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DECOMPOSITION OF AN INFINITE COMPLETE GRAPH INTO COMPLETE BIPARTITE SUBGRAPHS

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In this note we prove a theorem on decompositions of complete graphs into edge-disjoint complete bipartite subgraphs. For a finite complete graph K_n such a decomposition contains at least $n - 1$ graphs; this was proved by R. L. Graham and H. O. Pollak [1] and later a simpler proof was given by H. Tverberg [2]. In [2] the author also suggested to study the infinite case.

Theorem. *Let $\mathfrak{p}$ be a transfinite cardinal number, $\mathfrak{q} = \exp \mathfrak{p}$, and let $\mathfrak{r}$ be the cardinality of the set of all subsets of a set of cardinality $\mathfrak{p}$ which have cardinalities less than $\mathfrak{p}$. Let $K(\mathfrak{q})$ be the complete graph with the vertex set of cardinaliy $\mathfrak{q}$. Then there exists a set of complete bipartite subgraphs of $K(\mathfrak{q})$ which has cardinality $\mathfrak{r}$ and possesses the property that each edge of $K(\mathfrak{q})$ belongs to exactly one graph of this set.*

Proof. Let U be a set of cardinality $\mathfrak{p}$, let $\mathscr{P}(U)$ be the set of all subsets of U, let $\mathscr{P}_0(U)$ be the set of all subsets of U which have cardinalities less than $\mathfrak{p}$. We have $|\mathscr{P}(U)| = \mathfrak{q}$, $|\mathscr{P}_0(U)| = \mathfrak{r}$. The vertex set of $K(\mathfrak{q})$ may be identified with $\mathscr{P}(U)$; thus the vertices of $K(\mathfrak{q})$ are subsets of U.

Consider a well-ordering $<$ of the set U whose ordinal number is the least ordinal number of cardinality $\mathfrak{p}$. For each $x \in U$ let $J(x) = \{y \in U \mid y < x\}$. Now let $a \in U$, $M \subseteq J(a)$. Denote $\mathscr{A}(M, a) = \{X \in P(U) \mid X \cap J(a) = M\}$ and further, $\mathscr{A}_0(M, a) = \{X \in \mathscr{A}(M, a) \mid a \notin X\}$ and $\mathscr{A}_1(M, a) = \{X \in \mathscr{A}(M, a) \mid a \in X\}$. Then $G(M, a)$ will be the graph whose vertex set is $\mathscr{A}(M, a)$ and in which two vertices are adjacent if and only if one is in $\mathscr{A}_0(M, a)$ and the other is in $\mathscr{A}_1(M, a)$; it is evidently a complete bipartite graph.

Let e be an edge of $K(\mathfrak{q})$; denote its end vertices by C, D. The vertices C, D are subsets of U. As C, D are different sets, their symmetric difference is non-empty. As $<$ is a well-ordering, there exists a uniquely determined element a which is the least element of this symmetric difference. We have $C \cap J(a) = D \cap J(a)$; otherwise

$J(a)$ would contain an element of the symmetric difference of C and D, which is not possible. Denote $M = C \cap J(a)$; then $C \in \mathscr{A}(M, a)$, $D \in \mathscr{A}(M, a)$. As a belongs to the symmetric difference of C and D, exactly one of the sets C, D contains a and thus one of them belongs to $\mathscr{A}_0(M, a)$ and the other to $\mathscr{A}_1(M, a)$; the edge e belongs to $G(M, a)$. We have proved that each edge of $K(\mathfrak{q})$ belongs to at least one of the graphs $G(M, a)$.

Now suppose that there exist two graphs $G(M_1, a_1)$, $G(M_2, a_2)$ with a common edge e and such that either $M_1 \neq M_2$, or $a_1 \neq a_2$. Let again C, D be the end vertices of e. Then both C, D belong to $\mathscr{A}(M_1, a_1) \cap \mathscr{A}(M_2, a_2)$, i.e. $C \cap J(a_1) = D \cap \cap J(a_1) = M_1$, $C \cap J(a_2) = D \cap J(a_2) = M_2$. If $a_1 \neq a_2$, we may suppose without loss of generality that $a_1 < a_2$. As e is an edge of $G(M_1, a_1)$, one of the sets C, D belongs to $\mathscr{A}_0(M_1, a_1)$ and the other to $\mathscr{A}_1(M_1, a_1)$; this implies that exactly one of the sets C, D contains a_1 and hence also exactly one of the sets $C \cap J(a_2)$, $D \cap J(a_2)$ contains a_1. But then $C \cap J(a_2) \neq D \cap J(a_2)$, which is a contradiction. Thus we must have $a_1 = a_2$. But then we have $M_1 = C \cap J(a_1) = C \cap J(a_2) = M_2$, which is a contradiction. We have proved that each edge of $K(\mathfrak{q})$ belongs to exactly one of the graphs $G(M, a)$ for $a \in U$, $M \subseteq J(a)$.

As the ordinal number of $<$ is the least ordinal number of cardinality p, the set $J(a) \in \mathscr{P}_0(U)$ for each $a \in U$ and also $M \in \mathscr{P}_0(U)$ for $M \subseteq J(a)$. Thus the cardinality of the set of all graphs $G(M, a)$ is $\mathfrak{r} \cdot \mathfrak{p} = \mathfrak{r}$ (because obviously $\mathfrak{r} \geqq \mathfrak{p}$), which was to be proved.

Remark. In the proof of this theorem, Axiom of Choice was used (when the existence of the well-ordering of U was assumed).

If $\mathfrak{p} = \aleph_0$, then $\mathfrak{q} = \mathfrak{c}$ (the power of continuum) and $\mathfrak{r} = \aleph_0$. Thus we have a corollary.

Corollary. *Let $K(\mathfrak{c})$ be a complete graph with the vertex set of the power of continuum. Then there exists a countable set of complete bipartite subgraphs of $K(\mathfrak{c})$ with the property that each edge of $K(\mathfrak{c})$ belongs to exactly one graph of this set.*

References

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[2] *H. Tverberg:* On the decomposition of K_n into complete bipartite graphs. J. Graph Theory 6 (1982), 493—494.

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