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ON ALGEBRAS HAVING AT MOST TWO ALGEBRAIC OPERATIONS DEPENDING ON n VARIABLES

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INTRODUCTION

Let $\mathfrak{A} = (X, \mathbf{F})$ be an algebra in the sense of Professor Marczewski. In this paper it is supposed that card $X \geqq 2$. Let ω_n denote the number of essentially n-ary algebraic operations in $\mathfrak{A}$, it means, n-ary algebraic operations depending on each variable. Observe that $\omega_1 \geqq 1$ in view of the existence of the trivial unary operation $f(x) = x$.

Professor MARCZEWSKI suggested the examination of possible sequences $\{\omega_n\}$ in general algebras. In this paper we give a complete description of possible sequences $\{\omega_n\}$ under the condition $\omega_n \leqq 2$ for all n (Section 2). In Section 1 we look for a representation of algebras in which $\omega_n = 1$. This concrete topic has been suggested to me by J. PŁONKA in connection with my attending of seminar of Prof. G. GRÄTZER in Winnipeg.

Sometimes we shall omit the word "algebraic", when we speak about algebraic operation. The representability of an algebra is defined as in [6].

1.

In the first part we shall be interested in an algebra $\mathfrak{A} = (X; \mathbf{F})$, where $\omega_n = 1$ for $n = 0, 1, 2, \ldots$ So, if not stated otherwise, $\mathfrak{A}$ means such an algebra.

Remark 1. *It is clear that an algebra with $\omega_n = 1$ for all n possesses at least two elements.*

Lemma 1. *If $x . y$ is essentially binary in $\mathfrak{A}$, then it must be symmetric and associative.*

Proof. The first part follows from $\omega_2 = 1$. If $(x . y) . z$ is essentially ternary, then $x . (y . z) = (y . z) . x$ is also essentially ternary and in view of $\omega_3 = 1$ it must be $(x . y) . z = x . (y . z)$. If $(x . y) . z$ is not essentially ternary, it can be equal to one of the following functions: $x, y, z, x \,.\, y, x \,.\, z, y \,.\, z$ or to the constant c, which exists in view of $\omega_0 = 1$.

It cannot be $(x \,.\, y) \,.\, z = x$ or y because the operation . is symmetric and cannot be equal to $x \,.\, z$ or $y \,.\, z$, because in this case we get $x \,.\, z = y \,.\, z$, which contradicts to the assumption that . is essentially binary. If it were $(x \,.\, y) \,.\, z = z$ or $(x \,.\, y) \,.\, z = x \,.\, y$, then putting $z = u \,.\, v$ we would get in the first case $x \,.\, y = u \,.\, v$, hence $x \,.\, y = c$ – a contradiction. The second case is analogous. Thus it is $(x \,.\, y) \,.\, z = c$. And similarly we prove $x \,.\, (y \,.\, z) = c$. It means $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)$.

We have two possibilities: either $x \,.\, x = x$ or $x \,.\, x = c$.

Lemma 2. *If $x \,.\, x = x$, then $\mathfrak{A}$ can be represented as an at least two-element semilattice with* 0 *or* 1.

Proof. It follows from Lemma 1, that . satisfies the axioms of semilattice. Because $\omega_0 = \omega_1 = 1$ it must be $x \,.\, c = x$ or $x \,.\, c = c$. Every operation $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n$ for $n \geqq 3$ is essentially n-ary. In fact, in the opposite case we have $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n = c$. Then putting $x_3 = x_4 = \ldots = x_n = x_2$ we get $x_1 \,.\, x_2 = c$ – a contradiction. Then every essentially n-ary operation is of the form $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n$, which proves our lemma.

Lemma 3. *If $x \,.\, x = c$ and $x \,.\, c = x$, then $\mathfrak{A}$ can be represented as an at least two-element Boolean group.*

Proof. First we use Lemma 1. Each of the operations $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n$ is essentially n-ary. Otherwise it would be $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n = c$. Putting c on each variable x_i for $i \geqq 3$ we get $x_1 \,.\, x_2 = c$ – a contradiction. So any operation in $\mathfrak{A}$ is of the form $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n$.

Remark 2. *If $x \,.\, c = c$ and*

$$x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n = c \quad \text{for some} \quad n \geqq 2\,, \tag{1}$$

then $x \,.\, x = c$ and $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_m = c$ for $m > n$.

In fact, it cannot be $x \,.\, x = x$, because in this case, identifying all variables in (1), we get $x = c$, which is a contradiction to Remark 1. Further in view of (1) we have $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_m = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n \,.\, x_{n+1} \,.\, \ldots \,.\, x_m = c \,.\, x_{n+1} \,.\, \ldots \,.\, x_m = c$.

Theorem 1. *If $\mathfrak{A}$ is a groupoid (it means the operation $x \,.\, y$ can be taken as fundamental), then $\mathfrak{A}$ can be represented as an at least two-element semilattice with* 0 *or* 1, *an at least two-element Boolean group or a semigroup fulfilling the equalities $x \,.\, x = x \,.\, c = c$, $x \,.\, y = y \,.\, x$, $(x \,.\, y) \,.\, z = x \,.\, (y \,.\, z)$ and the equality $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_n = c$ does not hold for any $n \geqq 1$.*

It follows from Lemma 2, 3 and Remark 2.

Example of the last algebra is the following algebra: we take an infinite set X and let $\mathbf{A}$ be the set of all non-void subset of X. For $A, B \in \mathbf{A}$ we define an operation $.$ as follows: if $A \cap B = \emptyset$, then $A . B = A \cup B$ and $A . B = X$ otherwise.

Lemma 4. *For any natural number $n > 1$ and for any sequence $\{a_k\}$, where $k = 0, 1, 2, \ldots, a_k = 1$ for $k < n$ and $a_k = 0$ or 1 for $k \geqq n$, there exists an algebra $\mathfrak{A} = (X; ., f_i(x_1, x_2, \ldots, x_i)$ for $i \geqq n$, $a_i = 1)$ such that in $\mathfrak{A}$ we have $\omega_k = a_k$ and $x_1 . x_2 . \ldots . x_k$ is essentially k-ary for $1 < k < n$, $n > 2$ and $x_1 . x_2 = c$ for $n = 2$.*

Proof. Let G be the set of elements of a free algebra $(\{g_i\}_{i<\aleph_0}; ., c)$ with $\aleph_0$ free generators g_i in the equational class defined by the equalities: $(x . y) . z = x . (y . z)$,

(2)
$$x . y = y . x ,$$
$$x . x = c . x = c ,$$
$$x_1 . x_2 . \ldots . x_n = c .$$

Let us denote by $X = G \cup \{b\}$, $b \notin G$. We define the operation $x . y$ as follows: if $x, y \in G$, then the operation $.$ coincides with $.$ in our free algebra. Let us put $x . y = c$ otherwise. Let $a_i \neq 0$. We define the operation $f_i(x_1, x_2, \ldots, x_i)$ as follows: for distinct $g_1, g_2, \ldots, g_i$ it is $f_i(g_1, g_2, \ldots, g_i) = b$ and $f_i(x_1, \ldots, x_i) = c$ otherwise. It is easy to check that each operation $f_i(x_1, x_2, \ldots, x_i)$ is essentially i-ary and all required conditions are satisfied.

Remark 3. *The functions f_i in previous proof satisfy the equation $f_i(x, x, \ldots, x) = c$. The following proposition wtll show that this property is necessary.*

Proposition 1. *Let $\mathfrak{A}$ be an algebra, $c \in \mathfrak{A}$. Let $.$ be an essentially binary operation in $\mathfrak{A}$, satisfying equations $x . x = x . c = c$ and let in $\mathfrak{A}$ be $\omega_i \leqq 1$ for all i. Let $n \geqq 2$. Let $f(x_1, \ldots, x_n)$ be an essentially n-ary operation in $\mathfrak{A}$. Then it must be $f(x, x, \ldots, x) = c$.*

Proof. Let us suppose

(3)
$$f(x, x, \ldots, x) = x$$

and n is even. Putting in the operation f $x = x_i$ for $i = 1, 2, \ldots, n/2$ and $x_i = y$ for remaining variables we get $f(x, \ldots, x, y, \ldots, y)$, which is binary, symmetric because f is symmetric, and idempotent. So, it is essentially binary and different from $.$, which contradicts $\omega_2 = 1$.

Let us suppose now (3) and n is odd. It must be

(4)
$$f(x_1, x_2, x_2, \ldots, x_2) = x_1 .$$

In fact $f(x_1, x_2, \ldots, x_2)$ cannot be binary in view of $\omega_2 = 1$. If it were $f(x_1, x_2, \ldots, x_2) = x_2$, then we would have

(5)
$$f(f(x_1, x_2, \ldots, x_2), x_1, \ldots, x_1) = x_1 .$$

But the operation $f(f(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1})$ is essentially $(2n - 1)$-ary and therefore symmetric in view of $\omega_{2n-1} \leqq 1$. Namely let $f(f(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots$ $\ldots, x_{2n-1})$ be independent on $x_1, \ldots, x_n$. Then $f(f(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) =$ $= f(f(x_1, x_1, \ldots, x_1), x_{n+1}, \ldots, x_{2n-1}) = f(x_1, x_{n+1}, \ldots, x_{2n-1})$. But $f(x_1, x_{n+1}, \ldots$ $\ldots, x_{2n-1})$ depends on each variable, also on x_1. Thus $f(f(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots$ $\ldots, x_{2n-1})$ depends on $x_1, \ldots, x_n$. Let us suppose now, that the function $f(f(x_1, \ldots$ $\ldots, x_n), x_{n+1}, \ldots, x_{2n-1})$ does not depend on $x_{n+1}, \ldots, x_{2n-1}$. So $f(x_1, x_{n+1}, \ldots$ $\ldots, x_{2n-1}) = f(f(x_1, x_1, \ldots, x_1), x_{n+1}, \ldots, x_{2n-1}) = f(f(x_1, \ldots, x_1), x_1, \ldots, x_1) =$ $= f(x_1, \ldots, x_1) = x_1$, which contradicts the fact, that $f(x_1, \ldots, x_n)$ is essentially n-ary.

Thus $f(f(x_1, x_2, \ldots, x_2), x_1, \ldots, x_1) = f(f(x_1, x_1, \ldots, x_1), x_2, \ldots, x_2) = f(x_1, x_2,$ $\ldots, x_2) = x_2$. And in view of (5) we get $x_2 = x_1$. Then (4) must hold.

The operation $f(x_1, x_2, c, \ldots, c)$ is symmetric and binary, so it must be equal either to $x_1 \,.\, x_2$ or c. Putting $x_2 = c$ we get in both cases by (4) $x_1 = c$. So formula (3) leads to a contradiction and it must be $f(x, x, \ldots, x) = c$.

From Lemma 4 we get

Theorem 2. *For every $n \geqq 2$ there exists an algebra with $\omega_k = 1$, $k = 0, 1, 2, 3, \ldots$, for which a set of fundamental operations can be chosen as $\{\,.\,, f_n, f_{n+1}, \ldots\}$, where $.$ is a binary operation and f_i for $i \geqq n$ an essentially i-ary operation different from $x_1 \,.\, \ldots \,.\, x_i$.*

Proof. To prove this theorem it is enough to form algebras $\mathfrak{A}_n$ from Lemma 4 for $n = 2, 3, 4, \ldots$ and $\omega_k = 1$, $k = 0, 1, 2, \ldots$ Observe that any algebra $\mathfrak{A}_n$ satisfies equalities (2) and all equalities of the form $f_i(x_{p_1}, \ldots, x_{p_i}) = c$, where some of variables $x_{p_1}, \ldots, x_{p_i}$ are the same, or equalities of the form $\varphi = c$, where φ is some proper superposition in which operations f_i appear.

2.

Now we start to examine all possible representable sequences $\omega_0, \omega_1, \omega_2, \ldots$ such that $\omega_n \leqq 2$ for $n = 0, 1, 2, \ldots$ A sequence $a_0, a_1, a_2, \ldots$ is called *representable*, if there exists an algebra $\mathfrak{A}$ in which $\omega_k = a_k$ for every k. From Theorem of [1] it follows that:

Proposition 1. *Every sequence $\omega_0, \omega_1, \ldots, \omega_n, \ldots$, where $\omega_0, \omega_1 > 0$, is representable.*

Thus let us suppose from now that $\omega_0 = 0$. Further we shall use the following proposition:

Proposition 2. *Let $n \geqq 2$. The alternative subgroup A_n (i.e. the subgroup consisting of all even permutations) of a symmetric group S_n is the only subgroup in S_n with index 2.*

Lemma 1. *Let for* $\mathfrak{A}$ $\omega_0 = 0$ *and* $0 < \omega_{2k} \leqq 2$ *for some* $k > 1$. *Then there exists in* $\mathfrak{A}$ *a symmetric binary operation.*

Proof. Let $f(x_1, x_2, \ldots, x_{2k})$ be an essentially $2k$-ary operation. As $\omega_{2k} \leqq 2$, it must be either symmetric or it fulfils equalities $f(x_1, x_2, \ldots, x_{2k}) = f(x_{i_1}, x_{i_2}, \ldots \ldots, x_{i_{2k}})$, where $(i_1, i_2, \ldots, i_{2k})$ runs over all even permutations of the numbers $1, 2, \ldots, 2k$ (see Prop. 2). One of the equalities $f(x_1, x_2, \ldots, x_{2k}) = f(x_{2k}, x_{2k-1}, \ldots \ldots, x_2, x_1)$, $f(x_1, x_2, \ldots, x_{2k}) = f(x_{2k-1}, x_{2k}, x_{2k-2}, \ldots, x_2, x_1)$ must be fulfilled, because one of the permutations $(2k, 2k-1, \ldots, 2, 1)$ and $(2k-1, 2k, 2k-2, \ldots \ldots, 2, 1)$ is even. Now putting $x_1 = x_2 = \ldots = x_k = x$, $x_{k+1} = \ldots = x_{2k} = y$ we get a symmetric binary operation.

From Theorem 1 of [3] we have:

Proposition 3. *If* $\omega_0 = 0$ *and there exists in* $\mathfrak{A}$ *an essentially binary symmetric operation, then* $\omega_n > 0$ *for* $n \geqq 2$.

E. Marczewski calls (see [2]) a k-ary operation f *quasi-symmetrical*, if for each pair l, m integers such that $1 \leqq l \leqq m \leqq k$ there exists a permutation $p_1, p_2, \ldots, p_k$ of the numbers $1, 2, \ldots, k$ such that $p_l = m$, $p_m = l$ and $f(x_1, \ldots, x_k) = f(x_{p_1}, \ldots \ldots, x_{p_k})$. He proved

Proposition 4. *If there are no algebraic constants and* f *is a* k*-ary quasi-symmetrical operation in an algebra* $\mathfrak{A}$, *then every iteration of the form* $f(f(x_1, \ldots, x_k), y_2, \ldots, y_k), f(f(f(x_1, \ldots, x_k), y_2, \ldots, y_k), z_2, \ldots, z_k), \ldots$ *depends on each variable.*

J. Płonka called my attention to the fact, that Proposition 4 is valid for the functions $f(x_1, x_2, \ldots, x_n)$, for which the group of all permutations φ, such that

$$f(x_1, \ldots, x_n) = f(x_{\varphi(1)}, \ldots, x_{\varphi(n)}),$$

is transitive. One can prove this assertion only by formal alternations in the proof of Proposition 4.

Remark 1. *In our algebras with* $\omega_n = 2$ *there hold* $f(x_1, x_2, \ldots, x_k) = f(x_{i_1}, \ldots \ldots, x_{i_k})$, *where* $i_1, \ldots, i_k$ *runs over all even permutations of the numbers* $1, 2, \ldots, k$ *and* $f(x_2, x_1, x_3, \ldots, x_k) = f(x_{j_1}, \ldots, x_{j_k})$, *where* $j_1, \ldots, j_k$ *runs over all odd permutations of the numbers* $1, 2, \ldots, k$. *For* $k > 3$ *every operation* f *with this property is quasi-symmetrical.*

Lemma 2. *If* $\omega_0 = 0$, $\omega_n \leqq 2$ *for every* n, $\omega_{2n} = 0$ *for* $n > 1$ *and there exists an essentially* $(2k+1)$*-ary operation* $f(x_1, x_2, \ldots, x_{2k+1})$ *for* $k \geqq 1$ *in* $\mathfrak{A}$, *then there exists an essentially ternary operation in* $\mathfrak{A}$ *and it is* $\omega_{2n+1} > 0$ *for every* n.

Proof. In view of $0 < \omega_{2k+1} \leqq 2$ and Proposition 2 we have

$$f(x_1, x_2, \ldots, x_{2k+1}) = f(x_3, x_1, x_2, x_4, \ldots, x_{2k+1}). \tag{6}$$

First we prove that there exists an essentially ternary operation in $\mathfrak{A}$. If $k = 1$, the given operation f is essentially ternary. Suppose $k > 1$. Consider the operation

$$(7) \qquad f(x_1, x_2, x_3, y, \ldots, y)\,.$$

In view of (6) the operation (7) depends on each variable x_1, x_2, x_3 or none of them. If it depends, the operation (7) is essentially ternary, because of $\omega_4 = 0$. If (7) does not depend on x_1, x_2, x_3, then it must be $f(x_1, x_2, x_3, y, \ldots, y) = y$ or $g(y)$, where $g(y)$ is a non-trivial function. Consider the operation

$$(8) \qquad f(f(x_1, x_2, \ldots, x_{2k+1}), x_{2k+2}, \ldots, x_{4k+1})\,.$$

By Proposition 4 and Remark 1 it follows that (8) is essentially $(4k + 1)$-ary. Thus every even permutation of variables of (8) is admitted. So we can write one of the following equalities:

$$f(f(x_1, x_2, \ldots, x_{2k+1}), x_{2k+2}, \ldots, x_{4k+1}) =$$
$$= f(f(x_1, x_2, x_3, x_{2k+4}, \ldots, x_{4k+1}), x_{2k+2}, x_{2k+3}, x_4, \ldots, x_{2k+1})$$

or

$$f(f(x_1, x_2, \ldots, x_{2k+1}), x_{2k+2}, \ldots, x_{4k+1}) =$$
$$= f(f(x_1, x_2, x_3, x_{2k+4}, \ldots, x_{4k+1}), x_{2k+2}, x_{2k+3}, x_5, x_4, x_6, \ldots, x_{2k+1})\,.$$

Putting $x_4 = x_5 = \ldots = x_{2k+1} = y$ and $x_{2k+4} = x_{2k+5} = \ldots = x_{4k+1} = z$ we get $z = y$ or $g(z) = g(y)$, which contradicts $\omega_0 = 0$. Thus the operation (7) is essentially ternary. Let us denote it $f^*(x_1, x_2, x_3)$. By (6) it is cyclic, so, by Propos. 4 and Remark 1 $\omega_{2n+1} > 0$ for every $n > 0$.

Lemma 3. *If for an algebra $\mathfrak{A}$ one has $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 \leqq 2$ and $.$ is an essentially binary operation, then $.$ is diagonal, it means it fulfils equalities:*

$$x \,.\, x = x\,, \quad x \,.\, (y \,.\, z) = (x \,.\, y) \,.\, z = x \,.\, z\,.$$

Proof. Operation $.$ is not symmetric. Otherwise there would exist two symmetric essentially binary operations and at least eight essentially ternary operations (see (4), Lemma 4), which contradicts our assumption $\omega_3 \leqq 2$.

If one of the operations $(x \,.\, y) \,.\, z$ and $x \,.\, (y \,.\, z)$ is essentially ternary, say the first of them, then we have by the assumption $\omega_3 \leqq 2$ and Proposition 2

$$(9) \qquad (x \,.\, y) \,.\, z = (z \,.\, x) \,.\, y = (y \,.\, z) \,.\, x\,.$$

Thus $x \,.\, y = (x \,.\, y) \,.\, (x \,.\, y) = [(x \,.\, y) \,.\, x] \,.\, y = [(x \,.\, x) \,.\, y] \,.\, y = (x \,.\, y) \,.\, y = (y \,.\, y) \,.\, x = y \,.\, x$ against the first part of this proof. Hence none of the operations $(x \,.\, y) \,.\, z$, $x \,.\, (y \,.\, z)$ is essentially ternary and from the assumption of our Lemma and Theorem 2 from (5) it follows that $.$ is diagonal.

Lemma 4. *If $\omega_0 = 0$, $\omega_1 = 1$, $\omega_3 \leqq 2$, . is not symmetric, essentially binary operation and there exists an essentially ternary operation $f(x_1, x_2, x_3)$, then there exist at least* 20 *essentially 6-ary operations.*

Proof. As $\omega_3 \leqq 2$, by Proposition 2 $f(x_1, x_2, x_3)$ satisfies

$$f(x_1, x_2, x_3) = f(x_3, x_1, x_2) = f(x_2, x_3, x_1)\,. \tag{10}$$

Consider $g(x_1, x_2, \ldots, x_6) = f(x_1, x_2, x_3) \,.\, f(x_4, x_5, x_6)$. If g does not depend on x_1, then by (10) it does not depend on x_2, x_3. Putting $x_1 = x_2 = x_3 = x$, $x_4 = x_5 = x_6 = y$ and then $x_1 = x_2 = x_3 = z$, $x_4 = x_5 = x_6 = y$ we obtain $x \,.\, y = z \,.\, y$. Hence . is not essentially binary. Thus g must depend on x_1, x_2, x_3 and analogously it must depend on x_4, x_5, x_6. Thus g is an essentially 6-ary operation. If we had $f(x_1, x_2, x_3) \,.\, f(x_4, x_5, x_6) = f(x_4, x_2, x_3) \,.\, f(x_1, x_5, x_6)$, then it would be $g(x_1, x_2, \ldots, x_6) = f(x_3, x_4, x_2) \,.\, f(x_6, x_1, x_5) = f(x_6, x_4, x_2) \,.\, f(x_3, x_1, x_5) = f(x_2, x_6, x_4) \,.\, f(x_5, x_3, x_1) = f(x_5, x_6, x_4) \,.\, f(x_2, x_3, x_1) = f(x_4, x_5, x_6) \,.\, f(x_1, x_2, x_3)$.

Putting $x_1 = x_2 = x_3 = x$, $x_4 = x_5 = x_6 = y$ we get $x \,.\, y = y \,.\, x$ against the assumption about .. So we have at least $\binom{6}{3} = 20$ essentially 6-ary operations $f(x_{i_1}, x_{i_2}, x_{i_3}) \,.\, f(x_{i_4}, x_{i_5}, x_{i_6})$, where all i_k are different and for different choices of numbers i_1, i_2, i_3 we get different operations.

Theorem 1. *If $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 2$ and $\omega_n \leqq 2$ for $n > 2$, then $\omega_n = 0$ for $n > 2$ and the sequence $0, 1, 2, 0, \ldots$ is representable by a non-trivial diagonal semigroup and reversely every algebra representing this sequence is non-trivial diagonal semigroup.*

Proof. By Lemma 3 there exists a non-trivial diagonal operation $x \,.\, y$ and it is clearly not symmetric. By Lemma 1 we get $\omega_{2n} = 0$ for $n > 1$. Lemma 4 implies $\omega_3 = 0$ and Lemma 2 implies $\omega_{2n+1} = 0$ for $n \geqq 1$.

Corollary 1. *If $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 0$, $\omega_n \leqq 2$ for $n > 2$, then the only possible sequences are the following: $0, 1, 0, \ldots$ and $0, 1, 0, 1, 0, 1, \ldots$, where a trivial algebra gives a realisation of the sequence $0, 1, 0, 0, \ldots$ and an at least two-element Boolean group with the operation $x_1 + x_2 + x_3$ taken as fundamental gives the unique realisation of the other sequence.*

Proof. From Lemma 1 it follows that $\omega_{2n} = 0$. So the corollary follows form Theorem 1 and 2 of Urbanik (see [7]).

Lemma 5. *Let $\omega_0 = 0$, $\omega_1 = 1$ and let there exist in $\mathfrak{A}$ a symmetric and associative binary operation .. Suppose that we have a sequence of operations $f^{2^n}(x_1, x_2, \ldots, x_{2^n})$, such that every $f^{2^n}(x_1, x_2, \ldots, x_{2^n})$ is symmetric, $f^{2^n}(f^{2^{n-1}}(x_1^1, x_2^1, \ldots, x_{2^{n-1}}^1), f^{2^{n-1}}(x_1^2, x_2^2, \ldots, x_{2^{n-1}}^2), \ldots, f^{2^{n-1}}(x_1^{2^n}, x_2^{2^n}, \ldots, x_{2^{n-1}}^{2^n}))$ is symmetric and $f^{2^n}(\underbrace{x, x, \ldots, x}_{2^{n-1}-times}, \underbrace{y, y, \ldots, y}_{2^{n-1}-times}) = x \,.\, y$. Then $f^{2^n}(x_1, x_2, \ldots, x_{2^n}) = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n}$.*

Proof. For $n = 1$ the lemma is obvious. Suppose that it is true for some n. Then we have

$$f^{2^{n+1}}(x_1, x_2, \ldots, x_{2^{n+1}}) = f^{2^{n+1}}(f^{2^n}(x_1, x_1, \ldots, x_1), f^{2^n}(x_2, x_2, \ldots, x_2), \ldots$$
$$\ldots, f^{2^n}(x_{2^{n+1}}, x_{2^{n+1}}, \ldots, x_{2^{n+1}})) = f^{2^{n+1}}(f^{2^n}(x_1, x_2, \ldots, x_{2^n}),$$
$$f^{2^n}(x_1, x_2, \ldots, x_{2^n}), \ldots, f^{2^n}(x_{2^n+1}, x_{2^n+2}, \ldots, x_{2^{n+1}}), \ldots) =$$
$$= f^{2^{n+1}}(x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n},\ x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n}, \ldots, x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n},$$
$$x_{2^n+1} \,.\, x_{2^n+2} \,.\, \ldots \,.\, x_{2^{n+1}}, \ldots, x_{2^n+1} \,.\, x_{2^n+2} \,.\, \ldots \,.\, x_{2^{n+1}}) =$$
$$= x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n} \,.\, x_{2^n+1} \,.\, x_{2^n+2} \,.\, \ldots \,.\, x_{2^{n+1}} \,.$$

Theorem 2. *If $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 1$, $\omega_n \leqq 2$ for $n > 2$, then $\omega_n = 1$ for $n > 2$ and an at least two-element semilattice gives the unique realisation of the sequence* 0, 1, 1, 1, ... *between groupoids.*

Proof. Suppositions $\omega_2 = 1$ and $\omega_n \leqq 2$ imply that all operations are symmetric. Namely $x \,.\, y$ is symmetric, $x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k$ is then an essentially k-ary symmetric operation. So if there exists further essentially k-ary algebraic operation $f(x_1, x_2, \ldots \ldots, x_k)$ it must be symmetric, too. Further the operation . is associative because of $\omega_3 \leqq 2$. Namely by Proposition 2, it fulfils (9). Thus we have $(x \,.\, y) \,.\, z = (y \,.\, z) \,.\, x = x \,.\, (y \,.\, z)$, hence . is associative.

Now let for every n f^{2^n} be an arbitrary essentially 2^n-ary operation of $\mathfrak{A}$. Then the sequence $\{f^{2^n}\}$ satisfies the assumptions of Lemma 5 and hence $f^{2^n}(x_1, x_2, \ldots, x_{2^n}) = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n}$. Thus $\omega_{2^n} = 1$. Consider the operation $f(x_1, x_2, \ldots, x_k) \,.\, x_{k+1} \,.\, \ldots \,.\, x_{2^n}$. This operation si essentially 2^n-ary what can be checked by suitable identification. So it must be $f(x_1, x_2, \ldots, x_k) \,.\, x_{k+1} \,.\, \ldots \,.\, x_{2^n} = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_{2^n}$. Put $x_{k+1} = x_{k+2} = \ldots = x_{2^n} = y$, we have $f(x_1, x_2, \ldots, x_k) \,.\, y = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k \,.\, y$. Thus $f(x_1, x_2, \ldots, x_k) = f(x_1, x_2, \ldots, x_k) \,.\, f(x_1, x_2, \ldots, x_k) = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k \,.\, f(x_1, x_2, \ldots, x_k) = f(x_1, x_2, \ldots, x_k) \,.\, x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k \,.\, x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k = x_1 \,.\, x_2 \,.\, \ldots \,.\, x_k$. Hence $\omega_n = 1$ for every $n > 2$.

Theorem 3. *If $\omega_n \leqq 2$ for $n = 0, 1, 2, \ldots$, then sequences having algebraic realisation are exactly one the following forms:*

1) $\omega_0 > 0$, $\omega_1 > 0$, ω_n *arbitrary for* $n > 1$;
2) $\omega_0 = 0$, $\omega_1 = 1$, $\omega_n = 0$ *for* $n > 1$;
3) $\omega_0 = 0$, $\omega_n = 1$ *for* $n \geqq 1$;
4) $\omega_{2n} = 0$, $\omega_{2n+1} = 1$ *for* $n \geqq 0$;
5) $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_n = 0$ *for* $n > 2$;
6) $\omega_0 = 0$, $\omega_1 = 2$, $\omega_n \geqq 1$ *for* $n > 1$;
7) $\omega_0 = 0$, $\omega_1 = 2$, $\omega_n = 0$ *for* $n > 1$;
8) $\omega_0 = 0$, $\omega_1 = 2$, $\omega_2 = 2$, $\omega_n = 0$ *for* $n > 2$;

9) $\omega_0 = 0$, $\omega_1 = 2$, $\omega_{2n} = 0$ *for* $n \geqq 1$ *and* $\omega_{2n+1} > 0$ *for* $n \geqq 1$;
10) $\omega_0 = 0$, $\omega_1 = 2$, $\omega_2 = 2$, $\omega_{2n} = 0$ *for* $n > 1$ *and* $\omega_{2n+1} > 0$ *for* $n \geqq 1$.

Proof. First we prove that sequences 1) – 10) are the only possible. Let us recall that it is $\omega_1 > 0$ in any algebra because of existence of trivial unary operation. If $\omega_0 = 0$, $\omega_1 = 1$, then sequences 2), 3), 4), 5) are the only possible, which follows from Theorem 1, Corollary 1 and Theorem 2. If $\omega_0 = 0$, $\omega_1 = 2$ then sequences 6) – 10) are the only possible, which follows from Lemma 1, Proposition 3 and Lemma 2 and from the observation, that if $\omega_2 = 1$, then by Proposition 3 $\omega_m \geqq 1$ for $n \geqq 2$.

Sequence 1) has a realisation by Proposition 1, sequences 2), 3), 4), 5) by Theorem 1, Corollary 1 and Theorem 2. Sequence 6) and sequences 7), 8), 9), 10) have realisations by Theorem of (1).

For several sequences we have got representations in such sense that we can show equational classes of algebras realizing given sequences. It is illustrated in the following table:

sequence	representation
1, 1, 1, 1, ...	if $x . x = x$, semillatice with 0 or 1
	if $x . x = c$, $c . x = x$, at least two-element Boolean group
0, 1, 0, 0, ...	trivial algebra
0, 1, 1, 1, ...	at least two-element semilattice
0, 1, 0, 1, 0, 1, 0, ...	idempotent reduct of at least two-element Boolean group
0, 1, 2, 0, 0, ...	diagonal semigroup
0, 2, 0, 0, ...	algebra $(X, f(x))$, where $f(f(x)) = f(x)$ or x.

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