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Časopis pro pěstování matematiky, Vol. 106 (1981), No. 4, 347--353

Persistent URL: http://dml.cz/dmlcz/108496

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ON SOME MAPPINGS GENERATING VECTOR L-MEASURES

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(Received November 2, 1977)

1. NOTATION AND INTRODUCTORY REMARKS

1.1. In this paper the letter L will be reserved for an *orthomodular σ-poset*, that is a partially ordered set (with an ordering relation $\leqq$) with the greatest element 1 and with a binary relation $\perp$ (the so called *orthogonality*) satisfying the conditions:

(i) $\perp$ is a symmetric relation;
(ii) $a \perp b$, $c \leqq a$ implies $c \perp b$;
(iii) for every at most countable family $\{a_n\}_{n \in I}$ $(I \subset N)$ such that $a_n \perp a_m$ for $n \neq m$, there exists $\sup_n a_n$.

In this case we write $\sup_n a_n = \sum_n a_n$ and we call this supremum the *orthosum of* $\{a_n\}_n$ and $\{a_n\}_n$ an *orthofamily*; if $I = \{1, 2, \ldots, m\}$ then we also write $a_1 + \ldots + a_m$.

(iv) $a \leqq b + c$, $a \perp b$ implies $a \leqq c$;
(v) for every pair a, b such that $a \leqq b$, there exists one and only one element c such that $b = a + c$.

We write $c = b - a$ and we call c the (*relative*) *orthocomplement of a in b*.

1.2. Remarks. a) Since $a \leqq 1$ for every $a \in L$, the element $1 - a =_{\mathrm{df}} a^{\perp}$ exists according to (v); we call this element the *orthocomplement of a*. It is easy to show that the function $a \mapsto a^{\perp}$ is involutory and antitone. Further, the least element 0 of L exists and $0 = 1^{\perp}$.

b) The least upper bound or the greatest lower bound of a family $\{a_i\}_{i \in I}$ (which need not be orthogonal) will be denoted by $\bigvee\{a_i \mid i \in I\}$ or $\bigwedge\{a_i \mid i \in I\}$, respctively; for $I = N$ we shall also use the notation

$$\bigvee_{n=1}^{\infty} a_n \quad \text{or} \quad \bigwedge_{n=1}^{\infty} a_n\,,$$

respectively.

c) The abbreviation (σOP) will be used for the term "orthomodular σ-poset". For some properties of (σOP) see e.g. [1]. Boolean σ-algebra is a special case of (σOP); $a^{\perp}$ is then the Boolean complement of a. (σOP) is sometimes called an abstract logic of a physical system. The standard logic of quantum mechanics is the lattice of all closed subspaces of a separable Hilbert space H; this lattice is (σOP), where the ordering is given by the set inclusion and the orthogonality is the usual orthogonality of subspaces.

1.3. Definition. A set $M \subset L$ is called *compatible in* L if for each finite subset $\{a_1, \ldots, a_k\} \subset M$ there exists a finite orthogonal family in L such that every element a_i $(i = 1, \ldots, k)$ is the orthosum of a subfamily of this family.

From Tukey's lemma it follows that for every compatible set $M \subset L$ there exists a maximal compatible set B in L containing M.

We call every maximal compatible set in L a *block of* L. In the paper [1] the following theorem was proved.

Theorem. *Every block B in L is a Boolean σ-subalgebra of L.*

If L is a Boolean σ-algebra, then every subset of L is compatible; one and only one block of L is L in this case.

1.4. Definition. Let L, L' be (σOP). A mapping $h : L \to L'$ is called the *σ-orthohomomorphism* if it has the following properties:

(a) $h(0) = 0$, $h(1) = 1$,

(b) $a \perp b$ implies $h(a) \perp h(b)$,

(c) if $a_i \perp a_k$ for $i, k = 1, 2, \ldots$; $i \neq k$, then

$$h\left(\sum_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} h(a_n) .$$

The class of all σ-orthohomomorphisms of L into L' will be denoted by hom (L, L').

Remark. If M is compatible in L, then $h(M)$ is compatible in L'.

1.5. The letter E will be reserved for a separable Banach space. We denote by $\mathscr{G}$ the family of all open sets in E, by $\mathscr{B}$ the σ-algebra of all Borel sets in E. A *multiplicative base in E* is such an open base which is closed under finite intersections and which includes $\emptyset$ and E; we denote it by $\mathscr{G}^{\wedge}$. It is clear that every (countable) open base in E can be extended to a (countable) multiplicative open base.

2. L-MEASURES AND L-SCALES IN E

2.1. Definition. Each element of hom $(\mathscr{B}, L)$ is called an *L-measure in E.*

The class of all L-measures in E will be denoted by $\mathscr{L}_E$.

Remarks. 1) If x is an L-measure in E, then R_x is a σ-Boolean subalgebra in L. (By R_x we mean the range of the mapping x.)

2) If $\mathscr{S} \subset \mathscr{B}$ generates $\mathscr{B}$ and x, y are L-measures in E such that $x \mid \mathscr{S} = y \mid \mathscr{S}$, then $x = y$.

2.2. Definition. An *L-scale in E* is a mapping $f : \mathscr{G}^\wedge \to L$, such that

(1) $\mathscr{G}^\wedge$ is a multiplicative base in E,

(2) R_f is a compatible set in L,

(3) $f(G_1 \cap G_2) = f(G_1) \wedge f(G_2)$ for every pair $G_1, G_2 \in \mathscr{G}^\wedge$,

(4) for every $r > 0$ there exists a countable r-cover $\mathscr{P}_r \subset \mathscr{G}^\wedge$ of E (i.e. $E = \bigcup\{G \mid G \in \mathscr{P}_r\}$, diam $G < r$) such that $\bigvee\{f(G) \mid G \in \mathscr{P}_r\} = 1$.

If instead of (4) the following stronger condition holds:

(4*) a) $G = \bigcup_{n=1}^{\infty} G_n$, $G_n \in \mathscr{G}^\wedge$, $G \in \mathscr{G}^\wedge$ implies

$$f(G) = \bigvee_{n=1}^{\infty} f(G_n),$$

b) $f(E) = 1$,

then we call f a *σ-additive scale.*

The class of all L-scales in E will be denoted by $\mathscr{S}_E$.

2.3. Remarks. 1) It is clear that for every L-measure x in E and for every multiplicative base $\mathscr{G}^\wedge$ the restriction $x \mid \mathscr{G}^\wedge$ is a σ-additive L-scale in E.

2) *Real L-scale of Caratheodory* (cf. [2]) is a mapping $\tilde{f} : R \to L$ with the following properties:

(α) $\tilde{f}$ is isotonic, i.e. $p \leqq q$ implies $\tilde{f}(p) \leqq \tilde{f}(q)$,

(β) for every real sequence (p_n), $p_n \nearrow +\infty$,

$$\bigvee_{n=1}^{\infty} \tilde{f}(p_n) = 1,$$

(γ) for every real sequence (p_n), $p_n \searrow -\infty$,

$$\bigwedge_{n=1}^{\infty} \tilde{f}(p_n) = 0.$$

If we define $f : \mathcal{G}^\wedge \to L$, where $\mathcal{G}^\wedge$ is the base of open intervals in R,

$$f((p, q)) = \tilde{f}(q) - \tilde{f}(p), \quad p < q,$$

then f is an L-scale in R from Definition 2.2. On the other hand, every L-scale in R defined on the base of open intervals induces a scale of Caratheodory.

3) We shall try to motivate physically the definition of the L-scale. Suppose that we have a set $\mathcal{O}_E$ of objects, which we shall call observable vectors in E of a physical system. For every observable vector $f \in \mathcal{O}_E$ and for every open set $G \in \mathcal{G}^\wedge$ we shall interpret the pair (f, G) as the hypothesis that the "value" of f lies in the set G. Let us assume that the set of all pairs (f, G), $f \in \mathcal{O}_E$, $G \in \mathcal{G}^\wedge$ is (σOP), where $a \leqq b$ means that the hypothesis b is a consequence of the hypothesis a and $a \perp b$ means that the hypotheses a, b exclude each other. For a fixed observable vector $f \in \mathcal{O}_E$ the family $\{(f, G)\}_{G \in \mathcal{G}^\wedge}$ is a family of experimental hypotheses associated with the vector f. Now, if we define a mapping $\hat{f} : G \mapsto (f, G)$, then the conditions (1)–(4) from Definition 2.2 seem to be quite natural.

Let us remark that L-measure in R are sometimes called *observables* (see e.g. [4]). In Section 3 we will show that L-measures are generated (in a certain sense) by L-scales.

2.4. Definition. Let $f_1 : \mathcal{G}_1^\wedge \to L$, $f_2 : \mathcal{G}_2^\wedge \to L$ be two L-scales in E. We say that f_1 *is equivalent to* f_2 (and we write $f_1 \sim f_2$) if

1) $G_1 \in \mathcal{G}_1^\wedge$, $G_2 \in \mathcal{G}_2^\wedge$, $\bar{G}_1 \subset G_2$ implies $f_1(G_1) \leqq f_2(G_2)$,

2) $G_1' \in \mathcal{G}_1^\wedge$, $G_2' \in \mathcal{G}_2^\wedge$, $\bar{G}_2' \subset G_1'$ implies $f_2(G_2') \leqq f_1(G_1')$.

3. A THEOREM ON GENERATING

3.1. Theorem. *Let $f : \mathcal{G}^\wedge \to L$ be an L-scale in E. Then there exists one and only one L-measure x in E such that*

(i) $x(G) \leqq f(G)$ *for all* $G \in \mathcal{G}^\wedge$,

(ii) $f(G_1) \leqq x(G)$ *whenever* $\bar{G}_1 \subset G$, $G_1 \in \mathcal{G}^\wedge$, $G \in \mathcal{G}$.

If f is σ-additive, then $f = x \mid \mathcal{G}^\wedge$.

This theorem immediately implies

Corollary. *The equivalence of L-scales from Definition 2.4 is an equivalence relation on $\mathcal{S}_E$. Let $\mathcal{V}_E$ be the factor set $\mathcal{S}_E/\sim$. Then there exists one to one function s from $\mathcal{V}_E$ onto $\mathcal{L}_E$ such that $s(X) \mid \mathcal{G}^\wedge \in X$ for every $X \in \mathcal{V}_E$ and every multiplicative base $\mathcal{G}^\wedge$ in E.*

3.2. Proof of uniqueness of L-measure. It suffices to prove that the L-measure x from Theorem 3.1 satisfies

$$\text{(iii)} \qquad x(G) = \bigvee_{n=1}^{\infty} f(G_n)\,, \quad \text{where} \quad \bigcup_{n=1}^{\infty} G_n = G\,, \quad G_n \in \mathscr{G}^\wedge\,, \quad \bar{G}_n \subset G\,,$$

so it is uniquely defined on $\mathscr{G}$ and since $\mathscr{G}$ generates the σ-algebra $\mathscr{B}$, the uniqueness follows from Remark 2, Section 2.1.

If

$$G = \bigcup_{n=1}^{\infty} G_n\,, \quad G_n \in \mathscr{G}^\wedge\,, \quad \bar{G}_n \subset G\,,$$

then (i) implies

$$x(G_n) \leqq f(G_n)\,, \quad \text{so} \quad x(G) = \bigvee_{n=1}^{\infty} x(G_n) \leqq \bigvee_{n=1}^{\infty} f(G_n)\,;$$

on the other hand, it follows from (ii) that $f(G_n) \leqq x(G)$, hence

$$\bigvee_{n=1}^{\infty} f(G_n) \leqq x(G)$$

and (iii) holds. If f is σ-additive, then obviously $f(G) = x(G)$ for all $G \in \mathscr{G}^\wedge$.

The existence of L-measure x from Theorem 3.1 will be proved for special cases of (σOP) in the following sections 3.3–3.6. We may assume that the base $\mathscr{G}^\wedge$ is countable.

3.3. The assertion of Theorem 3.1 holds provided that L is a σ-field of sets.

Namely, there exists a map $g : 1 \to E$ such that $g^{-1} : M \mapsto g^{-1}(M)$, $M \subset E$, has properties (i), (ii) and therefore $x = g^{-1} \mid \mathscr{B}$ is the L-measure from Theorem 3.1.

Proof. Let t be any element of 1 and let us denote

$$\mathscr{B}_t = \{G \in \mathscr{G}^\wedge \mid t \in f(G)\}\,;$$

obviously $\mathscr{B}_t \neq \emptyset$ (see (4) of Definition 2.2). Further, $G_1, G_2 \in \mathscr{B}_t$ implies $G_1 \cap G_2 \in \mathscr{B}_t$ (this follows from (3), Definition 2.2). So $\mathscr{B}_t$ is a base of a filter $\mathscr{F}_t$, which is a Cauchy filter. Indeed, in view of (4), Definition 2.2, for every $r > 0$ there exists $G \in \mathscr{G}^\wedge$, diam $G < r$, such that $G \in \mathscr{B}_t$. Since E is a complete space, we have $\mathscr{F}_t \to s$ and this s is unique. We put $g(t) = s$, so a map $g : 1 \to E$ is defined. If $t \in g^{-1}(G)$, $G \in \mathscr{G}^\wedge$, then $g(t) = s \in G$, thus G is a neighbourhood of s and therefore $G \in \mathscr{F}_t$. Thus $t \in f(G)$, hence $g^{-1}(G) \subset f(G)$ and (i) holds. Let $\bar{G}_1 \subset G$ ($G_1 \in \mathscr{G}^\wedge$, $G \in \mathscr{G}$); we will prove that $f(G_1) \subset g^{-1}(G)$. Let $t \in f(G_1)$ and let us assume that $t \notin g^{-1}(G)$, so $g(t) = s \notin G$. Since $\bar{G}_1 \subset G$, there exists $G_0 \in \mathscr{G}^\wedge$ such that $s \in G_0$, $G_0 \cap G_1 = \emptyset$. Hence $f(G_0) \cap f(G_1) = \emptyset$ (according to (3)). Now $s \in G_0$ implies $t \in g^{-1}(G_0) \subset f(G_0)$, a contradiction. Therefore (ii) holds.

3.4. The assertion of Theorem 3.1 holds provided that L is a factor σ-algebra $\mathcal{M}/I$, where $\mathcal{M}$ is a σ-field of sets and I is a σ-ideal in $\mathcal{M}$ (we denote the greatest element of $\mathcal{M}$ by M).

Namely, there exists $g : M \to E$ such that the L-measure x from Theorem 3.1 is defined by $x(A) = [g^{-1}(A)]$ for every $A \in \mathcal{B}$. ($[C]$ is the equivalence class of $\mathcal{M}/I$ such that $C \in [C]$, $C \in \mathcal{M}$; the greatest element in $\mathcal{M}/I$ will be denoted by 1.)

Proof. For every $G \in \mathcal{G}^\wedge$ let us choose one and only one element $\hat{f}(G) \in f(G)$. We put

$$F_0 = \bigcup_{G_1, G_2 \in \mathcal{G}^\wedge} \{(\hat{f}(G_1 \cap G_2) - (\hat{f}(G_1) \cap \hat{f}(G_2)) \cup \cup ((\hat{f}(G_1) \cap \hat{f}(G_2)) - \hat{f}(G_1 \cap G_2))\} .$$

Since

$$[\hat{f}(G_1 \cap G_2)] = [\hat{f}(G_1)] \wedge [\hat{f}(G_2)] = [\hat{f}(G_1) \cap \hat{f}(G_2)] ,$$

it is clear that $F_0 \in I$. We have

$$\hat{f}(G_1 \cap G_2) - F_0 = (\hat{f}(G_1) \cap \hat{f}(G_2)) - F_0$$

for every $G_1, G_2 \in \mathcal{G}^\wedge$, so if we put $\tilde{f}(G) = \hat{f}(G) - F_0$, then $\tilde{f}(G_1 \cap G_2) = \tilde{f}(G_1) \cap \cap \tilde{f}(G_2)$, thus $\tilde{f}$ has the property (3) from Definition 2.2. Moreover, $[\tilde{f}(G)] = = [\hat{f}(G)] = f(G)$. For every rational $r > 0$ we denote by $\mathcal{P}_r$ such an r-cover of E, $\mathcal{P}_r \subset \mathcal{G}^\wedge$, that $\bigvee\{f(G) \mid G \in \mathcal{P}_r\} = 1$ (property (4) from Definition 2.2). Let us put

$$F_1 = \bigcup_{r \in Q^+} (M - \bigcup_{G \in \mathcal{P}_r} \tilde{f}(G)) ,$$

where Q^+ is the set of positive rational numbers. For every rational $r > 0$

$$1 = [M] = \bigvee\{[\tilde{f}(G)] \mid G \in \mathcal{P}_r\} = [\bigcup\{\tilde{f}(G) \mid G \in \mathcal{P}_r\}] ,$$

so $F_1 \in I$. We put $f^0(G) = \tilde{f}(G) \cup F_1$, thus

$$[f^0(G)] = [\tilde{f}(G)] = f(G) .$$

It is clear that

$$\bigcup\{f^0(G) \mid G \in \mathcal{P}_r\} = M ,$$

so f^0 has the property (4) (with respect to the σ-field $\mathcal{M}$). Obviously, f^0 also preserves intersections, so in view of 3.3 there exists

$$x^0 \in \hom(\mathcal{B}, \mathcal{M}) \quad (x^0 = g^{-1} \mid \mathcal{B} , \quad \text{where} \quad g : M \to E)$$

such that the conditions (i), (ii) from Theorem 3.1 are fulfilled for x^0 and f^0. We then define

$$x : \mathcal{B} \to \mathcal{M}/I \quad \text{by } x(A) = [x^0(A)] ,$$

so $x \in \hom(\mathcal{B}, \mathcal{M}/I)$ and the conditions (i), (ii) from Theorem 3.1 are fulfilled for x and f.

3.5. The assertion of Theorem 3.1 holds provided that L is an arbitrary σ-algebra.

Proof. According to Loomis representation theorem of Boolean σ-algebras (see e.g. [3]) there exists a σ-field $\mathcal{M}$ of sets, a σ-ideal I in $\mathcal{M}$ and an isomorphism h from L onto $\mathcal{M}/I$. If f is an L-scale in E, then $f^1 = h \circ f$ is an $\mathcal{M}/I$-scale in E, thus the preceding section gives that there exists $x^1 \in \hom(\mathcal{B}, \mathcal{M}/I)$ such that (i), (ii) hold for x^1 and f^1. If we put $x = h^{-1} \circ x^1$, then $x \in \hom(\mathcal{B}, L)$ and (i), (ii) hold for x and f.

3.6. Now it is easy to complete the proof of Theorem 3.1. Let L be any (σOP). Since R_f is compatible in L, there exists a Boolean σ-sublagebra $B \subset L$, $B \supset R_f$ (see Section 1.3) and $f : \mathcal{G}^\wedge \to B$ is also a B-scale in E. Therefore, in view of 3.5 there exists $x \in \hom(\mathcal{B}, B)$ such that (i), (ii) hold for x and f. Simultaneously, of course, $x \in \hom(\mathcal{B}, L)$, so x is an L-measure in E and thus Theorem 3.1 is proved.

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