[f(t) + \int_0^t r^{1-p'}(y) \, \mathrm{d}y \right]^{(q/p'+1)} \mathrm{d}t \, , \tag{1.29}$$

where $f(x)$ is a positive measurable function.

Corollary 1.2. *Let* B *be defined by*

$$B = \sup_{0<x<a} \int_x^a s(t) \, \mathrm{d}t \left[\int_0^x r^{1-p'}(t) \, \mathrm{d}t \right]^{q/p'} . \tag{1.30}$$

Then

$$B \leqq C \leqq K \leqq p(p')^{q/q'} B \, , \tag{1.31}$$

$$K \leqq p(p')^{q/p'} C \, . \tag{1.32}$$

Remark 1.2. The estimate $B \leqq C \leqq p(p')^{q/p'} B$ can be found also in [2], [3], [6].

Proof of Corollary 1.2. The inequality (1.32) is an easy consequence of (1.31). We shall prove (1.31). Consider the function

$$v(x) = \begin{cases} \left[\int_0^\xi r^{1-p'}(y)\,\mathrm{d}y\right]^{-1/p} r^{1-p'}(x) & \text{if } 0 < x < \xi\,, \\ 0 \quad \text{if } \xi \leqq x < a\,, \end{cases}$$

where ξ is fixed in the interval $]0, a[$. This function satisfies (1.23); hence

$$C \geqq \left(\int_0^\xi r^{1-p'}(t)\,\mathrm{d}t\right)^{q/p'} \int_\xi^a s(t)\,\mathrm{d}t\,.$$

Consequently we have $B \leqq C$. Putting

$$f(x) = p' B^{p'/q} \left(\int_x^a s(t)\,\mathrm{d}t\right)^{-p'/q} - \int_0^x r^{1-p'}(y)\,\mathrm{d}y\,.$$

in (1.24) we complete the proof of the inequality $K \leqq p(p')^{q/p'} B$. The inequality $C \leqq K$ follows from Theorem 1.2.

Theorem 1.3. *Let* $r(x)$, $s(x)$ *be functions from Notation* 1.1. *Moreover, let us suppose that the first derivative* $r'(x)$ *exists for all* $x \in]0, a[$. *Then the equation*

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x}\left(r^{q/p}(x)\,(y'(x))^{q/p'}\right) + s(x)\, y^{q/p'}(x) = 0 \tag{1.33}$$

(*with a certain* $\lambda > 0$) *has a solution* $y(x)$ (*with a locally absolutely continuous first derivative*) *such that*

$$y(x) > 0\,, \quad y'(x) > 0 \quad (0 < x < a) \tag{1.34}$$

iff there exists a constant $C_0 > 0$ *such that the inequality*

$$\left(\int_0^a |u(x)|^q s(x)\,\mathrm{d}x\right)^{1/q} \leqq C_0^{1/q} \left(\int_0^a |u'(x)|^p r(x)\,\mathrm{d}x\right)^{1/p} \tag{1.35}$$

holds for every function $u(x)$ *satisfying* (1.4).

Proof. Sufficiency follows from Lemma 1.1 after performing the change of variables inverse to (1.26).

We shall prove the necessity. If the inequality (1.35) holds for every function $u(x)$ satisfying (1.4), then $C \leqq C_0$, so $C < +\infty$ (C is a constant from Notation 1.1). From (1.32) it follows that $K < +\infty$ (K is defined by (1.24)). By Lemma 1.2 – (ii), after the change of variables inverse to (1.26) we get that there exists a solution of the problem (1.33)–(1.34) for arbitrary $\lambda > K$.

Remark 1.4. From (1.31) it follows that the inequality (1.21) holds for all functions $u(x)$ satisfying (1.4) iff

$$(1.36) \qquad B = \sup_{0<x<a} \int_x^a s(t)\,dt \left[\int_0^x r^{1-p'}(t)\,dt\right]^{q/p'} < +\infty\,.$$

(Cf. Introduction, condition (0.1).)

2. EXPLICIT FORMULAS FOR FUNCTIONS $r(x)$, $s(x)$ IN GENERALIZED HARDY'S INEQUALITY

In Section 1 we gave some characterizations of generalized Hardy's inequality (1.21). The first one, condition (1.36), is useful if we have two functions $s(x)$, $r(x)$ and want to know whether the inequality (1.21) holds. Another situation occurs when we have only one of these functions and would like to find the other one. One method how to solve this problem is suggested by Theorem 1.3, but the differential equation (1.33), which we have to solve, is in general nonlinear. Another method given in [5] for $1 < p = q < +\infty$ will be generalized here to the case $1 < p \leqq q < +\infty$ by using the results from Section 1.

First let us introduce some notation. Let $\varphi(x)$ be a continuously differentiable function defined on $]0, a[$ and such that

$$(2.1) \qquad \varphi'(x) > 0 \quad \text{if} \quad x \in]0, a[$$

and

$$(2.2) \qquad \lim_{x\to a} \varphi(x) = +\infty\,.$$

For brevity we set $\varphi(0) = \lim_{x\to 0} \varphi(x)$, where $\varphi(0) = -\infty$ is admitted. Let us define the functions $r(x)$, $s(x)$ by the formulas

$$(2.3) \qquad r(x) = e^{(1-p)\varphi(x)}[\varphi'(x)]^{1-p}\,,$$

$$(2.4) \qquad s(x) = e^{\varphi(x)}\,\varphi'(x)\left[e^{\varphi(x)} - e^{\varphi(0)}\right]^{(-q/p'-1)} = \frac{-p'}{q}\left(\left[e^{\varphi(x)} - e^{\varphi(0)}\right]^{-q/p'}\right)'\,.$$

Theorem 2.1. *Let $r(x)$, $s(x)$ be the functions defined by (2.3), (2.4). For every function $u(x)$ satisfying (1.4) and such that*

$$\int_0^a |u'(x)|^p\, r(x)\,dx < +\infty\,,$$

the following inequality holds:

$$(2.5)\qquad \left(\int_0^a |u(x)|^q s(x)\,dx\right)^{1/q} \leqq L\left(\int_0^a |u'(x)|^p r(x)\,dx\right)^{1/p},$$

where $L > 0$ does not depend on $u(x)$.

Proof. From (2.1), (2.3), (2.4) we easily see that $r(x)$, $s(x)$ are positive measurable functions on $]0, a[$. From (2.3) we have

$$r^{1/(1-p)}(x) = e^{\varphi(x)}\varphi'(x) = (e^{\varphi(x)})',$$

consequently

$$(2.6)\qquad \int_0^x r^{1-p'}(t)\,dt = e^{\varphi(x)} - e^{\varphi(0)}.$$

Hence we can see that the condition (1.20) is satisfied. To complete the proof it is sufficient to verify the condition (1.36).

Remark 2.1. The identity (2.6) implies

$$(2.7)\qquad \varphi(x) = \log\left[c + \int_0^x r^{1-p'}(t)\,dt\right].$$

Hence, if $r(x)$ is a given function and $c \geqq 0$ a given number, then the corresponding φ-function can be determined by (2.7). We can express $s(x)$ in terms of $r(x)$ (and conversely). From (2.4) and (2.6) we get

$$(2.8)\quad s(x) = -\frac{p'}{q}\left(\left[\int_0^x r^{1-p'}(t)\,dt\right]^{-q/p'}\right)' = r^{1-p'}(x)\left[\int_0^x r^{1-p'}(t)\,dt\right]^{(-q/p'-1)}.$$

From (2.3), (2.4) we get

$$(2.9)\qquad r(x) = (q/p')^{(p-1+p/q)}\,s^{-p}(x)\left[\int_x^a s(t)\,dt\right]^{(p-1+p/q)}.$$

Example 2.1. Let $a = +\infty$, $\alpha < p - 1$. Put $r(x) = x^\alpha$, then

$$\int_0^x r^{1-p'}(t)\,dt < +\infty \quad \text{for all} \quad x \in\,]0, +\infty[\,.$$

After standard calculations we get from (2.8)

$$s(x) = \left(\frac{p-1}{p-1-\alpha}\right)^{(-q/p'-1)} x^{(-q/p'-1+\alpha q/p)}.$$

This means that there exists a constant $C_0 > 0$ such that the inequality

$$\left(\int_0^{+\infty} |u(x)|^q\, x^{(-q/p'-1+\alpha q/p)}\,dx\right)^{1/q} \leqq C_0\left(\int_0^{+\infty} |u'(x)|^p\, x^\alpha\,dx\right)^{1/p}$$

holds for every absolutely continuous function $u(x)$ on $]0, +\infty[$ such that $u(0) = 0$.

If we put $p = q$, we get the classical Hardy's inequality

$$\int_0^{+\infty} |u(x)|^p x^{\alpha-p} \, dx \leqq C_0^{1/p} \int_0^{+\infty} |u'(x)|^p x^{\alpha} \, dx .$$

(Cf. [4], Theorem 330.)

References

[1] *P. R. Beesack:* Hardy's inequality and its extensions. Pacific J. Math., XI, 1961.

[2] *J. S. Bradley:* Hardy inequality with mixed norms. Canad. Math. Bull. *21* (1978), 405—408.

[3] *V. M. Kokilašvili:* On Hardy's inequalities in weighted spaces (Russian). Bull. Acad. Sci. Georgian SSR, *96* (1979), No. 1.

[4] *G. H. Hardy, J. E. Littlewood, G. Polya:* Inequalities, Cambridge University Press, 1934.

[5] *A. Kufner, H. Triebel:* Generalization of Hardy's inequality. Conf. Sem. Mat. Univ. Bari (1978), 156.

[6] *V. Maz'ja:* Einbettungssätze für Sobolewsche Räume I, II. Teubner-Texte zur Mathematik, Bd. *21*, *28*, Teubner Leipzig 1979, 1981.

[7] *B. Muckenhoupt:* Hardy's inequality with weights, Studia Math. *44* (1972), 31—38.

[8] *G. Tomaselli:* A class of inequalities, Boll. UMI (4), *6* (1969).

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).