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KERNELS OF LATTICE ORDERED GROUPS DEFINED BY PROPERTIES OF SEQUENCES

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If p is a property related to convex l-subgroups of a lattice ordered group G, then the kernel of G with respect to the property p is defined to be the largest element of the system of all convex l-subgroups of G possessing the property p (if such a largest element does exist).

Several types of kernels of lattice ordered groups have been studied (cf., e.g., [1], [4] and the articles quoted there).

In this paper we investigate two types of kernels of abelian lattice ordered groups which are defined by means of properties of sequences. The first one is defined by means of the o-convergence, the second by means of the lateral convergence (cf. [5]; in the latter case we assume that the lattice ordered groups under consideration are completely representable.

1. PRELIMINARIES

Let G be an abelian lattice ordered group. We recall some notions and notations concerning the o-convergence of sequences in G (the definitions are the same as for the case of vector lattices, cf. Luxemburg-Zaanen [6]).

Let $\{u_n: n = 1, 2, \ldots\}$ be a sequence in G. If $u_1 \geqq u_2 \geqq u_3 \ldots$ and $\bigwedge_n u_n = u$, then we write $u_n \downarrow u$. A sequence $\{f_n: n = 1, 2, \ldots\}$ in G is said to be order convergent to an element $f \in G$ whenever there exists a sequence $u_n \downarrow 0$ in G such that $|f - f_n| \leqq u_n$ holds for all n. This will be denoted by $f_n \to f$. Clearly $f_n \to f \Leftrightarrow f_n - f \to 0 \Leftrightarrow |f_n - f| \to 0$.

A sequence $\{g_n: n = 1, 2, \ldots\}$ in G is called fundamental if, whenever $\{g_{n(j)}: j = 1, 2, \ldots\}$ and $\{g_{m(j)}: j = 1, 2, \ldots\}$ are subsequences of $\{g_n: n = 1, 2, \ldots\}$, then $g_{n(j)} - g_{m(j)} \to 0$ $(j = 1, 2, \ldots)$. A subset G_1 of G is called Cauchy complete if each fundamental sequence in G_1 order converges to an element of G_1.

An element $e > 0$ of G is an order unit in G if for each $0 < g \in G$ we have $e \wedge g > 0$. We write $u_n \searrow 0$ (cf. [5]) if $u_n \downarrow 0$, u_1 is an order unit in G and

$$(u_{n-1} - u_n) \wedge u_{n+1} = 0 \quad \text{for each} \quad n > 1 .$$

For a sequence $\{f_n: n = 1, 2, \ldots\}$ in G and $f \in G$ we write $f_n \to_L f$ (and we say that the sequence $\{f_n: n = 1, 2, \ldots\}$ laterally converges to f) if there exists a sequence $\{u_n: n = 1, 2, \ldots\}$ in G such that $u_n \searrow 0$ and

$$|f - f_n| \leqq u_n \quad \text{for each} \quad n .$$

A sequence $\{g_n: n = 1, 2, \ldots\}$ in G will be called l-fundamental, if there is a sequence $\{u_n: n = 1, 2, \ldots\}$ in G such that $u_n \searrow 0$ and $|g_{n+p} - g_n| \leqq u_n$ holds for each pair of positive integers n and p.

A subset G_1 of G will be called l-complete if for each l-fundamental sequence $\{g_n: n = 1, 2, \ldots\}$ in G_1 there is $g \in G_1$ such that $g_n \to_L g$.

The lateral convergence in archimedean lattice ordered groups was studied in [5].

2. AUXILIARY RESULTS ON o-CONVERGENCE

Let c be a fixed element of G. For each $x \in G$ we denote $x' = x \wedge c$, $x'' = x \vee c$.

2.1. Lemma. *Let $u, v \in G$, $u \leqq v$. Then $v - u = (v' - u') + (v'' - u'')$.*

Proof. Put $t = v' \vee u$. Since $v' \wedge u = u'$, we obtain $t - u = v' - u'$. Next we have $u'' \wedge v = (c \vee u) \wedge v = v' \vee u = t$ and clearly $u'' \vee v = v''$, thus $v - t = v'' - u''$. Therefore $v - u = v' - u' + v'' - u''$.

2.2. Lemma. *Let $p, q \in G$. Then $|p - q| = |p' - q'| + |p'' - q''|$.*

Proof. We have $|p - q| = p \vee q - p \wedge q$; further,

$$|p' - q'| = (p \wedge c) \vee (q \wedge c) - (p \wedge c) \wedge (q \wedge c) = (p \vee q)' - (p \wedge q)' ,$$

and similarly

$$|p'' - q''| = (p \vee q)'' - (p \wedge q)'' .$$

Thus in view of 2.1, $|p - q| = |p' - q'| + |p'' - q''|$.

2.3. Lemma. *Let $x_n \in G$ $(n = 1, 2, \ldots)$, $x \in G$, $x_n \to x$. Then $x'_n \to x'$ and $x''_n \to x''$.*

Proof. This is a consequence of the fact that the operations $\wedge$ and $\vee$ are continuous with respect to the o-convergence [3] (cf. also [6] for the case of vector lattices).

2.4. Lemma. *Let $x_n \in G$ $(n = 1, 2, \ldots)$, $s_1, s_2 \in G$, $x'_n \to s_1$, $x''_n \to s_2$. Then $x_n \to s_1 + s_2 - c$.*

Proof. For each n we have $x''_n - x_n = c - x'_n$, hence $x_n = x'_n + x''_n - c$. Since the operations $+$ and $-$ are continuous with respect to the o-convergence (cf., e.g. [3]), we obtain $x_n \to s_1 + s_2 - c$.

2.5. Lemma. *Let* $\{x_n: n = 1, 2, \ldots\}$ *be a fundamental sequence in* G. *Then* $\{x'_n: n = 1, 2, \ldots\}$ *and* $\{x''_n: n = 1, 2, \ldots\}$ *are fundamental sequences in* G.

Proof. Let $\{n(j): j = 1, 2, \ldots\}$ and $\{m(j): j = 1, 2, \ldots\}$ be increasing sequences of positive integers. Since $\{x_n: n = 1, 2, \ldots\}$ is a fundamental sequence, we have $x_{n(j)} - x_{m(j)} \to 0$ $(j = 1, 2, \ldots)$. Hence there is a sequence $\{u_j: j = 1, 2, \ldots\}$ in G such that $u_j \downarrow 0$ and $|x_{n(j)} - x_{m(j)}| \leqq u_j$ for each j. In view of 2.2, the relations

$$|x'_{n(j)} - x'_{m(j)}| \leqq u_j \quad \text{and} \quad |x''_{n(j)} - x''_{m(j)}| \leqq u_j$$

are valid for each j; hence $x'_{n(j)} - x'_{m(j)} \to 0$ and $x''_{n(j)} - x''_{m(j)} \to 0$ $(j = 1, 2, \ldots)$. Thus both $\{x'_n: n = 1, 2, \ldots\}$ and $\{x''_n: n = 1, 2, \ldots\}$ are fundamental sequences.

2.6. Lemma. *Let* $a, b, c \in G$, $a \leqq c \leqq b$. *Assume that both* $[a, c]$ *and* $[c, b]$ *are Cauchy complete subsets of* G. *Then* $[a, b]$ *is also a Cauchy complete subset of* G.

Proof. Let $\{x_n: n = 1, 2, \ldots\}$ be a fundamental sequence in G, $x_n \in [a, b]$ $(n = 1, 2, \ldots)$. Then $x'_n \in [a, c]$ and $x''_n \in [c, b]$ for each n. In view of 2.5, both $\{x'_n: n = 1, 2, \ldots\}$ and $\{x''_n: n = 1, 2, \ldots\}$ are fundamental sequences. Thus in view of the assumption there are elements $s_1 \in [a, c]$ and $s_2 \in [c, b]$ such that $x'_n \to s_1$ and $x''_n \to s_2$. According to 2.4, $x_n \to s_1 + s_2 - c$. Clearly $s_1 + s_2 - c \in [a, b]$. Hence $[a, b]$ is a Cauchy complete subset of G.

2.7. Lemma. *Let* $a, b, d \in G$, $a \leqq b$. *Assume that* $[a, b]$ *is a Cauchy complete subset of* G. *Then* $[a + d, b + d]$ *is also a Cauchy complete subset of* G.

Proof. This follows from the continuity of the operation $+$ with respect to the o-convergence.

From 2.6 and 2.7 we obtain:

2.8. Corollary. *Let* $a, b \in G$, $a \geqq 0$, $b \geqq 0$. *If the intervals* $[0, a]$ *and* $[0, b]$ *are Cauchy complete subsets of* G, *then* $[0, a + b]$ *is also a Cauchy complete subset of* G.

3. C_b-KERNEL OF A LATTICE ORDERED GROUP

Again, let G be an abelian lattice ordered group. A subset G_1 of G will be called Cauchy b-complete in G if each fundamental sequence in G_1 which is bounded in G_1, order converges to an element of G_1.

3.1. Lemma. *Let* $0 < a \in G$ *and let* G_1 *be the convex l-subgroup of* G *generated by the element* a. *Assume that* $[0, a]$ *is a Cauchy complete subset of* G. *Then* G_1 *is Cauchy b-complete in* G.

Proof. We have $G_1 = \bigcup[-na, na]$ $(n = 1, 2, \ldots)$. Let $X = \{x_n: n = 1, 2, \ldots\}$ be a fundamental sequence in G such that $X \subseteq G_1$ and X is bounded in G_1. Hence there is a positive integer n with $X \subseteq [-na, na]$. From 2.8 and by induction we infer that $[-na, na]$ is a Cauchy complete subset of G, hence there is $x \in [-na, na]$ such that $x_n \to x$. Thus G_1 is Cauchy b-complete in G.

Let us denote by C the system of all convex l-subgroups of G; the system C is partially ordered by inclusion. Then C is a complete lattice and the operation $\bigvee$ in C coincides with the operation of the join in the system of all subgroups of the group G; hence for $\{G_i\}_{i \in I} \subseteq C$, $\bigvee_{i \in I} G_i$ is the subgroup of G generated by the set $\bigcup_{i \in I} G_i$.

3.2. Theorem. *Let G be an lattice ordered group. Let $S = \{G_i\}_{i \in I}$ be the system of all convex Cauchy b-complete l-subgroups of G. Then the system S possesses a largest element.*

Proof. Put $H = \bigvee_{i \in I} G_i$. It suffices to verify that H is a Cauchy b-complete subset of G. Hence we have to verify that each interval of H is a Cauchy complete subset of G. In view of 2.7, it suffices to show that for each $0 < v \in H$, the interval $[0, v]$ is a Cauchy complete subset of G. There are elements $a_1, \ldots, a_n \in \bigcup_{i \in I} G_i$ such that $0 < a_i$ $(i = 1, 2, \ldots, n)$ and $v \leqq v' = a_1 + a_2 + \ldots + a_n$. Hence in view of 2.8 (and by induction) the interval $[0, v']$ is a Cauchy complete subset of G (since all intervals $[0, a_i]$ are Cauchy complete subsets of G); therefore $[0, v]$ is a Cauchy complete subset of G as well.

3.3. Theorem. *Let G be an abelian lattice ordered group. Let H_1 be the set of all $v \in G$ such that the interval $[0, |v|]$ is a Cauchy complete subset of G. Then H_1 is a convex l-subgroup of G.*

Proof. Let H be as in the proof of 3.2. It suffices to verify that $H_1 = H$. Let $v \in H$. Then $|v| \in H$ and hence $[0, |v|]$ is a Cauchy complete subset of G; thus $v \in H_1$. Conversely, let $v \in H_1$. Hence $[0, |v|]$ is a Cauchy complete subset of G. Let G_1 be the convex l-subgroup of G generated by $|v|$. In view of 3.1, G_1 is a Cauchy b-complete subset of G. Therefore $G_1 \subseteq H$ and thus $v \in H$.

The element f of a lattice ordered group G is said to have the Egoroff property if, given any double sequence $\{u_{nk}: n, k = 1, 2, \ldots\}$ in G such that, for every fixed n, the relation

$$|f| \geqq u_{nk} \downarrow 0 \quad (k = 1, 2, \ldots)$$

is valid, then there exists a sequence $v_m \downarrow 0$ in G with the property that for every pair (m, n) of positive integers there exists a positive integer $k(m, n)$ such that $v_m \geqq u_{n,k(m,n)}$ holds.

The assertion (i) of the following theorem can be considered analogous to Thm. 3.3.

3.4. Theorem. (Cf. [6], Thm. 67.3.) *Let G be an abelian lattice ordered group. Let A be the set of all elements of G which have the Egoroff property. Then* (i) *A is a convex l-subgroup of G;* (ii) *if $a_n \in A$ $(n = 1, 2, \ldots)$, $g \in G$ and if $\bigvee_{n=1,2,\ldots} a_n = g$ holds in G, then $g \in A$.*

(In [6], Thm. 67.3 it is assumed that G is a vector lattice, but the proof remains valid for abelian lattice ordered groups as well. Let us also remark that the l-subgroup H_1 from Thm. 3.3 need not fulfil the condition (ii) of Thm. 3.4; cf. Example 3.5 below.)

Again, let H be as in the proof of 3.2. Then H will be said to be the C_b-kernel of G. Let us remark that H need not be a Cauchy complete subset of G. This can be verified by the following example:

3.5. Example. Let R be the additive group of all reals (with the natural linear order) and let I be the set of all positive integers. For each $i \in I$ let $G_i = R$; put $G_0 = \Pi_{i \in I} G_i$. For each $g \in G_0$ we denote by $I(g)$ the set of all $i \in I$ such that $g(i)$ is irrational. Let G be the set of all $g \in G_0$ with the property that $I(g)$ is finite. Then G is an l-subgroup of G_0. The C_b-kernel H of G consists of all $g \in G$ with finite supports. For each $i \in I$ let $h_i \in G$ such that $h_i(j) = j$ for each $j \leqq i$ and $h_i(j) = 0$ for $j > i$; further, let $g \in G$ such that $g(i) = i$ for each $i \in I$. The relation $h_i \to g$ $(i = 1, 2, \ldots)$ holds in G, hence $\{h_i\colon i = 1, 2, \ldots\}$ is a fundamental sequence in G, and each h_i belongs to H. On the other hand, g does not belong to H, hence H fails to be a Cauchy complete subset of G. Also, H fails to fulfil the condition (ii) of Thm. 3.4.

3.6. Example. Let G be an abelian lattice ordered group and let $S_1 = \{K_i\}_{i \in I}$ be the system of all convex l-subgroups of G which are Cauchy complete subsets of G. Then S_1 need not have a largest element. In fact, let G be as in 3.5. By way of contradiction, assume that S_1 has a largest element K_0. We can identify G_i with the set of all $g \in G$ such that $g(j) = 0$ for each $j \in I$, $j \neq i$. Then $G_i \in S_1$ for each $i \in I$; thus $G' = \bigvee_{i \in I} G_i \subseteq K_0$. We have $G' = H$, where H is as in 3.5. In view of 3.5, H fails to be a Cauchy complete subset of G, hence the same holds for K_0, which is a contradiction.

4. LATERAL CONVERGENCE IN A COMPLETELY REPRESENTABLE LATTICE ORDERED GROUP

An l-subgroup G of a lattice ordered group G_0 is said to be a regular l-subgroup of G_0 if, given a subset $\{g_i\}_{i \in I}$ of G such that $\bigvee_{i \in I} g_i = g$ holds in G, then g is the supremum of the set $\{g_i\}_{i \in I}$ in G_0 as well. (Under this assumption the corresponding dual condition is also valid.)

In this section we shall asssume that $G \neq \{0\}$ is lattice ordered group which can be embedded as a regular l-subgroup into a direct product of linearly ordered groups; lattice ordered groups having this property are called completely representable. (It is well-known that an abelian lattice ordered group is completely representable if and only if it is completely distributive (cf. [2], Theorem 5.10).)

Hence without loss of generality we can assume that G is a regular l-subgroup of a group $G_0 = \Pi_{i \in I} G_i$, where each G_i is a non-zero abelian linearly ordered group.

In the definition of the relation $u_n \searrow 0$ in Section 1 above it was assumed (in view of [5]) that u_1 is an order unit in G; the following considerations remain valid without this assumption.

4.1. Lemma. *Let $\{u_n: n = 1, 2, \ldots\}$ be a sequence in G such that $u_n \searrow 0$. Then there are disjoint sets $T_n \subseteq I$ $(n = 0, 1, 2, \ldots)$ such that* (i) $I = \bigcup T_n$ $(n = 0, 1, 2, \ldots)$, (ii) for each pair of positive integers m, n with $m \geqq n + 2$ and for each $t \in T_n$ we have $u_m(t) = 0$, and (iii) $u_n(t) = 0$ for each positive integer n and for each $t \in T_0$.

Proof. For each positive integer n we denote

$$T_n = \{t \in I : u_1(t) = u_n(t) > u_{n+1}(t)\} .$$

Let $t \in T_n$ and let $m \geqq n + 2$. Because of $u_n \searrow 0$ we have

$$(u_n - u_{n+1}) \wedge u_{n+2} = 0$$

and $0 \leqq u_m \leqq u_{n+2}$, thus

$$(u_n - u_{n+1}) \wedge u_m = 0 .$$

Therefore

$$(u_n(t) - u_{n+1}(t)) \wedge u_m(t) = 0 .$$

Since G_t is linearly ordered and $u_n(t) - u_{n+1}(t) > 0$, we must have $u_m(t) = 0$.

Thus (ii) is valid.

Put $T_0 = I \setminus \bigcup T_n$ $(n = 1, 2, \ldots)$. In view of the definition, (i) holds. Let $t \in T_0$. Then $0 \leqq u_1(t) = u_2(t) = u_3(t) = \ldots$. Because of $\bigwedge u_n = 0$ and since G is a regular l-subgroup of G_0 we obtain $\bigwedge u_n(t) = 0$ $(n = 1, 2, \ldots)$, therefore $u_n(t) = 0$ for each positive integer n. If $m, k \in \{0, 1, 2, \ldots\}$, $m \neq k$, then clearly $T_m \cap T_k = \emptyset$.

4.2. Lemma. *Let $\{x_n: n = 1, 2, \ldots\}$ be an l-fundamental sequence in G. Then there are disjoint sets $T_n \subseteq I$ $(n = 0, 1, 2, \ldots)$ such that* (i) $I = \bigcup T_n$ $(n = 0, 1, 2, \ldots)$, *and* (ii) *if n is a non-negative integer, m and m_1 positive integers with $m \geqq n + 2$, $m_1 \geqq n + 2$ and $t \in T_n$, then $x_m(t) = x_{m_1}(t)$.*

Proof. This is a consequence of 4.1 (in view of the definition of l-fundamentality of the sequence $\{x_n: n = 1, 2, \ldots\}$).

4.3. Lemma. *Let $\{x_n: n = 1, 2, \ldots\}$ and $\{u_n: n = 1, 2, \ldots\}$ be sequences in G such that* (i) $u_n \searrow 0$, (ii) $|x_{n+p} - x_n| \leqq u_n$ *holds for each n and each p,* (iii) $y \in G$ *and $x_n \to_L y$. Then $|y - x_n| \leqq u_n$ is valid for each n.*

Proof. In view of $x_n \to_L y$ there exists a sequence $\{v_n: n = 1, 2, \ldots\}$ in G such that $v_n \searrow 0$ and $|x_n - y| \leqq v_n$ holds for each n. We apply Lemma 4.1 for the sequence $\{v_n: n = 1, 2. \ldots\}$; we obtain disjoint subsets S_n $(n = 0, 1, 2, \ldots)$ of I fulfilling (i), (ii) and (iii) from 4.1 (where u_n is replaced by v_n). From the assumptions (i) and (ii) of the present lemma it follows that the sequence $\{x_n: n = 1, 2, \ldots\}$ is l-fundamental, hence the assertion of 4.2 is valid. Let T_n $(n = 0, 1, 2, \ldots\}$ be as in 4.2. We construct $x \in G_0$ as follows. For each $t \in I$ there exists $n \in \{0, 1, 2, \ldots\}$ with $t \in T_n$ (and this n is uniquely determined); now we put $x(t) = x_{n+2}(t)$.

Let $t \in T$ and let m be an arbitrary element of the set $\{0, 1, 2, \ldots\}$. Let k be a positive integer, $k \geqq \max\{n + 2, m\}$, where $t \in T_n$. In view of 4.2 (ii) we have $x(t) = x_{n+2}(t) = x_k(t)$, thus from the assumption (ii) of the present lemma we infer that the relation

$$|x(t) - x_m(t)| = |x_k(t) - x_m(t)| \leqq u_m(t)$$

holds, hence

$$(*) \qquad |x - x_m| \leqq u_m$$

is valid for each m.

Let $t' \in I$. There are non-negative integers $n(1)$ and $n(2)$ such that $t' \in T_{n(1)}$ and $t' \in S_{n(2)}$. Let $n(3) \geqq \max\{n(1), n(2)\} + 2$. Hence $x(t') = x_{n(3)}(t')$. Further we have $v_{n(3)}(t') = 0$, and thus the relation $|x_{n(3)} - y| \leqq v_{n(3)}$ yields $x_{n(3)}(t') = y(t')$, therefore $x = y$. Now $(*)$ implies $|y - x_n| \leqq u_n$ for each n.

5. THE KERNEL DEFINED BY THE LATERAL CONVERGENCE

Again, let G be a completely representable lattice ordered group. We adopt the same notations for G as in Section 4.

5.1. Lemma. *Let $\{u_n: n = 1, 2, \ldots\}$ be a sequence in G such that $u_n \searrow 0$ and let k be a positive integer. Then $ku_n \searrow 0$.*

This follows immediately from the definition of the relation $u_n \searrow 0$.

5.2. Lemma. *Let $a, b, c \in G$, $a \leqq c \leqq b$. Assue that both the intervals $[a, c]$ and $[c, b]$ are l-complete subsets of G. Then $[a, b]$ is also an l-complete subset of G.*

Proof. Let $\{x_n: n = 1, 2, \ldots\}$ be an l-fundamental sequence in G, $x_n \in [a, b]$ $(n = 1, 2, \ldots)$. As in Section 2, for each $z \in G$ we denote $z' = z \wedge c$, $z'' = z \vee c$. We have $x'_n \in [a, c]$ and $x''_n \in [c, b]$ for each n. There exists a sequence $\{u_n: n = 1, 2, \ldots\}$ in G such that $u_n \searrow 0$ and

$$|x_{n+p} - x_n| \leqq u_n \quad \text{for each } n \text{ and each } p\,.$$

Thus in view of 2.2,

$$(5.1) \qquad |x'_{n+p} - x'_n| \leqq u_n \quad \text{for each } n \text{ and each } p\,,$$

$$(5.2) \qquad |x''_{n+p} - x''_n| \leqq u_n \quad \text{for each } n \text{ and each } p\,.$$

Therefore both the sequences $\{x'_n: n = 1, 2, \ldots\}$ and $\{x''_n: n = 1, 2, \ldots\}$ are l-fundamental; hence there are elements $z_1 \in [a, c]$ and $z_2 \in [c, b]$ such that $x'_n \rightarrow_L z_1$ and

$x_n'' \to_L z_2$. Thus from (5.1), (5.2) and 4.3 we infer that the relations

$$|x_n' - z_1| \leqq u_n\,, \quad |x_n'' - z_2| \leqq u_n$$

are valid for each n. Put $x = z_1 + z_2 - c$. We have

$$|x_n - x| = |x_n' + x_n'' - c - (z_1 + z_2 - c)| =$$
$$= |(x_n' - z_1) + (x_n'' - z_2)| \leqq |x_n' - z_1| + |x_n'' - z_2| \leqq 2u_n\,.$$

Hence in view of 5.1, the relation $x_n \to_L x$ is valid. Thus the interval $[a, b]$ is l-complete.

If $\{y_n\colon n = 1, 2, \ldots\}$ is a sequence in G, $y \in G$, $a \in G$ and $y_n \to_L y$, then clearly $y_n + a \to_L y + a$ and $-y_n \to_L -y$. Hence from 5.2 we obtain as a corollary:

5.3. Lemma. *Let $a, b \in G$, $a \geqq 0$, $b \geqq 0$. Assume that the intervals $[0, a]$ and $[0, b]$ are l-complete. Then the intervals $[0, a + b]$ and $[-a, 0]$ are l-complete as well.*

From 5.3 we infer (by an analogous reasoning as in 3.1):

5.4. Lemma. *Let $0 < a \in G$ and let G_1 be the convex l-subgroup of G generated by the element a. Assume that $[0, a]$ is an l-complete subset of G. Then each interval of G_1 is an l-complete subset of G.*

5.5. Theorem. *Let G be a completely representable lattice ordered group. Let H be the set of all elements $a \in G$ such that the interval $[0, |a|]$ is an l-complete subset of G. Then H is a convex l-subgroup of G.*

The proof can be established by analogous arguments as those applied in the proofs of 3.2 and 3.3 with the distinction that instead of 3.1 we now apply Lemma 5.4.

The l-subgroup H of Theorem 5.4 is obviously the largest element of the system S' which consists of all convex l-subgroups G_i of G having the property that each interval of G_i is an l-complete subset of G. The lattice ordered group H will be said to be the l-kernel of G.

5.6. Example. In this example we show that if $x_n \to_L x$ and $y_n \to_L y$ hold in G, then the relation $x_n + y_n \to_L x + y$ need not be valid. (This is the reason why we cannot apply the same method as in Section 3 above when proving the existence of the l-kernel of G.)

Let N_0 be the additive group of all integers with the natural linear order and let I be the set of all non-negative reals. For each $i \in I$ let $G_i = A \circ B$ (the symbol $\circ$ denoting the operation of lexicographic product, A is an l-ideal in G_i), where $A = B = N_0$. Put $G^0 = \Pi_{i \in I} G_i$. For $g \in G$ the symbol g_i or $g(i)$ denotes the component of g in G_i. The elements g have the form $g_i = (g_i^1, g_i^2)$ with $g_i^1 \in A$, $g_i^2 \in B$.

Let G be the set of all $g \in G^0$ with the property that for each non-negative integer n, g_i^2 is a constant on the set

$$\{i \in I : n \leqq i < n + 1\}.$$

Then G is a regular l-subgroup of G^0.

We define elements x_n and y_n of G as follows:

$(x_n)_i^1 = 0$ for each n and each $i \in I$;
$(x_1)_i^2 = 1$ for each $i \in I$;
if $n > 1$, then $(x_n)_i^2 = 1$ whenever $i \geqq n - 1$, and $(x_n)_i^2 = 0$ otherwise;
$(y_n)_i^2 = 0$ for each n and each $i \in I$;
$(y_1)_i^1 = 1$ for each $i \in I$;
if $n > 1$, then $(y_n)_i^1 = 1$ whenever $0 < i \leqq 1/n$, and $(y_n)_i^1 = 0$ otherwise.

In view of this definition, both x_1 and y_1 are order units in G and $x_n \searrow 0$, $y_n \searrow 0$. Thus $x_n \to_L 0$ and $y_n \to_L 0$. Put $z_n = x_n + y_n$.

Assume that $z_n \to_L 0$. Then there is a sequence $\{v_n : n = 1, 2, \ldots\}$ in G with $v_n \searrow 0$ such that $z_n \leqq v_n$ is valid for each n. We have $\bigwedge v_n = 0$; because G is a regular subgroup of G^0, the relation $\bigwedge(v_n)_0 = 0$ is valid. Since $v_1 \geqq z_1 > x_1$, there exists a positive integer $n(1)$ such that $(v_{n(1)})_0^2 < (v_1)_0^2 = 1$ holds. Put $n(1) + 1 = m$. Thus we necessarily have

$$(v_m)_0 = 0,$$

and so $(v_m)_0^2 = 0$, implying $(v_m)_i^2 = 0$ for each $i \in I$ with $0 \leqq i < 1$. Hence for each such i, $(v_m)_i < (v_1)_i$, thus $(v_{m+1})_i = 0$. We obtain $v_{m+1} \geqq x_{m+1}$, which is a contradiction.

5.7. The question whether Thm. 5.5 remains valid without assuming that G is completely representable remains open.

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