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NOTE ON VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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This note is a supplement to the paper [2] which is devoted to the Volterra-Stieltjes integral equation in the space $BV_n[0, 1]$ of n-vector functions of bounded variation on the interval $[0, 1]$.

Assume that $\mathbf{K}(t, s)$ is an $n \times n$-matrix valued function defined on the square $[0, 1] \times [0, 1] = J$ such that

$$v(\mathbf{K}) < \infty \tag{1}$$

and

$$\operatorname{var}_0^1 \mathbf{K}(0, \cdot) < \infty \tag{2}$$

where $v(\mathbf{K})$ denotes the twodimensional Vitali variation of $\mathbf{K}$ on the square J and $\operatorname{var}_0^1 \mathbf{K}(0, \cdot)$ is the variation of $\mathbf{K}(0, s)$ in the second variable on the interval $[0, 1]$. The notions of variation are defined in the usual way by the norm in the space $L(R_n)$ of all $n \times n$-matrices which is the operator norm for linear operators on R_n (see [1], [2], [3]).

In [2], Theorem 3.1 asserts the following:

If $\mathbf{K} : J \to L(R_n)$ satisfies (1), (2) and for any $t \in (0, 1]$ the inverse matrix $[I - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))]^{-1}$ exists then the homogeneous Volterra-Stieltjes integral equation

$$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{0} \tag{3}$$

possesses only the trivial solution $\mathbf{x} = \mathbf{0}$ in $BV_n[0, 1]$.

This states that the condition

$$\mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-)) \text{ is a regular matrix for all } t \in (0, 1] \tag{4}$$

is sufficient for the equation (3) to have only the trivial solution $\mathbf{x} = \mathbf{0} \in BV_n$. Our aim is to prove that (4) is also a necessary condition for the equation (3) to have this property.

Note that the limit $\lim_{\tau \to t-} \mathbf{K}(t, \tau) = \mathbf{K}(t, t-)$ exists since (1) and (2) hold (see [1]).

1. Theorem. *If $K : J \to L(R_n)$ satisfies (1) and (2) then the homogeneous Volterra-Stieltjes integral equation (3) has only the trivial solution $x = 0$ in BV_n if and only if the condition (4) is satisfied.*

Proof. The sufficiency of (4) is stated in the above quoted theorem from [2]. It remains to prove the necessity. We show in the sequel that if (4) is not satisfied then (3) has a nonzero solution in the space BV_n.

It was shown in [2] that for the operator

$$x \in BV_n \to \int_0^t d_s[K(t, s)]\, x(s) \in BV_n$$

we have

$$\int_0^t d_s[K(t, s)]\, x(s) = \int_0^1 d_s[K^\Delta(t, s)]\, x(s) \tag{5}$$

where

$$K^\Delta(t, s) = K(t, s) - K(t, 0) \quad \text{if} \quad 0 \leqq s \leqq t \leqq 1\,, \tag{6}$$

$$K^\Delta(t, s) = K(t, t) - K(t, 0) = K^\Delta(t, t) \quad \text{if} \quad 0 \leqq t < s \leqq 1\,.$$

For the new "triangular" kernel K^Δ we have $\text{var}_0^1\, K^\Delta(0, \cdot) < \infty$, $v(K^\Delta) < \infty$, $K^\Delta(t, 0) = 0$ for $t \in [0, 1]$ if (1) and (2) is satisfied for the kernel K. Hence the equation (3) can be written in the Fredholm-Stieltjes form

$$x(t) - \int_0^1 d_s[K^\Delta(t, s)]\, x(s) = 0\,.$$

Since (1) and (2) hold we have $\text{var}_0^1\, H < \infty$ for the matrix valued function $H: [0, 1] \to L(R_n)$ defined by the relations

$$H(t) = K(t, t) - K(t, t-) \quad \text{for} \quad t \in (0, 1]\,, \quad H(0) = 0$$

and there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \in (0, 1]$ such that $H(t) = 0$ for $t \in [0, 1]$, $t \neq t_i$, $i = 1, 2, \ldots$ (see Lemma 3.1 in [2]). Hence $\sum_{i=1}^{\infty} \|H(t_i)\| < \infty$ because $\text{var}_0^1\, H = 2 \sum_{t_i \in (0,1)} \|H(t_i)\| + \|H(1)\|$. This implies that $\|H(t)\| < \frac{1}{2}$ for $t \in [0, 1]$ except for a finite set of points in (0, 1). Hence the matrix $I - H(t)$ can be singular only at a finite set of points T_i, $i = 1, \ldots, k$, $0 < T_1 < T_2 < \ldots < T_k \leqq 1$.

Let us assume that the condition (4) is not satisfied. Then by the facts shown above there is a point $T_1 \in (0, 1]$ such that $I - H(t) = I - (K(t, t) - K(t, t-))$ is a regular matrix for $t \in [0, T_1)$ but $I - H(T_1) = I - (K(T_1, T_1) - K(T_1, T_1-))$ is not regular. Hence there exists $z \in R_n$ such that the linear algebraic equation

$$[I - (K(T_1, T_1) - K(T_1, T_1-))]\, x = z \tag{7}$$

has no solution in R_n. If we define the function $\boldsymbol{y}^{\wedge} : [0, 1] \to R_n$ by the relations $\boldsymbol{y}^{\wedge}(t) = \boldsymbol{0}$ for $t \in [0, 1]$, $t \neq T_1$ and $\boldsymbol{y}^{\wedge}(T_1) = \boldsymbol{z}$ then $\boldsymbol{y}^{\wedge} \in BV_n$. Let us now consider the Volterra-Stieltjes integral equation

$$\boldsymbol{x}(t) - \int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{y}^{\wedge}(t)\,. \tag{8}$$

Since $\boldsymbol{I} - (\boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-))$ is regular for $t \in [0, T_1)$, every solution $\boldsymbol{x}$ of (8) vanishes on the interval $[0, T_1)$ by the first part of the theorem and for $t = T_1$ we have

$$\boldsymbol{x}(T_1) - \int_0^{T_1} \mathrm{d}_s[\boldsymbol{K}(T_1, s)]\, \boldsymbol{x}(s) = \boldsymbol{z}\,.$$

Using the relation

$$\int_0^{T_1} \mathrm{d}_s[\boldsymbol{K}(T_1, s)]\, \boldsymbol{x}(s) = (\boldsymbol{K}(T_1, T_1) - \boldsymbol{K}(T_1, T_1-))\, \boldsymbol{x}(T_1)$$

(see [1]) we get

$$\boldsymbol{x}(T_1) - (\boldsymbol{K}(T_1, T_1) - \boldsymbol{K}(T_1, T_1-))\, \boldsymbol{x}(T_1) = \boldsymbol{z}$$

but the value $\boldsymbol{x}(T_1)$ cannot be determined since the linear algebraic equation (7) has no solution. Hence there is no $\boldsymbol{x} \in BV_n[0, 1]$ satisfying the equation (8), i.e. the range of the operator

$$\boldsymbol{x} \in BV_n \to \boldsymbol{x}(t) - \int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) \in BV_n$$

is a proper subspace in $BV_n[0, 1]$.

Since the Volterra-Stieltjes integral equation is a special case of the Fredholm-Stieltjes integral equation we obtain by the Fredholm Theorem (see Theorem 6 in [3]) that there exists in BV_n a nonzero solution of the homogeneous equation (3) and our theorem is completely proved.

2. Corollary. *Let $\boldsymbol{K} : J \to L(R_n)$ satisfy* (1) *and* (2). *Then the nonhomogeneous Volterra-Stieltjes integral equation*

$$\boldsymbol{x}(t) - \int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{y}(t) \tag{9}$$

has a unique solution $\boldsymbol{x} \in BV_n[0, 1]$ for any $\boldsymbol{y} \in BV_n[0, 1]$ if and only if the condition (4) *is satisfied.*

Proof. Since (5) holds the equation (9) can be written in the Fredholm-Stieltjes form

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}^{\Delta}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{y}(t)$$

where $\boldsymbol{K}^{\Delta} : J \to L(R_n)$ is given by (6). By Theorem 1 the corresponding homogeneous equation has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ in BV_n and consequently by the Fredholm Theorem (see Theorem 6. in [3]) we obtain the statement of the corollary.

3. Theorem. *Let $\boldsymbol{K} : J \to L(R_n)$ satisfy (1) and (2). If the condition (4) is satisfied then for every $\boldsymbol{y} \in BV_n[0, 1]$ the unique solution of the equation (9) is given by the formula*

$$\boldsymbol{x}(t) = \boldsymbol{y}(t) + \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \boldsymbol{y}(s) \qquad t \in [0, 1] \tag{10}$$

where $\boldsymbol{\Gamma}(t, s)$, $0 \leqq s \leqq t \leqq 1$ is a uniquely determined $n \times n$ – matrix valued function such that

$$\boldsymbol{\Gamma}(t, s) = \boldsymbol{K}(t, s) - \boldsymbol{K}(t, 0) + \int_0^t \mathrm{d}_r[\boldsymbol{K}(t, r)]\, \boldsymbol{\Gamma}(r, s) \tag{11}$$

if $0 \leqq s \leqq t \leqq 1$. If we define $\boldsymbol{\Gamma}(t, s) = \boldsymbol{\Gamma}(t, t)$ for $0 \leqq t < s \leqq 1$ then $v(\boldsymbol{\Gamma}) < \infty$ and $\operatorname{var}_0^1 \boldsymbol{\Gamma}(t, \cdot) < \infty$ for every $t \in [0, 1]$.

Proof. Since the equation (9) can be rewritten in the form of a Fredholm-Stieltjes integral equation

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}^{\Delta}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{y}(t)$$

we obtain by Theorem 8. from [3] that the unique solution of this equation can be given by the formula

$$\boldsymbol{x}(t) = \boldsymbol{y}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \boldsymbol{y}(s) \tag{12}$$

where $\boldsymbol{\Gamma} : J \to L(R_n)$ satisfies the equality

$$\boldsymbol{\Gamma}(t, s) = \boldsymbol{K}^{\Delta}(t, s) - \boldsymbol{K}^{\Delta}(t, 0) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}^{\Delta}(t, r)]\, \boldsymbol{\Gamma}(r, s)$$

for all $t, s \in [0, 1]$, $\operatorname{var}_0^1 \boldsymbol{\Gamma}(0, \cdot) < \infty$, $\boldsymbol{\Gamma}(t, 0) = 0$ for all $t \in [0, 1]$, and $v(\boldsymbol{\Gamma}) < \infty$. Using the definition (6) of the "triangular" kernel $\boldsymbol{K}^{\Delta}$ and the relation (5) we obtain

$$\int_0^1 \mathrm{d}_r[\boldsymbol{K}^{\Delta}(t, r)]\, \boldsymbol{\Gamma}(r, s) = \int_0^t \mathrm{d}_r[\boldsymbol{K}(t, r)]\, \boldsymbol{\Gamma}(r, s)$$

and this yields the relation (11) for $0 \leqq s \leqq t \leqq 1$. Further, evidently $\boldsymbol{\Gamma}(t, s) = \boldsymbol{\Gamma}(t, t)$ for $0 \leqq t < s \leqq 1$ and also

$$\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \boldsymbol{y}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \boldsymbol{y}(s)$$

for every $\boldsymbol{y} \in BV_n$. Hence by (12) we obtain the representation (10) for the solution of the equation (7). Let us finally mention that by Theorem 8. in [3] the matrix valued function $\boldsymbol{\Gamma}(t, s)$ is uniquely determined on the square J.

References

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