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Časopis pro pěstování matematiky, Vol. 102 (1977), No. 3, 280--291

Persistent URL: `http://dml.cz/dmlcz/108456`

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IDEALS OF BINARY RELATIONAL SYSTEMS

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(Received May 20, 1976)

The concept of an ideal of a partially ordered set was introduced for the purpose of investigating systems with a partial ordering. This concept is a generalization of the lattice ideal (see [1], [7]). However, in [6] another definition of an ideal of a partially ordered set is given which is more general than the classical one and makes it possible to obtain deeper results for some partially ordered systems, especially for l-groups. The aim of this paper is to generalize this definition to the case of general binary relation and to show its applicability to some problems in binary relational systems.

1. ELEMENTARY PROPERTIES OF ϱ-IDEALS

Let ϱ be a binary relation on a set A. The pair $\langle A, \varrho \rangle$ is called a *binary relational system*. We introduce $U(a, b) = \{x \in A;\ a \varrho x,\ b \varrho x\}$ and $L(a, b) = \{x \in A;\ x \varrho a,\ x \varrho b\}$ for arbitrary $a, b \in A$. The system $\langle A, \varrho \rangle$ is said to be *ϱu-directed* (*ϱl-directed*) if $U(a, b) \neq \emptyset$($L(a, b) \neq \emptyset$, respectively) for each $a, b \in A$. If $\langle A, \varrho \rangle$ is both ϱu-directed and ϱl-directed, it will be called *ϱ-directed*. The set B is called a *ϱu-directed subset* of A if $\langle A, \varrho \rangle$ is a binary relational system, $B \subseteq A$ and $U(a, b) \cap B \neq \emptyset$ for each $a, b \in B$. Analogously we introduce *ϱl-directed* and *ϱ-directed subsets*.

Definition 1. Let $\langle A, \varrho \rangle$ be a binary relational system and I a non-void subset of A. If the conditions

(I_1) $a \in A,\ i \in I,\ a \varrho i$ imply $a \in I$,

(I_2) $i, j \in I$ implies $U(i, j) \cap I \neq \emptyset$

are satisfied, then I is called a *ϱ-ideal of* $\langle A, \varrho \rangle$.

An arbitrary subset I of A fulfilling the condition (I_1) is called a *semi ϱ-ideal of* $\langle A, \varrho \rangle$.

A non-void subset D of A is called a *dual ϱ-ideal of* $\langle A, \varrho \rangle$ if the following conditions (dual to (I_1), (I_2)) are satisfied:

(D_1) $b \in A,\ d \in D,\ d \varrho b$ imply $b \in D$,

(D_2) $d, g \in D$ implies $L(d, g) \cap D \neq \emptyset$.

The set of all ϱ-ideals of $\langle A, \varrho \rangle$ will be denoted by $\mathscr{I}(A)$. It is clear that $\langle \mathscr{I}(A), \subseteq \rangle$ is a partially ordered set.

Definition 2. A ϱ-ideal I of $\langle A, \varrho \rangle$ is called *maximal*, if the conditions $I \subseteq J$, $I \neq J$ are fulfilled by no ϱ-ideal J of $\langle A, \varrho \rangle$. A ϱ-ideal I of $\langle A, \varrho \rangle$ is called *prime*, if

$$\text{(P)} \quad a, b \in A\,, \quad \emptyset \neq L(a, b) \subseteq I \quad \text{imply} \quad a \in I \quad \text{or} \quad b \in I\,.$$

Dually we obtain the concept of a *dual prime ϱ-ideal.*

An arbitrary subset C of A is called a *ϱ-convex subset of* $\langle A, \varrho \rangle$, if $a, b \in C$, $x \in A$, $a \varrho x$, $x \varrho b$ imply $x \in C$.

Notation. Let ϱ be a binary relation on the set A. The transitive hull of ϱ is denoted by the symbol $t(\varrho)$; i.e. for $a, b \in A$ we have $a\, t(\varrho)\, b$ if and only if there exist $a_0, \ldots, a_n \in A$ with $a_0 = a$, $a_n = b$, $a_{i-1}\, \varrho\, a_i$ for $i = 1, \ldots, n$.

Example 1. If ϱ is a partial ordering on A, Definition 1 introduces the concept of an o-ideal from [6]. Moreover, if $\langle A, \varrho \rangle$ is a lattice, the concept of a ϱ-ideal coincides with that of a lattice ideal. If ϱ is an equivalence relation on A, then $\mathscr{I}(A) = A/\varrho$.

Proposition 1. *Let ϱ be a binary relation on a set A. Then*

(a) *Each ϱ-ideal of $\langle A, \varrho \rangle$ is a ϱ-convex and ϱu-directed subset of A.*

(b) *If $\langle A, \varrho \rangle$ is ϱl-directed, then each ϱ-ideal of $\langle A, \varrho \rangle$ is a ϱ-directed subset of A.*

(c) *$\langle A, \varrho \rangle$ is ϱu-directed if and only if $A \in \mathscr{I}(A)$.*

Proof. Let I be a ϱ-ideal of $\langle A, \varrho \rangle$. By (I_1), I is ϱ-convex and, by (I_2), I is ϱu-directed. If $\langle A, \varrho \rangle$ is ϱl-directed, then $L(a, b) \neq \emptyset$ for each $a, b \in I$. Let $t \in L(a, b)$. Then $t\, \varrho\, a$, hence by (I_1) it is $t \in I$. Thus $\emptyset \neq L(a, b) \subseteq I$, i.e. I is also ϱl-directed; (a) and (b) are proved. If A is a ϱ-ideal of $\langle A, \varrho \rangle$, then $\emptyset \neq U(a, b) \cap A = U(a, b)$ for ach $a, b \in A$, thus $\langle A, \varrho \rangle$ is ϱu-directed. Conversely, if $\langle A, \varrho \rangle$ is ϱu-directed, then $\emptyset \neq U(a, b) = U(a, b) \cap A$. As (I_1) is satisfied automatically, we obtain $A \in \mathscr{I}(A)$.

Proposition 2. *Let $\{I_\gamma; \gamma \in \Gamma\}$ be a chain of ϱ-ideals of $\langle A, \varrho \rangle$ (i.e. $I_\gamma \subseteq I_\delta$ or $I_\delta \subseteq I_\gamma$ for each $\gamma, \delta \in \Gamma$). Then $I = \bigcup_{\gamma \in \Gamma} I_\gamma$ is also a ϱ-ideal of $\langle A, \varrho \rangle$.*

Proof. Let $a \in A$, $i \in I$ and $a\, \varrho\, i$. Then $i \in I_\gamma$ for some $\gamma \in \Gamma$ and, by (I_1), $a \in I_\gamma$. Hence $a \in I$. If $i, j \in I$, then $i \in I_\gamma$, $j \in I_\delta$ for some $\gamma, \delta \in \Gamma$. Without loss of generality, suppose $I_\gamma \subseteq I_\delta$. Then $i, j \in I_\delta$, thus $U(i, j) \cap I_\delta \neq \emptyset$. As $I_\delta \subseteq I$, also $U(i, j) \cap I \neq \emptyset$, which completes the proof.

Corollary. *Each ϱ-ideal of $\langle A, \varrho \rangle$ is contained in a maximal ϱ-ideal of $\langle A, \varrho \rangle$.*

This follows directly from Proposition 2 by Kuratowski-Zorn theorem.

Proposition 3. *Let $\langle A, \varrho \rangle$ be a ϱl-directed binary relational system and I a prime ϱ-ideal of $\langle A, \varrho \rangle$. If $A - I \neq \emptyset$, then $D = A - I$ is a dual prime ϱ-ideal of $\langle A, \varrho \rangle$.*

Proof. Let $D = A - I \neq \emptyset$. Let $b \in A$, $d \in D$ and $d \varrho b$. If $b \notin D$, then $b \in I$ and, by (I_1), $d \in I$, a contradiction. Thus (D_1) is satisfied.

Let $c, d \in D$ and $L(c, d) \cap D = \emptyset$. As $\langle A, \varrho \rangle$ is ϱl-directed, we have $\emptyset \neq L(c, d) \subseteq I$. By the assumptions, I is a prime ϱ-ideal of $\langle A, \varrho \rangle$, thus $c \in I$ or $d \in I$, also a contradiction. Thus also (D_2) is satisfied and D is a dual ϱ-ideal of $\langle A, \varrho \rangle$.

Suppose $a, b \in A$ and $\emptyset \neq U(a, b) \subseteq D$. If $a \in I$ and $b \in I$, by (I_2) we have $\emptyset \neq U(a, b) \cap I$, which is a contradiction to $U(a, b) \subseteq D$. Thus either $a \in D$ or $b \in D$, i.e. D is a prime dual ϱ-ideal of $\langle A, \varrho \rangle$.

Proposition 4. *Let $\langle A, \varrho \rangle$ be a ϱl-directed binary relational system and I a prime ϱ-ideal of $\langle A, \varrho \rangle$. Then $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for each two ϱ-ideals I_1, I_2 of $\langle A, \varrho \rangle$.*

Proof. The assertion is evident for $I = A$. Let $I \neq A$. By Proposition 4, $D = A - I$ is a dual prime ϱ-ideal of $\langle A, \varrho \rangle$. If $x_1 \in I_1 - I$, $x_2 \in I_2 - I$, then $x_1, x_2 \in D$ and, by (D_2), $L(x_1, x_2) \cap D \neq \emptyset$. If $t \in L(x_1, x_2) \cap D$, then $t \varrho x_1$, $t \varrho x_2$ and by (I_1) we have $t \in I_1 \cap I_2 \subseteq I$, which is a contradiction. Thus $I_1 - I = \emptyset$ or $I_2 - I = \emptyset$, which implies the assertion.

2. PRINCIPAL ϱ-IDEALS AND SUPREMAL RELATIONS

Definition 3. Let $\langle A, \varrho \rangle$ be a binary relational system and $\emptyset \neq M \subseteq A$. If the intersection of all ϱ-ideals of $\langle A, \varrho \rangle$ containing M is also a ϱ-ideal of $\langle A, \varrho \rangle$, we denote it by $I(M)$ and call it a *ϱ-ideal generated by M*. If $M = \{a_1, \ldots, a_n\}$ is a finite set, $I(M)$ is denoted briefly by $I(a_1, \ldots, a_n)$ and called *a finitely generated ϱ-ideal*. For $M = \{a\}$, $I(a)$ is called a *principal ϱ-ideal generated by a*. If $I(a)$ exists for each $a \in A$, $\langle A, \varrho \rangle$ is called *principal*.

Notation. If $\langle A, \varrho \rangle$ is principal, $\mathscr{I}_0(A)$ denotes the set of all principal ϱ-ideals of $\langle A, \varrho \rangle$.

Lemma 1. *Let ϱ be a binary relation on A, $a, b \in A$ and let $I(a)$, $I(b)$ exist. If $a\ t(\varrho)\ b$, then $I(a) \subseteq I(b)$.*

Proof. By Definition 3, $b \in I(b)$. If $a\ t(\varrho)\ b$, then there exist $a_0, \ldots, a_n \in A$, $a_0 = a$, $a_n = b$ and $a_{i-1} \varrho a_i$ for $i = 1, \ldots, n$; thus by (I_1) also $a_{n-1} \in I(b)$ and inductively $a = a_0 \in I(b)$. Hence $I(a) \subseteq I(b)$.

Definition 4. A binary relation ϱ is called *supremal* on A, if for each $a, b \in A$ there exists at least one element $s(a, b) \in U(a, b)$ such that $x \in U(a, b)$ implies $s(a, b) =$

$= x$ or $s(a, b)\ \varrho\ x$. Each element $s(a, b)$ with this property is called a *ϱ-supremum of a, b*.

It is clear that the ϱ-supremum of a, b need not be determined uniquely. If for example $A = \{a, b\}$ and $a\ \varrho\ a$, $a\ \varrho\ b$, $b\ \varrho\ a$, $b\ \varrho\ b$, then a is a ϱ-supremum of a, b as well as b is. However, if $s(a, b) \neq s'(a, b)$ are two ϱ-suprema of a, b, then $s(a, b)\ \varrho\ s'(a, b)$ and $s'(a, b)\ \varrho\ s(a, b)$.

If ϱ is supremal on A and each $a, b \in A$ has just one ϱ-supremum, ϱ is called *uniquely supremal* on A. Clearly, each antisymmetrical supremal relation on A is uniquely supremal on A. The dual concepts are *infimal* and *uniquely infimal relation* on A.

The following examples show that for a uniquely supremal binary relation ϱ the system $\langle A, \varrho \rangle$ need not be a semilattice.

Example 2. Let A be the set of all integers and $a\ \varrho\ b$ if and only if $b - a \geqq 1$. Then ϱ is uniquely supremal on A and $s(a, b) = \max\{a, b\} + 1$. However, $s(a, a) \neq a$, thus $\langle A, \varrho \rangle$ is not a semilattice.

Example 3. Let $\lesssim$ be a reflexive, uniquely supremal and uniquely infimal relation on A. Then $\langle A, \lesssim \rangle$ is a *weakly associative lattice* (see [3]). However, $\langle A, \lesssim \rangle$ is not generally a semilattice, since it is not necessarily transitive (see [2]).

Lemma 2. *Let ϱ be a supremal relation on A and J a ϱ-ideal of $\langle A, \varrho \rangle$. Then $s(a, b) \in J$ for each $a, b \in J$ and for an arbitrary ϱ-supremum $s(a, b)$ of a, b.*

Proof. Let $a, b \in J$, $s(a, b)$ be a ϱ-supremum of a, b and $s(a, b) \notin J$. As J is a ϱ-ideal of $\langle A, \varrho \rangle$, there exists $x \in U(a, b) \cap J$. Thus $x \neq s(a, b)$. By Definition 4 we have $s(a, b)\ \varrho\ x$, thus $x \in J$ implies $s(a, b) \in J$, a contradiction.

Proposition 5. *If ϱ is a supremal relation on A, then every set $\{I_\gamma; \gamma \in \Gamma\}$ of ϱ-ideals of $\langle A, \varrho \rangle$ has an infimum $I = \bigcap_{\gamma \in \Gamma} I_\gamma$ in $\langle \mathscr{I}(A), \subseteq \rangle$ provided $I \neq \emptyset$. Moreover, if $\langle A, \varrho \rangle$ is also ϱl-directed, then $\langle \mathscr{I}(A), \subseteq \rangle$ is a conditionally complete and join complete lattice.*

Proof. If ϱ is supremal on A, then $\langle A, \varrho \rangle$ is ϱu-directed and, by Proposition 1(c), A is the greatest element of $\langle \mathscr{I}(A), \subseteq \rangle$. Let $\{I\gamma; \gamma \in \Gamma\} \subseteq \mathscr{I}(A)$ and $\emptyset \neq I = \bigcap_{\gamma \in \Gamma} I_\gamma$. If $a \in A$, $i \in I$, $a\ \varrho\ i$, then $i \in I_\gamma$ for each $\gamma \in \Gamma$ and, by (I_1), also $a \in I_\gamma$ for each $\gamma \in \Gamma$. Hence $a \in I$. If $i, j \in I$, then, by Lemma 2, $s(i, j) \in U(i, j) \cap I_\gamma$ for each $\gamma \in \Gamma$ and an arbitrary ϱ-supremum $s(i, j)$ of i, j. Hence $s(i, j) \in U(i, j) \cap I$. Accordingly, I is a ϱ-ideal of $\langle A, \varrho \rangle$. It is evident that I is the infimum of $\{I_\gamma; \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq \rangle$.

Let $\langle A, \varrho \rangle$ be ϱl-directed and $I_1, I_2 \in \mathscr{I}(A)$. Then $I_1 \cap I_2 \neq \emptyset$, since the relations $a \in I_1$, $b \in I_2$ imply $x \in I_1 \cap I_2$ for each $x \in L(a, b) \neq \emptyset$. By the former result, $I_1 \cap I_2$ is the infimum of $\{I_1, I_2\}$ in $\langle \mathscr{I}(A), \subseteq \rangle$. Let $\{I_\gamma; \gamma \in \Gamma\} \subseteq \mathscr{I}(A)$. Denote by $\mathscr{S}$ the set of all ϱ-ideals of $\langle A, \varrho\rangle$ containing $\bigcap_{\gamma\in\Gamma} I_\gamma$. By the first result, $A \in \mathscr{S}$, thus $\mathscr{S} \neq \emptyset$. Then $J = \bigcap\mathscr{S}$ is a ϱ-ideal of $\langle A, \varrho\rangle$. Clearly J is the supremum of $\{I_\gamma; \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq\rangle$. The proof is complete.

Corollary. *Let ϱ be a supremal relation on A. Then $\langle A, \varrho\rangle$ is principal and, moreover, there exists $I(M)$ for each $\emptyset \neq M \subseteq A$.*

Proposition 6. *Let ϱ be a supremal relation on A. If $\langle \mathscr{I}(A), \subseteq\rangle$ contains the least element, then it is an algebraic lattice and the finitely generated ϱ-ideals are its compact elements.*

Proof. If $\langle \mathscr{I}(A), \subseteq\rangle$ contains the least element, then by Proposition 5 it is a complete lattice. By Corollary of Proposition 5, $\langle A, \varrho\rangle$ is principal and $I(M)$ exists for each $\emptyset \neq M \subseteq A$.

Let $I \in \mathscr{I}(A)$. Then clearly $I(x) \subseteq I$ for each $x \in I$. Hence $\bigcup_{x\in I} I(x) \subseteq I$. As $x \in I(x)$, also $I \subseteq \bigcup_{x\in I} I(x)$, thus $I = \bigcup_{x\in I} I(x)$. Now $\bigcup_{x\in I} I(x)$ is a ϱ-ideal of $\langle A, \varrho\rangle$, hence $I = \bigcup_{x\in I} I(x) = \bigvee_{x\in I} I(x)$ (where $\bigvee$ stands for the supremum in the lattice $\langle \mathscr{I}(A), \subseteq\rangle$).

Let $a \in A$ and $I(a) \subseteq \bigvee_{\gamma\in\Gamma} I_\gamma$ for some $I_\gamma \in \mathscr{I}(A)$, $\gamma \in \Gamma$. By the proof of Proposition 5, $\bigvee_{\gamma\in\Gamma} I_\gamma = I(\bigcup_{\gamma\in\Gamma} I_\gamma)$, i.e. $a \in I(a) \subseteq \bigvee_{\gamma\in\Gamma} I_\gamma = I(\bigcup_{\gamma\in\Gamma} I_\gamma)$. By Proposition 2 and Proposition 5, $\mathscr{I}(A)$ is the algebraic closure system with $M \to I(M)$ as an algebraic closure operator on A (see [4], Theorem 1.2). This means that there exists a finite subset M of $\bigcup_{\gamma\in\Gamma} I_\gamma$, such that $a \in I(M)$. Now there exists a finite subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ with $M \subseteq \bigcup_{i=1}^{n} I_{\gamma_i}$. This yields $a \in I(M) \subseteq I(\bigcup_{i=1}^{n} I_{\gamma_i}) = \bigvee_{i=1}^{n} I_{\gamma_i}$, i.e. $I(a) \subseteq \bigvee_{i=1}^{n} I_{\gamma_i}$. Thus $I(a)$ is a compact element in $\langle \mathscr{I}(A), \subseteq\rangle$ for each $a \in A$. As ϱ is supremal, each finitely generated ϱ-ideal is principal, which completes the proof.

Notation. Let ϱ be a binary relation on A. We introduce operators

$$\mathscr{L},\ L : 2^A - \{\emptyset\} \to 2^A$$

by the rules

$$\mathscr{L}(X) = \{a \in A;\ a\ \varrho\ x \text{ for some } x \in X\},$$

$$L(X) = \mathscr{L}(X) \cup X.$$

If ϱ is supremal on A, we introduce operators $\mathscr{S}, S : 2^A - \{\emptyset\} \to 2^A$ by

$$\mathscr{S}(X) = \{a \in A;\ a = s(x, y) \text{ for some } x \in X,\ y \in X \text{ and } \varrho\text{-supremum } s(x, y)\},$$

$$S(X) = \mathscr{S}(X) \cup X.$$

Lemma 3. *Let ϱ be a binary relation on A and $\emptyset \neq X \subseteq Y \subseteq A$. Then*

$$X \subseteq L(X) \subseteq L(Y)\,.$$

If ϱ is also supremal on A, then

$$X \subseteq S(X) \subseteq S(Y)\,.$$

The proof is clear.

Notation. Let ϱ be supremal on A and $\emptyset \neq X \subseteq A$. Define $(SL)^1(X) = (SL)(X) = S(L(X))$ and for any integer n recursively

$$(SL)^{n+1}(X) = (SL)((SL)^n(X))\,.$$

Analogously, for the operators $\mathscr{S}$ and $\mathscr{L}$ let us write $(\mathscr{S}\mathscr{L})^1(X) = (\mathscr{S}\mathscr{L})(X) = \mathscr{S}(\mathscr{L}(X))$ if $\mathscr{L}(X) \neq \emptyset$ and $(\mathscr{S}\mathscr{L})^{n+1}(X) = (\mathscr{S}\mathscr{L})((\mathscr{S}\mathscr{L})^n(X))$ if $\mathscr{L}((\mathscr{S}\mathscr{L})^n(X)) \neq \emptyset$.

Proposition 7. *Let ϱ be a supremal relation on A. Then $I(M) = \bigcup_{n=1}^{\infty}(SL)^n(M)$ for each $\emptyset \neq M \subseteq A$.*

Proof. Let M be a non-void subset of A. First we prove that $I_M = \bigcup_{n=1}^{\infty}(SL)^n(M)$ is a ϱ-ideal $\langle A, \varrho\rangle$.

Let $a \in A$, $x \in I_M$ and $a \mathrel{\varrho} x$. Then $x \in (SL)^n(M)$ for an integer n, thus $a \in L((SL)^n(M))$ and, by Lemma 3, $a \in (SL)((SL)^n(M)) = (SL)^{n+1}(M)$. Hence $a \in I_M$. If $i, j \in I_M$, then there exist integers n, m with $i \in (SL)^n(M)$, $j \in (SL)^m(M)$. By Lemma 3, for $k = \max\{n, m\}$ we have $i, j \in (SL)^k(M)$, thus $s(i, j) \in (SL)(SL)^k(M)) = (SL)^{k+1}(M) \subseteq I_M$ for each ϱ-upremum $s(i, j)$ of i, j. Hence $U(i, j) \cap I_M \neq \emptyset$, thus I_M is a ϱ-ideal of $\langle A, \varrho\rangle$. Clearly $M \subseteq I_M$.

It remains to prove $I_M = I(M)$. Let I be a ϱ-ideal of $\langle A, \varrho\rangle$ with $M \subseteq I$. From (I_1) and Lemma 2 we obtain $(SL)(M) \subseteq I$. By induction we can easily extend it to $(SL)^k(M) \subseteq I$ for each integer k, thus $I_M \subseteq I$, i.e. $I_M \subseteq I(M)$. The converse inclusion is evident, thus $I_M = I(M)$.

Corollary. *Let ϱ be a reflexive and supremal binary relation on A. Then $I(M) = \bigcup_{n=1}^{\infty}(\mathscr{S}\mathscr{L})^n(M)$ for each non-void subset M of A.*

Remark. From Proposition 7 we can derive an explicite description of the suprema of $\{I_\gamma;\, \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq\rangle$ in the case ϱ is supremal on A. Indeed,

$$\bigvee_{\gamma\in\Gamma} I_\gamma = \bigcup_{n=1}^{\infty}(SL)^n\Big(\bigcup_{\gamma\in\Gamma} I_\gamma\Big)\,.$$

For some special binary relations frequently used in mathematical investigations the set of ϱ-ideals can be characterized more precisely.

A binary relation ϱ on the set A is called *complete*, if either $a \varrho b$ or $b \varrho a$ is satisfied for each $a, b \in A$. Clearly, ϱ is complete if and only if its symmetrical hull is a universal relation on A.

Proposition 8. *If ϱ is a complete binary relation on a set A, then*
(a) *$\langle A, \varrho \rangle$ is principal and $I(a) = \{x \in A;\ x\, t(\varrho)\, a\}$ for each $a \in A$.*
(b) *Every finitely generated ϱ-ideal of $\langle A, \varrho \rangle$ is principal.*
(c) *Each ϱ-ideal of $\langle A, \varrho \rangle$ is prime.*
(d) *$\langle \mathscr{I}(A), \subseteq \rangle$ is a chain.*

Proof. (a) Le ϱ be a complete relation on A. Then $a \varrho a$ for each $a \in A$, i.e. ϱ is reflexive. If $a, b \in A$, then $a \varrho b$ or $b \varrho a$. As $a \varrho a$, $b \varrho b$, it implies $a \in U(a, b)$ or $b \in U(a, b)$. Suppose $a \in U(a, b)$. If $c \in U(a, b)$, then $a \varrho c$, $b \varrho c$, thus $a = s(a, b)$. For $b \in U(a, b)$ clearly $b = s(a, b)$. Thus ϱ is also supremal and, by Corollary of Proposition 5, $\langle A, \varrho \rangle$ is principal. For $a \in A$ fix denote $M = \{x \in A;\ x\, t(\varrho)\, a\}$. Clearly $a \in M$.

If $b \in A$, $x \in M$, $b \varrho x$, then there exist $a_0, \ldots, a_n \in A$ with $a_0 = x$, $a_n = a$ and $a_{i-1} \varrho a_i$ for $i = 1, \ldots, n$. Thus $b \varrho x$ implies $b\, t(\varrho)\, a$, i.e. $b \in M$. If $i, j \in M$, then either $i \in U(i, j)$ or $j \in U(i, j)$. Hence $U(i, j) \cap M \neq \emptyset$ and M is a ϱ-ideal of $\langle A, \varrho \rangle$ containing a.

Conversely, let I be a ϱ-ideal of $\langle A, \varrho \rangle$ containing a. If $t \in M$, then $t \varrho a_1, \ldots$ $\ldots, a_{n-1} \varrho a_n = a$ for some $a_1, \ldots, a_n \in A$. As $a \in I$, it is also $a_{n-1} \in I$ and, inductively by (I_1), $t \in I$. Hence $M \subseteq I$, i.e. $M = I(a)$. As $a \in A$ was chosen arbitrary, the statement (a) is proved.

(b) By Corollary of Proposition 5, there exists finitely generated ϱ-ideal $I(a_1, \ldots, a_n)$ for every finite subset $\{a_1, \ldots, a_n\}$ of A. Without loss of generality, suppose $a_1 \varrho a_2$. Then clearly $I(a_1, \ldots, a_n) = I(a_2, \ldots, a_n)$. With respect to the finiteness of $\{a_1, \ldots, a_n\}$, by $n - 1$ steps we obtain $I(a_1, \ldots, a_n) = I(a_i)$ for some $i \in \{1, \ldots, n\}$.

(c) Let I be a ϱ-ideal of $\langle A, \varrho \rangle$ and $i, j \in A$. As ϱ is complete, $i \in L(i, j)$ or $j \in$ $\in L(i, j)$ is fulfilled. Then $\emptyset \neq L(i, j) \subseteq I$ implies $i \in I$ or $j \in I$, thus I is prime.

(d) Let I, J be ϱ-ideals of $\langle A, \varrho \rangle$. By Proposition 5, $I \cap J$ is also a ϱ-ideal of $\langle A, \varrho \rangle$ and by (c) $I \cap J$ is prime. As $I \cap J \subseteq I \cap J$, by Proposition 4 we obtain $I \subseteq I \cap J \subseteq J$ or $J \subseteq I \cap J \subseteq I$, thus $\langle \mathscr{I}(A), \subseteq \rangle$ is a chain.

Remark. If ϱ is complete on A, clearly $S(X) = X$ for each $\emptyset \neq X \subseteq A$. As ϱ is also reflexive, we have $L = \mathscr{L}$. Then by Proposition 7 we have $I(M) = \bigcup_{n=1}^{\infty} \mathscr{L}^n(M)$ and by Proposition 8, $\{x \in A;\ x\, t(\varrho)\, a\} = \bigcup_{n=1}^{\infty} \mathscr{L}^n(\{a\})$.

Definition 5. Let ϱ be a binary relation on a set A, $c \in B \subseteq A$. We call c the *ϱ-greatest element of B*, if $b \varrho c$ is true for all $b \in B$.

An element $d \in B$ is called *ϱ-maximal of B*, if $d \varrho b$ is true for none of the elements $b \in B$, $b \neq d$.

We say that $\langle A, \varrho \rangle$ satisfies the *ϱ-maximal condition* if each non-void subset of A has a ϱ-maximal element.

Lemma 4. *Let B be a semi ϱ-ideal of $\langle A, \varrho \rangle$ with the ϱ-greatest element $b \in B$. Then B is the principal ϱ-ideal and $B = I(b)$.*

Proof. If $x, y \in B$, then $x \varrho b$, $y \varrho b$ and it means $b \in U(x, y) \cap B$. As B is a semi ϱ-ideal, B is a ϱ-ideal of $\langle A, \varrho \rangle$. Further, if I is a ϱ-ideal of $\langle A, \varrho \rangle$ containing b, then $t \varrho b$ implies $t \in I$ for each $t \in A$. However, $t \varrho b$ is true for each $t \in B$, thus $B \subseteq I$. Hence $B = I(b)$.

Lemma 5. *Every ϱu-directed subset B in a binary relational system (A, ϱ) has at most one ϱ-maximal element. If such an element exists in B, it is at the same time the ϱ-greatest element of B.*

Proof. If B is a ϱu-directed subset of A and $a, b \in B$ are ϱ-maximal elements of B, then $a \varrho t$, $b \varrho t$ for each $t \in U(a, b) \cap B \neq \emptyset$, thus it remains only $a = t = b$. Let B have a ϱ-maximal element m. If $x \in B$, then there exists $s \in U(x, m) \cap B$ since B is ϱu-directed, i.e. $x \varrho s$ and $m \varrho s$. As m is ϱ-maximal, we have $m = s$, thus $x \varrho m$. As x was chosen arbitrary, m is the ϱ-greatest element of B.

Proposition 9. *Let $\langle A, \varrho \rangle$ satisfy the ϱ-maximal condition. Then each ϱ-ideal of $\langle A, \varrho \rangle$ is principal and has a ϱ-greatest element.*

Proof. By Proposition 1, each ϱ-ideal I of $\langle A, \varrho \rangle$ is ϱu-directed and, by Lemma 5, I has the ϱ-greatest element because $\langle A, \varrho \rangle$ satisfies the ϱ-maximal condition. By Lemma 4, I is principal.

Definition 6. Let $\langle A, \varrho \rangle$, $\langle B, \sigma \rangle$ be binary relational systems. A *homomorphism of $\langle A, \varrho \rangle$ into $\langle B, \sigma \rangle$* is a mapping h of A into B such that $a \varrho b$ implies $h(a) \sigma h(b)$. If h is a surjective and injective homomorphism of $\langle A, \varrho \rangle$ onto $\langle B, \sigma \rangle$ and h^{-1} is also a homomorphism of $\langle B, \sigma \rangle$ onto $\langle A, \varrho \rangle$ we call h an *isomorphism of $\langle A, \varrho \rangle$ onto $\langle B, \sigma \rangle$* and wirte $\langle A, \varrho \rangle \cong \langle B, \sigma \rangle$. For this definition see e.g. to [5].

Notation. If $\langle A, \varrho \rangle$ is principal, then by Lemma 1 the mapping $J_0 : a \to I(a)$ is a homomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathscr{I}_0(A), \subseteq \rangle$. Denote by Θ_0 the equivalence relation induced by J_0 on A. By the notation introduced in [5], $\langle A, \varrho \rangle / \Theta_0$ means the binary relational system $\langle A', \varrho' \rangle$, the support A' of which is the factor set A/Θ_0 and the relation ϱ' on A/Θ_0 is defined by $X, Y \in A/\Theta_0$, $X \varrho' Y$ if and only if $x \varrho y$ for some $x \in X$, $y \in Y$.

Denote by $[a]$ the class of A/Θ_0 containing the element a.

Proposition 10. *Let $\langle A, \varrho \rangle$ be principal. If each principal ϱ-ideal of $\langle A, \varrho \rangle$ has the ϱ-greatest element, then $\langle \mathcal{J}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle / \Theta_0$.*

Proof. Clearly the mapping $[a] \to I(a)$ is a bijection of A/Θ_0 onto $\mathcal{J}_0(A)$. Suppose $a, b \in A$, $[a]\, \varrho'\, [b]$. Then there exist $a' \in [a]$, $b' \in [b]$ with $a'\, \varrho\, b'$. By Lemma 1, $I(a') \subseteq I(b')$, hence $I(a) \subseteq I(b)$ and the mapping $[a] \to I(a)$ is a homomorphism.

Let $I(a) \subseteq I(b)$. Denote by c the ϱ-greatest element of $I(b)$. Then $a\, \varrho\, c$, $b\, \varrho\, c$ and $c \in I(b)$, i.e. $I(b) \subseteq I(c)$. Clearly $I(c) \subseteq I(b)$, thus $I(b) = I(c)$. From $a\, \varrho\, c$ we have $[a]\, \varrho'\, [c]$ and from $I(b) = I(c)$ it follows that $[b] = [c]$, thus also $[a]\, \varrho'\, [b]$. Accordingly, also the converse mapping of $[a] \to I(a)$ is a homomorphism of $\langle A, \varrho \rangle / \Theta_0$ onto $\langle \mathcal{J}_0(A), \subseteq \rangle$, thus $\langle \mathcal{J}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle / \Theta_0$.

Corollary. *Let $\langle A, \varrho \rangle$ be a principal binary relational system satisfying the ϱ-maximal condition. Then $\langle \mathcal{J}(A), \subseteq \rangle$ is a lattice if and only if $\langle A, \varrho \rangle / \Theta_0$ is a lattice.*

This follows directly from Proposition 10, since by Proposition 9 each ϱ-ideal of $\langle A, \varrho \rangle$ is principal and has the ϱ-greatest element.

It is well-known (see e.g. [1]) that for a partial order $\leqq$ the mapping $a \to I(a)$ is an isomorphism of $\langle A, \leqq \rangle$ onto $\langle \mathcal{J}_0(A), \subseteq \rangle$. It can be proved that also the converse proposition is true. These facts show that partially ordered sets can be fully characterized by their sets of principal $\leqq$-ideals. This characterization is given by the following

Proposition 11. *Let $\langle A, \varrho \rangle$ be a binary relational system. The following conditions are equivalent:*

(a) *$\langle A, \varrho \rangle$ is principal and a is the ϱ-maximal element of $I(a)$ for each $a \in A$.*
(b) *J_0 is an isomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathcal{J}_0(A), \subseteq \rangle$.*
(c) *J_0 is an injective mapping of A onto $\mathcal{J}_0(A)$.*
(d) *ϱ is a partial ordering on A.*

Proof. Clearly (b) $\Rightarrow$ (c) and $(d) \Rightarrow (b)$. Prove (c) $\Rightarrow$ (a). The existence of J_0 implies that $\langle A, \varrho \rangle$ is principal. Let $a \in A$. Suppose the existence of $b \in I(a)$ with $a\, \varrho\, b$. By Lemma 1, $a\, \varrho\, b$ implies $I(a) \subseteq I(b)$, from $b \in I(a)$ we have $I(b) \subseteq I(a)$, thus $I(a) = I(b)$. From the injectivity of J_0 we have $a = b$. Thus a is the ϱ-maximal element of $I(a)$ for each $a \in A$.

It remains to prove (a) $\Rightarrow$ (d). Let $a \in A$ be the ϱ-maximal element of $I(a)$. As $I(a)$ is a ϱu-directed subset of A, by Lemma 5 a is the ϱ-greatest element of $I(a)$. Thus $a\, \varrho\, a$, i.e. ϱ is reflexive on A. Let $a, b \in A$ and $a\, \varrho\, b$, $b\, \varrho\, a$. By Lemma 1 we have $I(a) = I(b)$ and, by Lemma 5, $a = b$, since $I(a) = I(b)$ has just one ϱ-maximal element. Thus ϱ is also antisymmetrical. Suppose $a\, \varrho\, b$, $b\, \varrho\, c$ for $a, b, c \in A$. Then $I(a) \subseteq I(b) \subseteq I(c)$, i.e. $a \in I(c)$. As c is the ϱ-greatest element in $I(c)$ (by Lemma 5), we have $a\, \varrho\, c$. Accordingly, ϱ is also transitive, i.e. ϱ is a partial order on A.

Lemma 6. *Let ϱ be a transitive binary relation on A. If $a \in A$ and $a \varrho a$, then $I(a)$ exists and $I(a) = \{x \in A; x \varrho a\}$.*

Proof. Suppose $a \in A$ and $a \varrho a$. Denote $M = \{x \in A; x \varrho a\}$. Then $a \in M$ and $x, y \in M$ implies $x \varrho a$, $y \varrho a$, thus $a \in U(x, y) \cap M$. If $b \in M$, $x \in A$, $x \varrho b$, then $b \varrho a$ and the transitivity of ϱ implies $x \varrho a$ and hence $x \in M$. Accordingly, M is the ϱ-ideal of $\langle A, \varrho \rangle$ containing a. If I is also a ϱ-ideal of $\langle A, \varrho \rangle$ containing a, then $x \in M$ implies $x \varrho a$, thus, by (I_1), $x \in I$, i.e. $M \subseteq I$. Hence $I(a) = M$.

Proposition 12. *For an arbitrary binary relational system $\langle A, \varrho \rangle$ the following conditions are equivalent:*

(a) *$\langle A, \varrho \rangle$ is principal and $I(a) \subseteq I(b)$ if and only if $a \varrho b$;*

(b) *$\langle A, \varrho \rangle$ is principal and $I(a) = \{x \in A; x \varrho a\}$;*

(c) *ϱ is a quasiorder on A.*

Proof. If ϱ is a quasiorder, by Lemma 6 we obtain the implication (c) $\Rightarrow$ (b). Suppose (b). Then $I(a) \subseteq I(b)$ implies $a \varrho b$, the converse implication is given by Lemma 1, thus (b) $\Rightarrow$ (a). Suppose (a). $I(a) \subseteq I(a)$ for each $a \in A$, ϱ is reflexive. Let $a, b, c \in A$ and $a \varrho b$, $b \varrho c$. By Lemma 1 we obtain $I(a) \subseteq I(c)$ and the assumption (a) implies $a \varrho c$, thus ϱ is also transitive. Thus also (a) $\Rightarrow$ (c), which completes the proof.

Proposition 13. *Let ϱ be a quasiorder on A. If ϱ is uniquely supremal on A, then $\langle \mathcal{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle$.*

Proof. Let ϱ be uniquely supremal on A. As ϱ is reflexive and transitive, from unique supremality we have also the antisymmetry, thus ϱ is a partial ordering on A and, by Proposition 11, $\langle \mathcal{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle$.

Remark. Proposition 13 can be clearly dualized for ϱ uniquely infimal on A.

4. EMBEDDING OF RELATIONAL SYSTEMS INTO POSETS

The concept of a replica for the general case of algebraic structures is introduced in [5]. Its modification for the case of binary relational systems is given by

Definition 7. Let $\mathscr{C}$ be class of binary relational systems and let $\langle A, \varrho \rangle$ be an arbitrary system not necessarily from $\mathscr{C}$. A homomorphism h of $\langle A, \varrho \rangle$ onto a system $\langle D, \delta \rangle \in \mathscr{C}$ is called an *embedding of $\langle A, \varrho \rangle$ into $\mathscr{C}$* and $\langle D, \delta \rangle$ is called a *$\mathscr{C}$-replica*, if for each system $\langle B, \beta \rangle \in \mathscr{C}$ and an arbitrary homomorphism g of $\langle A, \varrho \rangle$ onto $\langle B, \beta \rangle$ there exists a homomorphism f of $\langle D, \delta \rangle$ onto $\langle B, \beta \rangle$ with $g = f \,.\, h$.

Denote by $\mathscr{P}$ the class of all partially ordered sets. It is known (see e.g. [5], § 11.3) that $\mathscr{P}$ forms a quasivariety of algebraic systems. Thus, by Theorem 5 from § 11.3

in [5], for an arbitrary binary relational system $\langle A, \varrho \rangle$ there exists an embedding into $\mathscr{P}$ and a $\mathscr{P}$-replica. In this section we shall give a condition for $\langle A, \varrho \rangle$ to have a $\mathscr{P}$-replica $\langle \mathscr{J}_0(A), \subseteq \rangle$.

Definition 8. A binary relational system $\langle A, \varrho \rangle$ is called *strictly principal,* if it is principal and $I(a) \subseteq I(b)$ implies $a\ t(\varrho)\ b$.

Example 4. If ϱ is a complete relation on A, then, by Proposition 8 (a), $\langle A, \varrho \rangle$ is strictly principal.

If ϱ is a quasiorder on A, then $\langle A, \varrho \rangle$ is strictly principal by Proposition 12 (a).

If $\langle A, \varrho \rangle$ is a *finite cycle,* i.e. $A = \{a_1, \ldots, a_n\}$ and $a_1\ \varrho\ a_2, \ldots, a_{n-1}\ \varrho\ a_n$, $a_n\ \varrho\ a_1$ (ϱ need not be transitive or reflexive), then $I(a) = A$ for each $a \in A$ and $a\ t(\varrho)\ b$ is also true for each $a, b \in A$, thus $\langle A, \varrho \rangle$ is strictly principal.

Proposition 14. *Let $\langle A, \varrho \rangle$ be a strictly principal binary relational system. Then $\langle \mathscr{J}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica and J_0 is an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$.*

Proof. By Lemma 1, J_0 is a homomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathscr{J}_0(A), \subseteq \rangle \in \mathscr{P}$. Let $\langle P, \leqq \rangle \in \mathscr{P}$ and let g be a homomorphism of $\langle A, \varrho \rangle$ onto $\langle P, \leqq \rangle$. Introduce the relation $\mathscr{J}_0(A) \to P$ by the rule $I(a) \to g(a)$ for each $a \in A$.

1°. If $I(a) = I(b)$, then $a\ t(\varrho)\ b$, $b\ t(\varrho)\ a$, i.e. there exist $a_0, \ldots, a_n, b_0, \ldots, b_m \in A$ such that $a_0 = a = b_m$, $b_0 = b = a_n$ and $a_{i-1}\ \varrho\ a_i$ $(i = 1, \ldots, n)$, $b_{j-1}\ \varrho\ b_j$ $(j = 1, \ldots, m)$. As g is a homomorphism, it follows that $g(a) \leqq g(b)$ and $g(b) \leqq g(a)$. As $\leqq$ is a partial order, $g(a) = g(b)$. Accordingly, the relation $\to$ is a mapping of $\mathscr{J}_0(A)$ onto P. Denote this mapping by f.

2°. If $I(a) \subseteq I(b)$, then $a\ t(\varrho)\ b$ because $\langle A, \varrho \rangle$ is strictly principal, i.e. there exist $a_0, \ldots, a_n \in A$ with $a_0 = a$, $a_n = b$, $a_{i-1}\ \varrho\ a_i$ for $i = 1, \ldots, n$. As g is a homomorphism, we have $g(a) \leqq g(b)$. Thus the mapping f is a homomorphism of $\langle \mathscr{J}_0(A), \subseteq \rangle$ onto $\langle P, \leqq \rangle$.

3°. Evidently, $f(J_0(a)) = f(I(a)) = g(a)$ for each $a \in A$, thus $\langle \mathscr{J}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica and J_0 an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$.

Corollary 1. *Let ϱ be a reflexive binary relation on a set A and let $\langle A, \varrho \rangle$ be principal. If a principal ϱ-ideal generated by $a \in A$ in $\langle A, \varrho \rangle$ is equal to the principal $t(\varrho)$-ideal generated by $a \in A$ in $\langle A, t(\varrho) \rangle$ for each $a \in A$, then J_0 is an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$ and $\langle \mathscr{J}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica.*

Proof. If ϱ is reflexive, then $\sigma = t(\varrho)$ is a quasiorder on A, and by Proposition 12, $\langle A, \varrho \rangle$ is principal and $I(a) \subseteq I(b) \Rightarrow a\ \sigma\ b$, i.e. $a\ t(\varrho)\ b$. As $I(a)$ is the same in $\langle A, \varrho \rangle$ as in $\langle A, \sigma \rangle$, it follows that $\langle A, \varrho \rangle$ is strictly principal and, by Proposition 14, we obtain the result.

Corollary 2. *Let ϱ be a complete relation on A. Then the $\mathscr{P}$-replica $\langle \mathscr{I}_0(A), \subseteq \rangle$ of $\langle A, \varrho \rangle$ is a chain.*

It follows directly from Proposition 14 and Proposition 8.

Corollary 3. *Let ϱ be an equivalence relation on a set A. Then the $\mathscr{P}$-replica of $\langle A, \varrho \rangle$ is the antichain (i.e. a complete unordered set) $\langle A/\varrho, \subseteq \rangle$.*

Proof. By example 1, $\mathscr{I}(A) = A/\varrho$ for an equivalence relation ϱ on A. Then clearly $I(a) = [a]$ for each $a \in A$, where $[a]$ denotes the class of the partition A/ϱ, $I(a) \subseteq \subseteq I(b)$ is equivalent to $[a] \subseteq [b]$, which is equivalent to $[a] = [b]$, i.e. $a \varrho b$. Hence $\langle A, \varrho \rangle$ is also strictly principal and, by Proposition 14, the assertion is obtained, because $\mathscr{I}_0(A) = \mathscr{I}(A) = A/\varrho$.

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