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FINITE GRAPHS AND THEIR SPANNING TREES

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Let us begin with the formula

$$\sum_{i=0}^{n-4} i!\binom{n-4}{i}(i+4)\,n^{n-i-5} = n^{n-4}\,, \tag{1}$$

valid for every integer $n \geqq 4$. This formula can be obtained as a by-product in solving the following problem concerning the graphs: Let a complete graph $\mathscr{U}_n$ with n vertices ($n \geqq 4$) and two of its edges h_1, h_2 without common vertices be given. We are to determine the number of all spanning trees of the graph $\mathscr{U}_n$ which contain both h_1 and h_2. Under a *spanning tree* we understand the maximal tree subgraph of a given graph. Some authors (for example O. ORE, [5]) use the term *skeleton* or *scaffolding* for denoting this concept, others use *frame* etc. Note also that the notation $\mathscr{G} = [U, H]$ means that the graph $\mathscr{G}$ has the vertex set U and the edge set H.

Theorem 1. *For every integer $n \geqq 4$, equality* (1) *holds.*

Proof. Let us choose a complete graph $\mathscr{U}_n$ with n vertices and two of its edges h_1 and h_2 which do not have a common vertex. Using two different procedures we are going to determine the number of those spanning trees of the graph $\mathscr{U}_n$ each of which contains both edges h_1 and h_2.

In the first consideration, let us proceed as follows: Obviously, each such spanning tree contains a unique simple path $C(h_1, h_2)$ with h_1 as the first edge and h_2 as the last one. Denote $i + 4$ the number of vertices on $C(h_1, h_2)$. Thus, let us first inquire how many simple paths $C(h_1, h_2)$ can be constructed in $\mathscr{U}_n$. There are four ways of choosing the first two and last two vertices on $C(h_1, h_2)$. In the graph $\mathscr{U}_n$ there still remain $n - 4$ vertices which are to be ordered in a sequence with i terms; this can be accomplished in $i!\binom{n-4}{i}$ ways. Consequently, the simple path $C(h_1, h_2)$ can be constructed in $4i!\binom{n-4}{i}$ ways. The vertices and edges of the simple path

$C(h_1, h_2)$ constitute a tree, and from the paper [7] it is known in how many ways it can be completed to a spanning tree[1]) of the graph $\mathscr{U}_n$. It turns out that this can be done in $(i + 4)\, n^{n-i-5}$ ways. Thus, we see that the number $4 \sum_{i=0}^{n-4} i! \binom{n-4}{i} (i + 4) \,.\, n^{n-i-5}$ is the amount of the sought spanning trees.

Next, let us turn to the second consideration. As known, the graph $\mathscr{U}_n$ has totally n^{n-2} spanning trees. Denote $k(h_1, h_2)$ and $k(\text{non } h_1, h_2)$ and $k(h_1, \text{non } h_2)$ and $k(\text{non } h_1, \text{non } h_2)$ the number of these spanning trees each of which contains both h_1 and h_2, does not contain h_1 but contains h_2, contains h_1 but does not contain h_2, contains neither h_1 nor h_2. Thus, we have $n^{n-2} = k(h_1, h_2) + k(\text{non } h_1, h_2) + k(h_1, \text{non } h_2) + k(\text{non } h_1, \text{non } h_2)$. Because $\mathscr{U}_n$ is complete, we have $k(\text{non } h_1, h_2) = k(h_1, \text{non } h_2)$. Consequently,

$$(2) \qquad k(h_1, h_2) = n^{n-2} - 2k(\text{non } h_1, h_2) - k(\text{non } h_1, \text{non } h_2)\,.$$

The numbers $k(\text{non } h_1, h_2)$ and $k(\text{non } h_1, \text{non } h_2)$ may be found by a well known determinant method (see [1] and [3]). We assume here that the vertices of the graph $\mathscr{U}_n$ have been enumerated by integers $1, 2, 3, \ldots, n$ so that, for example, h_1 joins 1 and 2 while h_2 joins $n - 1$ and n.

In order to establish $k(\text{non } h_1, h_2)$, denote $\mathscr{G}_1$ the graph obtained from $\mathscr{U}_n$ by removing the edge h_1. Next, replace each undirected edge xy of the graph $\mathscr{G}_1$ by a pair of edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$; thereby, we get a directed graph $\overrightarrow{\mathscr{G}_1}$. It is clear that $k(\text{non } h_1, h_2)$ is equal to the number of W-bases[2]) of the graph $\overrightarrow{\mathscr{G}_1}$ with sources $n - 1$ and n. Thus, we construct an $n \times n$ square matrix $A = (a_{ij})$ such that $a_{12} = a_{21} = 0$, while $a_{ij} = -1$ for the remaining $i \neq j$. The elements of the main diagonal are $n - 1$ with exception of $a_{11} = a_{22} = n - 2$. The principal minor obtained by deleting the last two columns and rows of the matrix A is then equal to the number $k(\text{non } h_1, h_2)$. The computation yields $k(\text{non } h_1, h_2) = 2(n - 2)\, n^{n-4}$.

The determining of the number $k(\text{non } h_1, \text{non } h_2)$ is analogous. Let $\mathscr{G}_2$ arise from $\mathscr{G}_1$ by removing the edge h_2. Replacing again each edge of the graph $\mathscr{G}_2$ by a pair of directed edges we get a graph $\overrightarrow{\mathscr{G}_2}$. It can be readily seen that $k(\text{non } h_1, \text{non } h_2)$ is equal to the number of connected W-bases of the graph $\overrightarrow{\mathscr{G}_2}$, where the source of these W-bases can be chosen arbitrarily in $\overrightarrow{\mathscr{G}_2}$. Thus, the consideration reduces to the computation of an $(n - 1)$-st degree determinant as in the preceding paragraph. Following this trend of thought, we get $k(\text{non } h_1, \text{non } h_2) = (n - 2)^2\, n^{n-4}$.

Substituting now into equation (2), we obtain $k(h_1, h_2) = 4n^{n-4}$. A comparison with the result obtained above yields relation (1) after cancelling a common factor. This finishes the proof.

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[1]) Theorem 4 in paper [7] reads as follows: Let $\mathscr{U}_n$ be a complete graph with n vertices, and let $\mathscr{S} = [U, H]$ be its subgraph with $|U| = s$. Furthermore, let $\mathscr{S}$ be a tree. Then the graph $\mathscr{U}_n$ contains exactly sn^{n-s-1} spanning trees each of which contains $\mathscr{S}$ as its subgraph.

[2]) The concept of a *W-base* is defined in paper [3].

It is evident that the method described above permits us to derive a series of further formulas analogous to (1), if we replace $\mathscr{U}_n$ by a different type of graph or if we require that the spanning trees contain some other chosen elements instead of h_1 and h_2. The author wishes also to note that formula (1) may easily be proved by elementary methods. An elementary proof, originated by J. Kaucký, was communicated to him by A. Rosa. Here, we consider the function

$$f(i) = -\binom{n-4}{i} i!\, n^{n-i-4}$$

defined for $i = 0, 1, 2, 3, \ldots$, and construct the difference

$$\Delta f(i) = f(i+1) - f(i) = i! \binom{n-4}{i} (i+4)\, n^{n-i-5}\,.$$

Thus, we have

$$\sum_{i=0}^{n-4} i! \binom{n-4}{i} (i+4)\, n^{n-i-5} = \sum_{i=0}^{n-4} \Delta f(i) = f(n-3) - f(0) = n^{n-4}\,.$$

Now, let us turn our attention to further problems. First, let us prove an auxiliary theorem.

Lemma 1. *Let x be a vertex of a finite connected graph $\mathscr{G} = [U, H]$ with $H \neq \emptyset$. Let $\mathscr{G}_i = [U_i, H_i]$ $(i = 1, 2, \ldots, r)$ be all lobe graphs[3]) of $\mathscr{G}$ such that $x \in U_i$. Furthermore, let A be a set of edges ending in x such that $A \cap H_i \neq \emptyset$ for all $i = 1, 2, \ldots, r$, and let B be the set of all remaining edges ending in x. Then the graph $\mathscr{G}^{(1)} = [U, H - B]$ is connected.*

Proof. Suppose that $\mathscr{G}^{(1)}$ is not connected. Thus, there exist two vertices u, v which cannot be connected by a path. However, in the graph $\mathscr{G}$ there exists a path S beginning at u and ending at v. Let $\varphi(S, B)$ be the number of edges from the path S which belong to B. Choose S such that $\varphi(S, B)$ is minimal. If several such paths exist choose an arbitrary one from them and denote it by $S_{\min}$. In $S_{\min}$ we choose an edge $h_1 \in B$ (assume that it is the m-th edge of this path) and find the lobe graph $\mathscr{G}_i$ containing h_1. Because we have $A \cap H_i \neq \emptyset$ we can find an edge $h_2 \in A$ such that $h_2 \in H_i$. Since both h_1 and h_2 belong to the same lobe graph $\mathscr{G}_i$, we can construct a circuit $\mathscr{O}$ containing these two edges. Obviously, the circuit $\mathscr{O}$ contains a unique edge from B. Next, construct a new path S^* between u and v as follows: We remove the m-th edge of the path $S_{\min}$ and instead of it we introduce into S^* the edges and vertices of the circuit $\mathscr{O}$ (except h_1). It is clear that $\varphi(S^*, B) = \varphi(S_{\min}, B) - 1$, which is a contradiction. This concludes the proof.

[3]) The definition of a lobe graph may be found in the book [5].

Now, let us still introduce two notations. Let $\varrho(x, \mathscr{G})$ denote the degree of a vertex x in the graph $\mathscr{G}$, and let $\alpha(x, \mathscr{G})$ be the number of all lobe graphs of $\mathscr{G}$ which contain the vertex x. In the bibliography the lobe graph concept is usually defined only for connected graphs with at least one edge. Then it is clear that $\alpha(x, \mathscr{G}) \leqq \varrho(x, \mathscr{G})$.

Theorem 2. *Let x be a vertex of a finite connected graph $\mathscr{G} = [U, H]$ with $H \neq \emptyset$. Furthermore, let s be a given positive integer. The necessary and sufficient condition for the existence of a spanning tree $\mathscr{K}$ of the graph $\mathscr{G}$ satisfying the condition $\varrho(x, \mathscr{K}) = s$ is*

$$\alpha(x, \mathscr{G}) \leqq s \leqq \varrho(x, \mathscr{G}) . \tag{3}$$

Proof. It can be easily verified that (3) is necessary. Actually, $\varrho(x, \mathscr{K}) > \varrho(x, \mathscr{G})$ cannot be true for any spanning tree $\mathscr{K} = [U, H^*]$. If we had $\varrho(x, \mathscr{K}) < \alpha(x, \mathscr{G})$, denote M the set of all edges from $\mathscr{K}$ ending in x. Then we could find a lobe graph $\mathscr{G}_0 = [U_0, H_0]$ such that $x \in U_0$ and $M \cap H_0 = \emptyset$. However, it is clear that $H^* \cap H_0$ is the set of edges of a spanning tree $\mathscr{K}_0$ of the graph $\mathscr{G}_0$, so that we can find an edge $h \in M$ satisfying simultaneously the condition $h \in H_0$ (contradiction).

Next, let us prove that condition (3) is sufficient. We construct an arbitrary set A described in Lemma 1 such that $|A| = s$. By Lemma 1 it follows that the graph $\mathscr{G}^{(1)}$ is connected. Thus, it remains to construct its spanning tree $\mathscr{K}_1 = [U, H_1]$ such that $A \subset H_1$. The construction can be carried out as follows: From all trees $\mathscr{S} = [U', H']$, where $U' \subset U$ and $A \subset H' \subset H$, we choose that which maximizes $|U'|$, and denote it by $\mathscr{S}_{\max} = [U'_{\max}, H'_{\max}]$. We are going to show that $U'_{\max} = U$. Actually, if a $y_1 \in U - U'_{\max}$ existed, choose $y_2 \in U'_{\max}$ and construct in $\mathscr{G}$ a path S between y_1 and y_2. On S we can find the last vertex belonging to $U - U'_{\max}$, say y_3. Consequently, the vertex y_4 following y_3 on S belongs to $U'_{\max}$. The graph $[U'_{\max} \cup \{y_3\}, H'_{\max} \cup \{y_3 y_4\}]$ is obviously a tree having more vertices than $S_{\max}$ (contradiction). Hence, the proof.

Finally, let us illustrate Theorem 2 by an example. If we choose a complete graph with n vertices ($n \geqq 2$) for $\mathscr{G}$, then, by Theorem 2, for every integer $s \in \langle 1, n - 1 \rangle$ and every vertex x a spanning tree $\mathscr{K}$ exists such that $\varrho(x, \mathscr{K}) = s$. A calculation shows that here there exist exactly $\binom{n-2}{s-1}(n - 1)^{n-s-1}$ such spanning trees – see also [2].

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Výtah

KONEČNÉ GRAFY A JEJICH KOSTRY

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V práci se nejprve ukazuje, že rovnici (1) a rovnice obdobné je možno dostat jako vedlejší výsledek, řeší-li se tato úloha: Je dán graf (vhodného typu) s n uzly a dvě jeho hrany h_1, h_2 bez společných uzlů; máme určit počet těch koster, jež obsahují h_1 i h_2. Dále je odvozena nutná a postačující podmínka k tomu, aby v konečném souvislém grafu $\mathcal{G}$ s daným uzlem x existovala kostra, ve které se stupeň uzlu x rovná danému přirozenému číslu s.

Резюме

КОНЕЧНЫЕ ГРАФЫ И ИХ ОСНОВЫ

ЙИРЖИ СЕДЛАЧЕК (Jiří Sedláček), Прага

В работе сначала показано, что уравнение (1) и аналогичные ему уравнения можно получить как второстепенный результат при решении следующей задачи: Дан граф (подходящего типа) с n вершинами и его два ребра h_1, h_2 без общих вершин. Требуется определить число всех основ, содержащих h_1 и h_2. Затем выведено необходимое и достаточное условие для того, чтобы в конечном связном графе $\mathcal{G}$ с данной вершиной x существовала основа, в которой степень вершины x равна заданному натуральному числу s.