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Časopis pro pěstování matematiky, roč. 109 (1984), Praha

ON SOME PROPERTIES OF SOLUTIONS OF THE CAUCHY PROBLEM FOR A QUASILINEAR PARABOLIC EQUATION

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In this paper we show that some properties of the initial function (e.g. monotonicity, boundedness) induce similar properties of the classical solution (its every t-cut) of the Cauchy problem for parabolic equations of a certain type. Further we discuss the asymptotic behaviour of the solution as $t \to \infty$, especially the convergence to a solution of an ordinary differential equation.

All proofs are based on the maximum principle which will be used in the form given by the following theorem.

Theorem 1. *Let* $Q_T = \Omega \times (0, T]$ (Ω *is an unbounded domain in* R^n), $S_T = \{(x, t) : x \in \partial\Omega,\ t \in (0, T]\} \cup \{(x, 0) : x \in \bar{\Omega}\}$. *Assume that the operator*

$$Lu \equiv -u_t + \sum_{i,j=1}^{n} a_{ij}(x, t)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t)\, u_{x_i} + c(x, t)\, u$$

satisfies in Q_T *the conditions*

$$(1) \qquad \sum_{i,j=1}^{n} a_{ij}(x, t)\, y_i y_j \geqq 0 \quad \text{for any} \quad y = (y_1, \ldots, y_n) \in R^n\,,$$

$$(2) \qquad |a_{ij}(x, t)| \leqq M\,, \quad |b_i(x, t)| \leqq M(\|x\| + 1)\,, \quad |c(x, t)| \leqq M(\|x\|^2 + 1)\,,$$

M being a positive constant. If $Lu \geqq 0$ *in* Q_T, $u|_{S_T} \leqq 0$ *and*

$$u(x, t) \leqq D \exp(d\|x\|^2) \quad \text{in} \quad \bar{Q}_T$$

for some positive constants D, d, then $u \leqq 0$ *in* $\bar{Q}_T$.

This theorem is a simple generalization of Theorem 9.4.II, [1], where the coefficients of L are supposed to be continuous, a_{ij} form a positive definite matrix and $\Omega = R^n$. The proof of Theorem 1 is practically the same as that of the above mentioned theorem.

For studying the asymptotic behaviour we shall need the following immediate corollary of Theorem 1.

Corollary 1. *Let* $Q = \Omega \times (0, \infty)$, $S = \{(x, t) : x \in \partial\Omega,\ t \in (0, \infty)\} \cup \{(x, 0) : : x \in \bar{\Omega}\}$. *Let L satisfy the assumptions* (1) *and* (2) *in* Q, *where the positive constant* M *is replaced by a positive continuous nondecreasing function* $M(t)$. *If* $Lu \geqq 0$ *in* Q, $u|_s \leqq 0$ *and*

$$u(x, t) \leqq D \exp\left(d(t) \|x\|^2\right) \quad in \quad \bar{Q}$$

($D > 0$, $d(t)$ *is a positive continuous nondecreasing function*), *then* $u \leqq 0$ *in* $\bar{Q}$.

Let us consider the equation

$$(3) \qquad u_t = \sum_{i,j=1}^{n} a_{ij}(x, t, u, u_x)\, u_{x_i x_j} + b(x, t, u, u_x) + B(t, u)$$

given in $P = R^n \times (0, T]$, where the coefficients satisfy for all $(x, t) \in P$, $u \in R$, $p \in R^n$ the conditions:

$$a_{ij}(x, t, u, p) \text{ are bounded and } \Sigma a_{ij}(x, t, u, p)\, y_i y_j \geqq 0\,,$$
$$|b(x, t, u, p)| \leqq M(\|x\| + 1)\, \|p\|\,,$$

B is Lipschitz continuous with respect to u, i.e.

$$|B(t, u) - B(t, v)| \leqq K|u - v|$$

for any $u, v \in R$, $t \in [0, T]$ and some $K > 0$.

We shall study the solutions of the Cauchy problem for the equation (3) with the initial condition

$$(4) \qquad u(x, 0) = u_0(x)\,, \quad x \in R^n\,.$$

We shall consider such solutions that the inequality

$$(5) \qquad |u(x, t)| \leqq D \exp\left(d\|x\|^2\right)$$

holds in $\bar{P}$.

Theorem 2. *Let* $n = 1$ $(P = R \times (0, T])$. *Let* u *be a solution of the Cauchy problem* (3), (4) *satisfying* (5). *If the initial function* u_0 *is nonincreasing* (*nondecreasing*), *then any t-cut of the solution* u *is nonincreasing* (*nondecreasing*).

Proof. Let u_0 be a nonincreasing function. Let $\Omega = \{(x, y) : x > y,\ x, y \in R\}$, $P_1 = \Omega \times (0, T]$. Let $w(x, y, t) = u(x, t) - u(y, t)$ in $\bar{P}_1$ and define in P_1

$$\begin{aligned}
A_{11}(x, y, t) &= a_{11}(x, t, u(x, t), u_x(x, t))\,,\\
A_{22}(x, y, t) &= a_{11}(y, t, u(y, t), u_y(y, t))\,,\\
A_1(x, y, t) &= M(|x| + 1) \operatorname{sgn} u_x(x, t)\,,\\
A_2(x, y, t) &= -M(|y| + 1) \operatorname{sgn} u_y(y, t)\,,\\
A(x, y, t) &= K \operatorname{sgn}(u(x, t) - u(y, t))\,.
\end{aligned}$$

Then

$$Lw \equiv -w_t + A_{11}w_{xx} + A_{22}w_{yy} + A_1w_x + A_2w_y + Aw =$$

$$= -u_t(x, t) + u_t(y, t) + A_{11}u_{xx}(x, t) - A_{22}u_{yy}(y, t) +$$

$$+ M(|x| + 1)\,|u_x(x, t)| + M(|y| + 1)\,|u_y(y, t)| + K|u(x, t) - u(y, t)| \geqq$$

$$\geqq -u_t(x, t) + a_{11}(x, t, u, u_x)\,u_{xx}(x, t) + b(x, t, u, u_x) + B(t, u(x, t)) +$$

$$+ u_t(y, t) - a_{11}(y, t, u, u_y)\,u_{yy}(y, t) - b(y, t, u, u_y) - B(t, u(y, t)) = 0 \quad \text{in} \quad P_1 .$$

L satisfies the conditions (1), (2) in P_1. Obviously $w(x, y, t) = 0$ for any $(x, y, t) \in \in \partial\Omega \times (0, T]$, $w(x, y, 0) = u_0(x) - u_0(y) \leqq 0$. Further,

$$|w(x, y, t)| = |u(x, t) - u(y, t)| \leqq De^{dx^2} + De^{dy^2} \leqq 2D \exp\left[d(x^2 + y^2)\right].$$

All assumptions of Theorem 1 being satisfied, we obtain that $w \leqq 0$ in $\bar{P}_1$, i.e. $u(x, t) \leqq u(y, t)$ for all $(x, t), (y, t) \in P$, $x > y$.

Theorem 3. *Let u be a solution of the problem* (3), (4) *satisfying* (5). *Assume that the function B in the equation* (3) *is continuous on* $[0, T] \times R$. *If*

$$m_1 \leqq u_0(x) \leqq m_2 \quad (m_1, m_2 \in R)$$

and v_1, v_2 are solutions of the equation

$$v' = B(t, v)\,, \quad t \in (0, T]\,,$$

with initial conditions

$$v_1(0) = m_1\,, \quad v_2(0) = m_2\,,$$

then $v_1(t) \leqq u(x, t) \leqq v_2(t)$ holds in $\bar{P}$.

Proof. Due to the properties of B, v_1, v_2 exist on the whole interval $[0, T]$, they are unique and $v_1(t) < v_2(t)$ $(t \in [0, T])$ if $m_1 < m_2$. Let $w(x, t) = v_1(t) - u(x, t)$ in $\bar{P}$ and denote in P

$$A_{ij}(x, t) = a_{ij}(x, t, u(x, t),\ u_x(x, t))\,,$$

$$A_i(x, t) = -M(\|x\| + 1)\operatorname{sgn} u_{x_i}(x, t)\,,$$

$$A(x, t) = K \operatorname{sgn}(v_1(t) - u(x, t))\,.$$

Then

$$Lw \equiv -w_t + \Sigma A_{ij}w_{x_ix_j} + \Sigma A_iw_{x_i} + Aw =$$

$$= -v_1' + u_t - \Sigma A_{ij}u_{x_ix_j} + M(\|x\| + 1)\,\Sigma|u_{x_i}| + K|v_1 - u| \geqq$$

$$\geqq -v_1' + u_t - \Sigma a_{ij}(x, t, u, u_x)\,u_{x_ix_j} - b(x, t, u, u_x) - B(t, u) + B(t, v_1) = 0\,,$$

$$w(x, 0) = m_1 - u_0(x) \leqq 0\,.$$

Using the maximum principle (Theorem 1) we get that $w(x, t) \leqq 0$ in $\bar{P}$. For proving the other part of the assertion let us define $\bar{w}(x, t) = u(x, t) - v_2(t)$. Then $\bar{L}\bar{w} \geqq 0$, $\bar{w}(x, 0) \leqq 0$, hence also $\bar{w}(x, t) \leqq 0$ in $\bar{P}$.

Corollary 2. *If the assumptions of Theorem 3 are fulfilled and the initial function u_0 is constant, then the problem (3), (4) has a solution which is unique in the class of functions satisfying (5). This solution coincides with the solution of the problem*

$$u' = B(t, u), \quad t \in (0, T], \tag{6}$$

$$u(0) = u_0. \tag{7}$$

Proof. The solution u of the problem (6), (7) obviously satisfies the problem (3), (4). That it is the unique solution satisfying (5) follows from Theorem 3.

Corollary 3. *If the assumptions of Theorem 3 are fulfilled, then u is a bounded function.*

Corollary 4. *Let the assumptions of Theorem 3 be fulfilled. If the function B is nonincreasing in the second variable, then*

$$|u(x, t) - u(y, t)| \leqq m_2 - m_1 \quad \textit{for any} \quad (x, t), (y, t) \in \bar{P}.$$

(The modulus of continuity of any t-cut of the solution u is bounded by the same number as the modulus of continuity of the initial function u_0.)

Proof. According to Theorem 3, the inequality $v_1(t) \leqq u(x, t) \leqq v_2(t)$ holds, where v_1, v_2 are solutions of $v' = B(t, v)$, $v_1(0) = m_1$, $v_2(0) = m_2$, hence

$$|u(x, t) - u(y, t)| \leqq v_2(t) - v_1(t) \equiv r(t).$$

Since $r' = B(t, v_2) - B(t, v_1) \leqq 0$, so the function r is nonincreasing on $[0, T]$, $r(t) \leqq r(0) = m_2 - m_1$.

A theorem analogous to Theorem 3 holds also in the case when the equation (3) is given in $Q = R^n \times (0, \infty)$ and its coefficients satisfy for any $(x, t) \in Q$, $u, v \in R$, p, $y \in R^n$ the conditions:

$$|a_{ij}(x, t, u, p)| \leqq M(t), \quad \Sigma a_{ij}(x, t, u, p)\, y_i y_j \geqq 0,$$

$$|b(x, t, u, p)| \leqq M(t)\,(\|x\| + 1)\,\|p\|,$$

where M is a positive continuous nondecreasing function on $(0, \infty)$ and B is continuous on $\langle 0, \infty) \times R$ and Lipschitz continuous in the second variable.

Theorem 4. *Let u be a solution of the equation (3) in Q with the initial condition (4). Let*

$$|u(x, t)| \leqq D \exp\left[d(t)\,\|x\|^2\right] \quad \textit{in} \quad \bar{Q} \tag{8}$$

for a positive constant D and a positive continuous nondecreasing function d. If

$$m_1 \leqq u_0(x) \leqq m_2 \quad (m_1, m_2 \in R)$$

and v_1, v_2 are solutions of the equation

$$v' = B(t, v), \quad t \in (0, \infty),$$

with initial conditions

$$v_1(0) = m_1, \quad v_2(0) = m_2,$$

then $v_1(t) \leqq u(x, t) \leqq v_2(t)$ in $\bar{Q}$.

This theorem can be proved in the same way as Theorem 3 only using Corollary 1 instead of Theorem 1.

Concerning the asymptotic behaviour of the solution, a natural problem arises: to describe such functions B that

$$\lim_{t \to \infty} [v_1(t) - v_2(t)] = 0,$$

or that $v_2(t) - v_1(t)$ does not increase. Sufficient conditions are given in the following lemmas.

Lemma 1. *Let a function B defined on $[t_0, \infty) \times R$ satisfy the condition*

$$B(t, y_2) - B(t, y_1) \leqq G(t)(y_2 - y_1) \quad \textit{for any} \quad y_1, y_2 \in R, \tag{9}$$

$y_2 > y_1$, $t \geqq t_0$, where G is a function satisfying

$$\int_{t_0}^{\infty} G(t)\,dt = -\infty.$$

If the equation $v' = B(t, v)$, $t \in (t_0, \infty)$ has the global uniqueness property and v_1, v_2 are solutions of this equation (defined on $[t_0, \infty)$), then

$$\lim_{t \to \infty} [v_2(t) - v_1(t)] = 0.$$

Proof. Let $v_1(t_0) = m_1$, $v_2(t_0) = m_2$, $m_1 < m_2$. The global uniqueness property implies that

$$r(t) = v_2(t) - v_1(t) > 0,$$

$$r' = B(t, v_2) - B(t, v_1) \leqq G(t)(v_2 - v_1) = G(t)\,r,$$

$$r(t_0) = m_2 - m_1,$$

hence

$$r(t) \leqq (m_2 - m_1) \exp\left(\int_{t_0}^{t} G(s)\,ds\right)$$

and $\lim_{t \to \infty} r(t) = 0$.

Remark 1. The condition (9) holds for example in the case that $B_u(t, u)$ exists and has a negative upper bound.

Lemma 2. *Let the assumptions of Lemma* 1 *except of the condition* (9) *hold. If B is nonincreasing in the second variable, then $r(t) = v_2(t) - v_1(t)$ does not increase on the interval $[t_0, \infty)$.*

Proof. $r' = B(t, v_2) - B(t, v_1) \leqq 0$.

Combining Theorem 4 and Lemma 1 we get

Corollary 5. *Let the assumptions of Theorem* 4 *hold, let B satisfy the condition* (9) *on $[0, \infty) \times R$. If v is a solution of*

$$v' = B(t, v), \quad t \in (0, \infty), \tag{10}$$

$$v(0) = v_0, \tag{11}$$

v_0 is an arbitrary real number, then

$$\lim_{t \to \infty} |u(x, t) - v(t)| = 0$$

uniformly with respect to x, i.e. for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $(x, t) \in Q$ the following implication holds:

$$t > t_0 \Rightarrow |u(x, t) - v(t)| < \varepsilon .$$

Proof. $|u(x, t) - v(t)| \leqq (u(x, t) - v_1(t)) + |v_1(t) - v(t)| \leqq (v_2(t) - v_1(t)) + |v_1(t) - v(t)|$, and the last expression converges to zero according to Lemma 1.

Remark 2. In Corollary 5 choose $v_0 \in [m_1, m_2]$ $(m_1 \leqq \inf_{R^n} u_0(x), m_2 \geqq \sup_{R^n} u_0(x))$. Let the function G from (9) be nonpositive. If we replace the solution of the problem (3), (4) by the solution of (10), (11) on an interval $[t_0, \infty)$, then the error is estimated by the number

$$(m_2 - m_1) \exp\left(\int_0^{t_0} G(s) \, ds\right).$$

(Recalling the proof of Lemma 1 we have

$$|u(x, t) - v(t)| \leqq v_2(t) - v_1(t) \leqq (m_2 - m_1) \exp\left(\int_0^t G(s) \, ds\right);$$

the last function is nonincreasing and converges to zero as $t \to \infty$.)

If the function B is only nonincreasing in the second variable, then applying Lemma 2 we obtain

$$|u(x, t) - v(t)| \leqq v_2(t) - v_1(t) \leqq m_2 - m_1 .$$

In some cases it is possible to make some conclusions about the character of the solution even if the condition (9) is not satisfied.

Example. Consider a solution u of the problem

$$Lu \equiv -u_t + \Sigma a_{ij}(x, t)\, u_{x_i x_j} + \Sigma b_i(x, t)\, u_{x_i} + c(t)\, u = f(t) \quad \text{in} \quad Q\,,$$
$$u(x, 0) = u_0(x)\,, \quad |u_0(x)| \leqq N \quad (N > 0)\,, \quad x \in R^n\,.$$

Assume that u satisfies (8). Let the coefficients of L satisfy in Q the conditions: a_{ij} form a positive semidefinite matrix, $|a_{ij}(x, t)| \leqq M(t)$,

$$|b_i(x, t)| \leqq M(t)\,(\|x\| + 1)\,, \quad c_1 \leqq c(t) \leqq c_2 \;\; (c_1, c_2 \in R,\; c_2 > 0)\,,$$

c, f are continuous on $[0, \infty)$.

Theorem 4 yields only that $v_1(t) \leqq u(x, t) \leqq v_2(t)$, where v_1, v_2 are solutions of $v' = c(t)\, v - f(t)$, $t \in (0, \infty)$, $v_1(0) = -N$, $v_2(0) = N$. Neither Lemma 1 nor Lemma 2 can be applied.

Introduce the substitution $w = u e^{-\lambda t}$, $\lambda = c_2 + \varepsilon$, $\varepsilon > 0$. w is a solution of the problem

$$w_t = \Sigma a_{ij} w_{x_i x_j} + \Sigma b_i w_{x_i} + (c - \lambda)\, w - f e^{-\lambda t} \quad \text{in} \quad Q\,,$$
$$w(x, 0) = u_0(x)\,.$$

$B(t, u) \equiv (c(t) - \lambda)\, u - e^{-\lambda t} f(t)$ satisfies the assumption (9) according to Remark 1 ($B_u(t, u) = c(t) - \lambda \leqq -\varepsilon$). Using Corollary 5 we obtain that $\lim_{t \to \infty} |w(x, t) - z(t)| = 0$, where z is the solution of $z' = B(t, z)$, $z(0) = 0$. If the relation $\lim_{t \to \infty} z(t) = 0$ does not hold, then $\lim_{t \to \infty} w(x_0, t) = 0$ does not hold for any $x_0 \in R^n$. (This means, for example, that there exists no polynomial P such that $|u(x_0, t)| \leqq P(t)$, $t \in [0, \infty)$.)

In what follows we shall consider the equation

$$u_t = a(x, t, u, u_x)\, u_{xx} + b(x, t, u, u_x) + B(t) \tag{12}$$

in $P = R \times (0, T]$, where coefficients $a(x, t, u, p)$, $b(x, t, u, p)$ for any $(x, t) \in P$, $u, p \in R$ satisfy the conditions:

$$a(x, t, u, p) \geqq m \quad \text{for some} \quad m > 0\,, \tag{13}$$

$a(x, t, u, p)$ is bounded, $|b(x, t, u, p)| \leqq M|p|$ for some $M > 0$. B is an arbitrary function defined on $(0, T]$.

Theorem 5. *Let u be a solution of the Cauchy problem for the equation (12) satisfying (5). If the initial function u_0 is bounded ($|u_0(x)| \leqq J$, $J > 0$) and Lipschitz continuous, then u is bounded and $|u_x(x, t)| \leqq C$, where C is a constant depending only on J, m, M and the Lipschitz constant K of u_0.*

Proof. Introduce the function $f(s) = \min\{Ks, 2J\}$, $s \in [0, \infty)$. Then $|u_0(x) - u_0(y)| \leqq f(|x - y|)$ for any $x, y \in R$. If M_1 is a positive number, then there exists

$k > 0$ such that $f(s) \leqq k - ke^{-M_1 s} \equiv g(s)$ holds for $s \in [0, \infty)$. The function g is concave ($g''(s) = -M_1^2 k e^{-M_1 s} < 0$), increasing ($g'(s) = M_1 k e^{-M_1 s} > 0$) for any $k > 0$, so it suffices to choose such a k that $g(2J/K) = 2J$, i.e.

$$k = 2J \left[1 - \exp\left(- \frac{2M_1 J}{K} \right) \right]^{-1} .$$

g satisfies the equation $g'' + M_1 g' = 0$ in $(0, \infty)$, $g(0) = 0$. Define P_1, w, A_{11}, A_{22} in the same way as in the proof of Theorem 2, set $z(x, y, t) = g(x - y)$ in $\bar{P}_1$ for $M_1 = M/m$. In P_1 we have

$$L_1 w \equiv -w_t + A_{11} w_{xx} + A_{22} w_{yy} + M(|w_x| + |w_y|) \geqq$$

$$\geqq -u_t(x, t) + u_t(y, t) + A_{11} u_{xx}(x, t) - A_{22} u_{yy}(y, t) +$$

$$+ b(x, t, u, u_x) - b(y, t, u, u_y) + B(t) - B(t) = 0 ,$$

$$L_1 z = -z_t + A_{11} z_{xx} + A_{22} z_{yy} + M(|z_x| + |z_y|) \leqq$$

$$\leqq A_{11} \left(z_{xx} + \frac{M}{m} |z_x| \right) + A_{22} \left(z_{yy} + \frac{M}{m} |z_y| \right) =$$

$$= (A_{11} + A_{22}) (g''(x - y) + M_1 g'(x - y)) = 0 ,$$

$$0 \leqq L_1 w - L_1 z = -(w - z)_t + A_{11}(w - z)_{xx} + A_{22}(w - z)_{yy} +$$

$$+ A_1(w - z)_x + A_2(w - z)_y = L_1'(w - z) ,$$

where

$$A_1(x, y, t) = M \frac{|u_x(x, t)| - |z_x(x, y, t)|}{u_x(x, t) - z_x(x, y, t)} , \quad u_x(x, t) \neq z_x(x, y, t) .$$

$$= 0 , \quad u_x(x, t) = z_x(x, y, t) .$$

A_2 is defined analogously. Further,

$$w(x, y, 0) = u_0(x) - u_0(y) \leqq f(x - y) \leqq g(x - y) = z(x, y, 0) ,$$

$$w(x, x, t) = 0 = g(0) = z(x, x, t) .$$

According to Theorem 1, $w \leqq z$ in $\bar{P}_1$, i.e.

$$u(x, t) - u(y, t) \leqq g(x - y) - g(0) ,$$

hence $u_x(x, t) \leqq g'(0) = M_1 k = C$. (Analogously $-w \leqq z$ in $\bar{P}_1$ and $-u_x(x, t) \leqq \leqq g'(0)$.) Furthermore, we get $|u(x, t) - u(y, t)| \leqq g(x - y)$, where g is a bounded function ($g(s) \leqq k$, $s \in [0, \infty)$). Take an arbitrary $x_0 \in R$, then $u(x, t) = u(x_0, t) + [u(x, t) - u(x_0, t)]$. The second member on the right-hand side is bounded as well as the first one, because we consider a classical solution u, so its x_0-cut is a function continuous on $[0, T]$.

Remark 3. The assumption that the equation (12) is not degenerate is not essential. The condition (13) can be replaced by the following one:

$$a(x, t, u, p) \geqq 0 \quad \text{for any} \quad (x, t) \in P\,, \quad u, p \in R\,,$$

if we assume that

$$|b(x, t, u, p) - b(y, t, v, q)| \leqq M[a(x, t, u, p)\,|p| + a(y, t, v, q)\,|q|]$$

for some $M > 0$ and any $t \in (0, T]$, $x, y, u, v, p, q \in R$. Then the constant C depends only on M, K, J. In the proof we set $z(x, y, t) = g(x - y)$ for $M_1 = M$. Instead of L_1 we introduce

$$\bar{L}_1 w \equiv -w_t + A_{11}(w_{xx} + M|w_x|) + A_{22}(w_{yy} + M|w_y|)\,.$$

We obtain

$$|u_x(x, t)| \leqq g'(0) = 2MJ\left[1 - \exp\left(-\frac{2MJ}{K}\right)\right]^{-1}.$$

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