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ON THE LATTICE OF SEMISIMPLE CLASSES OF LINEARLY ORDERED GROUPS

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Radical classes and semisimple classes of linearly ordered groups were studied by C. G. Chehata and R. Wiegandt [1]. The author [2] and G. Pringerová [4], [5] investigated radical classes and semisimple classes of abelian linearly ordered groups.

In the author's paper [3] some basic properties of the lattice $\mathscr{R}$ of all radical classes of linearly ordered groups were established. It was proved that $\mathscr{R}$ fails to be modular and has no atoms. Also it was shown that for each $X \in \mathscr{R}$ distinct from the least element R_0 of $\mathscr{R}$ there is a chain $C \subset [R_0, X]$ of principal elements of $\mathscr{R}$ such that $R_0 \notin C$, $\inf C = R_0$ and C is a proper class. The greatest element of $\mathscr{R}$ is inaccessible by means of chains of principal elements of $\mathscr{R}$.

In the present paper analogous questions for the lattice $\mathscr{R}$ of all semisimple classes of linearly ordered groups will be investigated. From [1], Thms. 3, 5 it follows that there exists a dual isomorphism φ of the lattice $\mathscr{R}$ onto $\mathscr{R}_s$. Hence, in view of the results of [3], the lattice $\mathscr{R}_s$ is not modular and has no dual atoms. It will be shown below that the lattice $\mathscr{R}_s$ has no atoms; thus $\mathscr{R}$ has no dual atoms.

In view of the quoted results of [3] concerning the principal radical classes the natural question arises whether for a principal radical class X the corresponding semisimple class $\varphi(X)$ must also be principal. (If this were be valid, then from the theorems of [3] concerning the principal elements of the lattice $\mathscr{R}$ we could immediately obtain the corresponding dual theorems concerning the principal elements of the lattice $\mathscr{R}_s$.) It will be proved that the answer to this question is negative: if X is principal, then $\varphi(X)$ fails to be principal.

1. PRELIMINARIES

Small greek letters will denote ordinals (if not otherwise stated). A collection C will be said to be proper if there exists an injective mapping of the class of all cardinals into C.

Let $\mathscr{G}$ be the class of all linearly ordered groups. When considering a subclass X

of $\mathscr{G}$ we always assume that X is closed with respect to isomorphisms and that $\{0\} \in X$.

Let $X \subseteq \mathscr{G}$. Let us denote by

Hom X – the class of all homomorphic images of linearly ordered groups belonging to X;

Sub X – the class of all convex subgroups of linearly ordered groups belonging to X;

Ext X – the class of all linearly ordered groups G having the property that there exists an ascending chain of normal convex subgroups of G

$$\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots (\alpha < \delta)$$

such that (i) $\bigcup_{\alpha<\delta} G_\alpha = G$, and (ii) for each $\beta < \delta$, the linearly ordered group $G_\beta / \bigcup_{\gamma<\beta} G_\gamma$ belongs to X;

co-Ext X – the class of all linearly ordered groups G having the property that there exists a descending chain of normal convex subgroups of G

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_\alpha \supseteq \ldots (\alpha < \delta)$$

such that (i) $\bigcap_{\alpha<\delta} G_\alpha = \{0\}$, and (ii) for each $\beta < \delta$, the linearly ordered group $(\bigcap_{\gamma<\beta} G_\gamma)/G_\beta$ belongs to X.

For each ordinal $\varkappa$ we define the class $\mathrm{Ext}_\varkappa X$ as follows. We put $\mathrm{Ext}_1 X = \mathrm{Ext}\, X$; for $\varkappa > 1$ we set

$$\mathrm{Ext}_\varkappa X = \mathrm{Ext}\,(\textstyle\bigcup_{\alpha<\varkappa} \mathrm{Ext}_\alpha X)\,.$$

Further, we denote

$$\mathrm{ext}\, X = \textstyle\bigcup_\gamma \mathrm{Ext}_\gamma X\,,$$

where γ runs over the class of all ordinals.

Similarly, we put $\text{co-Ext}_1 X = \text{co-Ext}\, X$ and for $\varkappa > 1$ we set

$$\text{co-Ext}_\varkappa X = \text{co-Ext}\,(\textstyle\bigcup_{\alpha<\varkappa} \text{co-Ext}_\alpha X)\,;$$

we denote

$$\text{co-ext}\, X = \textstyle\bigcup_\gamma \text{co-Ext}_\gamma X$$

(with γ running over the class of all ordinals).

The class X is *a radical class* of linearly ordered groups, if

$$\mathrm{Hom}\, X = X = \mathrm{Ext}\, X\,.$$

X is said to be *a semisimple class* of linearly ordered groups if

$$\mathrm{Sub}\, X = X = \text{co-Ext}\, X\,.$$

(Cf. [1].)

Let $\mathscr{R}$ and $\mathscr{R}_s$ be the collection of all radical classes of linearly ordered groups or the collection of all semisimple classes of linearly ordered groups, respectively. Both $\mathscr{R}$

and $\mathscr{R}_s$ are partially ordered by inclusion. Then $\mathscr{R}$ and $\mathscr{R}_s$ are complete lattices; the lattice operations in them will be denoted by $\wedge$, $\vee$.

The operation $\wedge$ in $\mathscr{R}$ and $\mathscr{R}_s$ coincides with the operation of forming the intersection of classes. In [3] (Thm. 2.3) it was shown that, whenever $J \neq \emptyset$ is a class and X_j is a radical class for each $j \in J$, then

$$\bigvee_{j \in J} X_j = \text{ext} \bigcup_{j \in J} X_j .$$

For the analogous result concerning the operation $\vee$ in the lattice $\mathscr{R}_s$ cf. Thm. 2.2 below.

Let $X \subseteq \mathscr{G}$. The intersection of all radical classes (or semisimple classes) Y with $X \subseteq Y$ will be denoted by $T(X)$ or $T_s(X)$. If $G \in \mathscr{G}$ and X is the class of all $H \in \mathscr{G}$ such that either H is isomorphic to G or $H = \{0\}$, then we write also $T(X) = T(G)$ or $T_s(X) = T_s(G)$ and put $\mathscr{R}_p = \{T(G) : G \in \mathscr{G}\}$, $\mathscr{R}_{sp} = \{T_s(G) : G \in \mathscr{G}\}$. $\mathscr{R}_p$ and $\mathscr{R}_{sp}$ is the collection of all principal radical classes or the collection of all principal semisimple classes, respectively.

2. THE OPERATION $\vee$ IN $\mathscr{R}$.

2.1. Theorem. *Let $X \subseteq \mathscr{G}$. Then $T_s(X)$* = co-ext Sub X.

Proof. According to the definition of a semisimple class we have co-Ext Sub $X \subseteq T_s(X)$ and hence by transfinite induction we infer that co-ext Sub $X \subseteq T_s(X)$. For proving the relation $T_s(X) \subseteq$ co-ext Sub X we have to verify that the class $Y =$ co-ext Sub X fulfils the conditions (a*) co-Ext $Y \subseteq Y$, and (b*) Sub $Y \subseteq Y$. The validity of (a*) is obvious. When investigating the validity of (b*) for the class Y we proceed as follows.

a) Let $G \in Y$ and let $H \neq \{0\}$ be a convex subgroup of G. There is an ordinal $\varkappa$ such that $G \in$ co-Ext$_\varkappa$ Sub X. Hence there is a descending chain of normal convex subgroups

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_\alpha \supseteq \ldots \ (\alpha < \delta)$$

of G such that (i) $\bigcap_{\alpha<\delta} G_\alpha = \{0\}$, and (ii) for each $\beta < \delta$, $(\bigcap_{\gamma<\beta} G_\gamma)/G_\beta$ belongs to

$$\bigcup_{\alpha<\varkappa} \text{co-Ext}_\alpha \text{ Sub } X .$$

Put $H_\alpha = G_\alpha \cap H$ for each $\alpha < \delta$. Then $\{H_\alpha\}_{\alpha<\delta}$ is a descending chain of convex normal subgroups of H and from (i) we infer that $\bigcap_{\alpha<\delta} H_\alpha = \{0\}$ is valid. Let τ be the first ordinal with $H \supseteq G_\tau$. For $\beta < \tau$, the linearly ordered group $(\bigcap_{\gamma<\beta} H_\gamma)/H_\beta$ is trivial; if $\beta < \tau$, then

$$(\bigcap_{\gamma<\beta} H_\gamma)/H_\beta = (\bigcap_{\gamma<\beta} G_\gamma)/G_\beta .$$

For $\beta = \tau$ we have

$$(\bigcap_{\gamma<\beta} H_\gamma)/H_\beta \in \text{Sub } (\bigcap_{\gamma<\beta} G_\gamma)/G_\beta \subseteq \text{Sub} \bigcup_{\alpha<\varkappa} \text{co-Ext}_\alpha \text{ Sub } X = $$
$$= \bigcup_{\alpha<\varkappa} \text{Sub co-Ext}_\alpha \text{ Sub } X .$$

Thus it suffices to verify that for each ordinal $\mu < \varkappa$ we have

$$\text{Sub co-Ext}_\mu \text{ Sub } X \subseteq \text{co-Ext}_\mu \text{ Sub } X . \tag{3.1}$$

b) We prove (3.1) by transfinite induction. If $\mu = 1$, then the validity of (3.1) can be easily established (by using analogous arguments as we did in part a) of this proof). Let $\mu > 1$. Assume that (3.1) is valid for each ordinal less than μ. Put co-Ext$_\mu$ Sub $X =$ $= Z$. Then

$$Z = \text{co-Ext} (\bigcup_{\alpha<\mu} \text{co-Ext}_\alpha \text{ Sub } X) ,$$

hence

$$\text{Sub } Z \subseteq \text{co-Ext} (\bigcup_{\alpha<\mu} \text{Sub co-Ext}_\alpha \text{ Sub } X) \subseteq$$
$$\subseteq \text{co-Ext} (\bigcup_{\alpha<\mu} \text{co-Ext}_\alpha \text{ Sub } X) \subseteq Z ,$$

which completes the proof.

From 2.1 we obtain as a corollary:

2.2. Theorem. *Let $J \neq \emptyset$ be a class and for each $j \in J$ let $X_j \in \mathscr{R}_s$. Then*

$$\bigvee_{j\in J} X_j = \text{co-ext} \bigcup_{j\in J} X_j .$$

We deduce some further consequences of 2.1 (these will be applied in § 3 below).

2.3. Lemma. *Let $\varkappa$ be an ordinal, $\varkappa > 1$, $X \subseteq \mathscr{G}$, $\{0\} \neq G \in$ co-Ext$_\varkappa$ Sub X. Then there is an ordinal $\tau_1 < \varkappa$ and a convex normal subgroup K_1 of G such that $K_1 \neq G$ and $G/K_1 \in$ co-Ext$_{\tau_1}$ Sub X.*

Proof. If there exists $\tau_1 < \varkappa$ such that $G \in$ co-Ext$_{\tau_1}$ Sub X, then we put $K_1 = \{0\}$. Now assume that

$$G \notin \text{co-Ext}_{\tau_1} \text{ Sub } X \quad \text{for each} \quad \tau_1 < \varkappa . \tag{$*$}$$

We have

$$G \in \text{co-Ext} (\bigcup_{\tau<\varkappa} \text{co-Ext}_\tau \text{ Sub } X) .$$

Hence there exists a descending chain of convex normal subgroups of G

$$G = G_1 \supseteq \ldots \supseteq G_\alpha \supseteq \ldots \quad (\alpha < \delta)$$

such that, for each $\beta < \delta$,

$$(\bigcap_{\alpha<\beta} G_\alpha)/G_\beta \in \text{co-Ext}_{\tau(\beta)} \text{ Sub } X ,$$

where $\tau(\beta) < \varkappa$. In view of $(*)$ we must have $2 < \delta$. It suffices to put $K_1 = G_2$, $\tau_1 = \tau(2)$.

2.4. Lemma. *Let the same assumptions as in* 2.3 *be satisfied. Let* τ_1 *and* K_1 *be as in* 2.3. *Assume that* $\tau_1 > 1$. *Then there is* $\tau_2 < \tau_1$ *and a convex subgroup* K_2 *of* G *with* $K_2 \neq G$ *such that* $G/K_2 \in \text{co-Ext}_{\tau_2} \text{Sub } X$.

Proof. By applying 2.3 we infer that there is $\tau_2 < \tau_1$ and a convex normal subgroup K_2' of G/K_1 such that $K_2' \neq G/K_1$ and $(G/K_1)/K_2' \in \text{co-Ext}_{\tau_2} \text{Sub } X$. There is a convex normal subgroup K_2 of G with $K_2 \neq G$ such that G/K_2 is isomorphic to $(G/K_1)/K_2'$. Hence $G/K_2 \in \text{co-Ext}_{\tau_2} \text{Sub } X$.

From 2.3 and 2.4 we infer:

2.5. Corollary. *Let* $X \subseteq \mathscr{G}$, $\{0\} \neq G \in \text{co-ext Sub } X$. *Then there is a normal subgroup* K *of* G *with* $K \neq G$ *such that* $G/K \in \text{Sub } X$.

3. NONEXISTENCE OF ATOMS IN $\mathscr{R}_s$

The trivial variety R_0 is the least element in both the lattices $\mathscr{R}$ and $\mathscr{R}_s$. In this section it will be shown that if $X \in \mathscr{R}_s$ and $X \neq R_0$, then the interval $[R_0, X]$ of $\mathscr{R}_s$ is a proper collection; in particular, $\mathscr{R}_s$ has no atoms. The construction for proving this is analogous to that applied in [2]; cf. also [5]. We use the same notations concerning lexicographic products of linearly ordered groups as in [2].

Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the first ordinal having the property that the set of all ordinals less than $\omega(\alpha)$ has the cardinality α. Let $I(\alpha)$ be the linearly ordered set dual to $\omega(\alpha)$.

Let $G \in \mathscr{G}$, $G \neq \{0\}$ and let α be a cardinal with $\alpha > \text{card } G$. We put

$$G_\alpha^1 = \Gamma_{i \in I(\alpha)} G_i ,$$

where G_i is isomorphic to G for each $i \in I$. Next, let G_α^2 be the subgroup of G_α^1 consisting of all $g \in G_\alpha^1$ such tha the set $\{i \in I(\alpha) : g(i) \neq 0\}$ is finite.

From the construction of G_α^2 we immediately obtain:

3.1. Lemma. *Let* $K \in \text{Sub}\{G_\alpha^2\}$, $K \neq \{0\}$. *Then* $\text{card } K = \alpha$.

3.2. Lemma. $G_\alpha^2 \in \text{co-Ext}\{G\}$. *If* β *is a cardinal with* $\beta > \alpha$, *then* $G_\beta^2 \in \text{co-Ext}\{G_\alpha^2\}$.
From 3.2 and 2.1 we conclude:

3.3. Lemma. $G_\alpha^2 \in T_s(G)$. *If* $\beta > \alpha$, *then* $G_\beta^2 \in T_s(G_\alpha^2)$.

3.4. Lemma. $G \notin T_s(G_\alpha^2)$. *If* $\beta > \alpha$, *then* $G_\alpha^2 \notin T_s(G_\beta^2)$.

Proof. This is a consequence of 3.1, 2.5 and 2.1.

From 3.3 and 3.4 we infer:

3.5. Lemma. $T_s(G^2_\alpha) < T_s(G)$. *If* $\beta > \alpha$, *then* $T_s(G^2_\beta) < T_s(G^2_\alpha)$.

3.6. Theorem. *Let* $X \in \mathcal{R}_s, X \neq R_0$. *Then there exists* $C \subset \mathcal{R}_{sp}$ *such that* (i) C *is a chain*, (ii) $R_0 \notin C$, (iii) C *is a proper collection*, (iv) $\inf C = R_0$, (v) $C \subset [R_0, X]$.

Proof. There exists $G \in X$ with $G \neq \{0\}$. Let C be the collection of all semisimple classes $T_s(G^2_\alpha)$, where α runs over the class of all cardinals α such that $\alpha >$ card G. Then $R_0 \notin C \subset \mathcal{R}_{sp}$. In view of 3.5, (i) and (iii) are valid. Assume that there is $H \neq \{0\}$ such that $H \in T_s(G^2_\alpha)$ for each $\alpha >$ card G. Hence in view of 3.1, 2.5 and 2.1 we have card $H \geqq \alpha$ for each $\alpha >$ card G, which is impossible. Therefore $\inf C = R_0$. The validity of (v) is a consequence of the fact that $G \in X$.

3.7. Corollary. *The lattice* $\mathcal{R}_s$ *has no atoms.*

Foı the analogous result concerning the lattice of semisimple classes of abelian linearly ordered groups cf. [5].

4. THE RELATION BETWEEN SEMISIMPLE CLASSES AND RADICAL CLASSES

Let us recall the following definitions introduced in [1].

Let $G \in \mathcal{G}$. A convex subgroup G_1 of G is said to be accessible in G if there are convex subgroups $G_2, \ldots, G_n$ of G such that

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n = G$$

and G_i is a normal subgroup of G_{i+1} for $i = 1, 2, \ldots, n - 1$.

Let $X \in \mathcal{R}$. The class of all $G \in \mathcal{G}$ such that no nonzero accessible convex subgroup of G belongs to X will be denoted by sX.

Let $Y \in \mathcal{R}_s$. The class of all $G \in \mathcal{G}$ having the property that no nonzero homomorphic image of G belongs to Y will be denoted by uY.

4.1. Proposition. (Cf. [1]. Propos. 7 and 9.) *Let* $X \in \mathcal{R}$ *and* $Y \in \mathcal{R}_s$. *Then*

(i) $sX \in \mathcal{R}_s$,

(ii) $uY \in \mathcal{R}$,

(iii) $usX = X$ and $suY = Y$.

Consider the mapping of the collection $\mathcal{R}$ into $\mathcal{R}_s$ defined by $X \to sX$ for each $X \in \mathcal{R}$. According to the definition of s and u, from $X_1, X_2 \in \mathcal{R}$, $X_1 \leqq X_2$ we conclude $sX_1 \geqq sX_2$, and similarly $Y_1, Y_2 \in \mathcal{R}_s$, $Y_1 \leqq Y_2$ implies that $uY_1 \geqq uY_2$. Thus in view of 4.1 (iii) we obtain the following result:

4.2. Lemma. *The mapping* s *is a dual isomorphism of the lattice* $\mathcal{R}$ *onto the lattice* $\mathcal{R}_s$ *and* $u = s^{-1}$.

Hence to each theorem concerning merely lattice properties of $\mathscr{R}$ there corresponds a dual theorem concerning $\mathscr{R}_s$, and conversely. For example, the fact that the lattice $\mathscr{R}$ has no atoms [3] implies:

4.3. Proposition. *The lattice $\mathscr{R}_s$ has no dual atoms.*

Similarly, 3.7 yields:

4.4. Proposition. *The lattice $\mathscr{R}$ has no dual atoms.*

Also, since the notion of modularity is self-dual and since $\mathscr{R}$ is not modular [3] we get:

4.5. Proposition. *The lattice $\mathscr{R}_s$ is not modular.*

Now we can ask whether the above correspondences concern also those properties of $\mathscr{R}$ which are expressed in terms of principal elements, i.e., whether for each principal element X of $\mathscr{R}$ the semisimple class sX is principal, and conversely. It will be shown below that the answer to this question is 'No'.

Let α be an infinite cardinal. Let $I(\alpha)$ be as in § 3 and let $I'(\alpha)$ be the linearly ordered set dual to $I(\alpha)$. For $G \in \mathscr{G}$ we put

$$G_\alpha = \Gamma_{i \in I'(\alpha)} G_i ,$$

where each G_i is isomorphic to G. Taking into account the structure of G_α we obtain:

4.6. Lemma. *Let $G \in \mathscr{G}, G \neq \{0\}$, $\alpha >$ card G. Let K be a convex normal subgroup of G_α, $K \neq G_\alpha$. Then* card $(G_\alpha/K) = \alpha$.

4.7. Lemma. *$\mathscr{R}_{sp}$ has no maximal element.*

Proof. Let $G \in \mathscr{G}$, $G \neq \{0\}$. Let α be a cardinal, $\alpha >$ card G. Put $H = G \circ G_\alpha$. Since $G \in \operatorname{Sub}\{H\}$ we have $G \in T_s(H)$, hence $T_s(G) \leqq T_s(H)$. From 4.6, 2.1 and 2.5 it follows that H does not belong to $T_s(G)$, therefore $T_s(G) < T_s(H)$, which completes the proof.

Let I and J be linearly ordered sets. We denote by $I \circ J$ the set of all pairs (i, j) with $i \in I, j \in J$ which is linearly ordered as follows: for $(i_1, j_1), (i_2, j_2) \in I \circ J$ we put $(i_1, j_1) < (i_2, j_2)$ if either $j_1 < j_2$, or $j_1 = j_2$ and $i_1 < i_2$.

4.8. Proposition. *Let X be a principal radical class. Then the semisimple class sX fails to be principal.*

Proof. Let $G \in \mathscr{G}$ and let X be the principal radical class generated by G. If $G = \{0\}$, then $sX = \mathscr{G}$ and hence in view of 4.7, sX fails to be principal. Let $G \neq \{0\}$. Assume that there is $H \in \mathscr{G}$ such that $sX = T_s(H)$. Then we must have $H \neq \{0\}$. Let $K \in \mathscr{G}$, $K \neq \{0\}$ and let α be a cardinal with $\alpha > \max\{\text{card } G, \text{card } H\}$. Put (under the above notations)

$$M(\alpha) = I(\alpha) \circ I'(\alpha)\,,$$
$$K_{[\alpha]} = \Gamma_{i \in M(\alpha)} K_i\,,$$

where each K_i is isomorphic to K. For each nonzero convex subgroup K_1 of $K_{[\alpha]}$ we have card K_1 > card G, hence in view of Lemma 4.1 in [3], K_1 does not belong to $R = T(G)$. Therefore $K_{[\alpha]} \in sX$. On the other hand, if K_2 is a nonzero homomorphic image of $K_{[\alpha]}$, then card K_2 > card H. Hence in view of 2.1 and 2.5, $K_{[\alpha]}$ does not belong to $T_s(H)$. Therefore the relation $sX = 1_s(H)$ cannot hold.

The proof of the following proposition is analogous to that of 4.8; it will be omitted.

4.9. Proposition. Let Y be a principal semisimple class. Then the radical class uY fails to be principal.

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