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PERMUTABLE TOLERANCES

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1. In the papers [1] and [2] the so-called compatible tolerances on algebras are introduced. In the papers [2] and [3], existence conditions for compatible tolerances which are not congruences are investigated. In the paper [4] it is proved that the set of all compatible tolerances on a given algebra forms a lattice, some of whose properties are the same as those of the lattice of all congruences on this algebra or are analogous to them.

In [6] the importance of the permutability of congruences at investigating the lattice of all congruences of a given algebra is shown. For example, if all congruences on an algebra $\mathfrak{A}$ are permutable, then this lattice is modular, and if $\mathfrak{A}$ has a one-element sublagebra, then a generalization of Schreier's theorem on refinements (see Theorem 88 in [6]) holds for the congruences. Thus it is a natural problem to study which analoga hold between permutable congruences and permutable compatible tolerances on a given algebra.

2. By the symbol $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ we denote an algebra $\mathfrak{A}$ with the support A and with the set $\mathscr{F}$ of fundamental operations. If $\mathfrak{A}$ is a lattice, then we shall not distinguish an algebra and its support, i.e. for a lattice L, the symbol L denotes also the support of this lattice.

Definition 1. Let A be a set. Each reflexive and symmetric binary relation on A is called a *tolerance* on A. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra and let T be a tolerance on A. The tolerance T is called *compatible with* $\mathfrak{A}$, if each n-ary operation $f \in \mathscr{F}$ and arbitrary $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of A for which $a_i T b_i$ for $i = 1, \ldots, n$ satisfy $f(a_1, \ldots, a_n)\, T f(b_1, \ldots, b_n)$.

In the paper [4] it is proved that the set of all compatible tolerances on an algebra $\mathfrak{A}$ forms a complete lattice with respect to the set inclusion. By $LT(\mathfrak{A})$ we denote the lattice of all compatible tolerances on the algebra $\mathfrak{A}$.

The lattice operations in $LT(\mathfrak{A})$ will be denoted by the symbols $\vee$ (join), $\wedge$ (meet). Further, $K(\mathfrak{A})$ denotes the lattice of all congruences on the algebra $\mathfrak{A}$ and the symbol $\uplus$ denotes the join in the lattice $K(\mathfrak{A})$. By the symbol $\cup$ we shall denote the set union of two tolerances (taken as subsets of the Cartesian power of the corresponding set).

From the definition it is evident that each congruence on an algebra $\mathfrak{A}$ is a tolerance compatible with $\mathfrak{A}$.

Definition 2. Let A be a set, let R_1, R_2 be two binary relations on A. The relations R_1, R_2 are called *permutable*, if $R_1 . R_2 = R_2 . R_1$, where the symbol "·" denotes the product of relations.

3. In [6], supplement of Theorem 86, it is proved that if C_1, C_2 are permutable congruences on an algebra $\mathfrak{A}$, then $C_1 . C_2 = C_1 \uplus C_2$. We shall study the interrelation between $T_1 . T_2$ and $T_1 \vee T_2$ for compatible tolerances T_1, T_2 on $\mathfrak{A}$.

Lemma. *Let A be a set, let T_1, T_2 be two tolerances on A. Then $T_1 . T_2$ is a tolerance on A if and only if $T_1 . T_2 = T_2 . T_1$, i.e. if T_1 and T_2 are permutable.*

Proof is straightforward.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let T_1, T_2 be two permutable tolerances from $LT(\mathfrak{A})$. Then $T_1 . T_2 \in LT(\mathfrak{A})$ and $T_1 \cup T_2 \subseteq T_1 \vee T_2 \subseteq T_1 . T_2 \subseteq \subseteq (T_1 \cup T_2)^2 \subseteq (T_1 \vee T_2)^2$.*

Proof. By Theorem 3 from [5] we have $T_1 . T_2 \in LT(\mathfrak{A})$. Since $U \subseteq V$ and $R \subseteq S$ implies $R . U \subseteq S . V$ for any binary relation on A, the proof of the assertion concerning the inclusions is immediate.

Corollary. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $T_1 \in LT(\mathfrak{A})$, $T_2 \in LT(\mathfrak{A})$. Let T_1, T_2 be permutable, let $T_1 \vee T_2$ be a congruence. Then $T_1 . T_2 = T_1 \vee T_2$.*

Proof. As $T_1 \vee T_2$ is a congruence, we have $(T_1 \vee T_2)^2 = T_1 \vee T_2$. Thus $T_1 \vee T_2 \subseteq T_1 . T_2 \subseteq T_1 \vee T_2$, which means $T_1 \vee T_2 = T_1 . T_2$.

4. In the paper [7] it is proved that for every distributive lattice L the lattice $K(L)$ is a sublattice of $LT(L)$, if and only if $K(L) = LT(L)$, i.e. if each compatible tolerance on L is a congruence (see Corollary 2 in [7]). Now we shall show that this condition follows from the condition of equality of product and join for congruences from $LT(L)$.

Theorem 2. *Let L be a distributive lattice. For any two congruences C_1, C_2 on L let $C_1 . C_2 = C_1 \vee C_2$. Then each compatible tolerance on L is a congruence.*

Proof. Let the condition be fulfilled. Then any two congruences on L are permutable, because $C_1 . C_2 = C_1 \vee C_2 = C_2 \vee C_1 = C_2 . C_1$ for any two congruences C_1, C_2 on L. The product $C_1 \,.\, C_2$ is a congruence on L (see for example [8]). But $C_1 \,.\, C_2 = C_1 \vee C_2$ is the least compatible tolerance on L which contains C_1 and C_2. As it is a congruence, we have $C_1 \vee C_2 = C_1 \uplus C_2$. The meet in both $K(L)$ and $LT(L)$ is the set intersection, thus $K(L)$ is a sublattice of $LT(L)$. By Corollary 2 from [7] this assertion is equivalent to the assertion of the theorem.

Remark. The assertion can be proved also by using the result from [9] saying that if all congruences on an algebra $\mathfrak{A}$ from a given variety are permutable, then each compatible reflexive relation on $\mathfrak{A}$ is a congruence.

5. Now we shall prove some theorems on permutable compatible tolerances and their transitive hulls. In [4] it is proved that the transitive hull of a tolerance compatible with an algebra $\mathfrak{A}$ is a congruence on $\mathfrak{A}$.

Theorem 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let T_1, T_2 be two permutable tolerances from $LT(\mathfrak{A})$. Let C_1, C_2 be the transitive hulls of T_1, T_2 respectively. Then C_1 and C_2 are permutable and $C_1 \,.\, C_2$ is the transitive hull of $T_1 \,.\, T_2$.*

Proof. We have $C_1 = \bigcup_{n=1}^{\infty} T_1^n$, $C_2 = \bigcup_{n=1}^{\infty} T_2^n$. Let $a \; C_1 \,.\, C_2 \; b$ for some $a \in A$, $b \in A$. This means that there exists $c \in A$ such that $a \; C_1 \; c$, $c \; C_2 \; b$. Now there exist positive integers m, n such that $a \; T_1^m \; c$, $c \; T^n \; b$. Thus $a \; T_1^m \,.\, T_2^n \; b$. As T_1, T_2 are permutable, so are T_1^m, T_2^n and we have $a T_2^n \,.\, T_1^m b$. There exists $d \in A$ such that $a \; T_2^n \; d$, $d \; T_1^m \; b$. But $T_2^n \subseteq C_2$, $T_1^m \subseteq C_1$ and we have $a \; C_2 \; d$, $d \; C_1 \; b$, which means $a \; C_2 \,.\, C_1 \; b$. As a, b were chosen arbitrarily, we have $C_1 \,.\, C_2 \subseteq C_2 \,.\, C_1$. Analogously we can prove the inverse inclusion and thus $C_1 \,.\, C_2 = C_2 \,.\, C_1$. Now as $T_1 \subseteq C_1$, $T_2 \subseteq C_2$, we have $T_1 \,.\, T_2 \subseteq C_1 \,.\, C_2$, thus also the transitive hull of $T_1 \,.\, T_2$ is contained in $C_1 \,.\, C_2$. Let $x \; C_1 \,.\, C_2 \; y$ for $x \in A$, $y \in A$. Then there exists $z \in A$ such that $x \; C_1 \; z$, $z \; C_2 \; y$. This means that there exist positive integers r, s such that $x \; T_1^r \; z$, $z \; T_2^s \; y$. Let $t = \max(r, s)$. As T_1, T_2 are reflexive, the inequalities $r \leqq t$, $s \leqq t$ imply the inclusions $T_1^r \subseteq T_1^t$, $T_2^s \subseteq T_2^t$. We have $x \; T_1^t \; z$, $z \; T_2^t \; y$, which means $x T_1^t \,.\, T_2^t y$. As T_1, T_2 are permutable, we have $T_1^t \,.\, T_2^t = (T_1 \,.\, T_2)^t$. Thus $C_1 \,.\, C_2 \subseteq \bigcup_{n=1}^{\infty} (T_1 \,.\, T_2)^n$, but the right-hand side of this inclusion is the transitive hull of $T_1 \,.\, T_2$. We have proved that this transitive hull is equal to $C_1 \,.\, C_2$.

Theorem 4. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $T \in LT(\mathfrak{A})$. Let C be the transitive hull of T. Then $C \,.\, T = T \,.\, C = C$.*

Proof. Let I denote the identity relation on A. Then $I \subseteq T \subseteq C$ and, by the remark above, we have

$$T = I \,.\, T \subseteq C \,.\, T \subseteq C \,.\, C = C = C \,.\, I \subseteq C \,.\, T .$$

Hence $C = C \,.\, T$ and, similarly, $C = T \,.\, C$.

Corollary 2. *Let* $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ *be an algebra. Let* $\mathfrak{S}$ *be the maximal (with respect to set inclusion) set of compatible tolerances on* $\mathfrak{A}$ *such that any two tolerances from this set are permutable. Then* $\mathfrak{S}$ *is a commutative semigroup with the property that each monogenous subsemigroup of* $\mathfrak{S}$ *either is infinite, or has the period one. The unit element of* $\mathfrak{S}$ *is the identity relation on A, the zero element of* $\mathfrak{S}$ *is the universal relation on A.*

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