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Časopis pro pěstování matematiky, Vol. 85 (1960), No. 2, 158--187

Persistent URL: http://dml.cz/dmlcz/108387

Časopis pro pěstování matematiky, roč. 85 (1960), Praha

ESTIMATES OF THE DISTANCE OF TWO SOLUTIONS BASED ON THE THEORY OF GENERALIZED DIFFERENTIAL EQUATIONS

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(Received May 16, 1959)

Estimates of the distance of two solutions based on the theory of generalized differential equations are presented. The new methods are compared with the classical ones.

Introduction. The presented paper is closely connected to that of J. KURZWEIL [1], where the foundations of the theory of generalized differential equations have been established and, especially, the theorem on the continuous dependence of solutions on a parameter has been proved for a rather general class of equations.

Let $x(t)$ be a vector solution of the classical equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)\,, \tag{0,1}$$

$f(x, t)$ being continuous. Then $x(t)$ fulfils the integral equation

$$x(t) - x(0) = \int_0^t f(x(\tau), \tau)\, \mathrm{d}\tau\,. \tag{0,2}$$

Let us denote $\int_0^t f(x, \tau)\, \mathrm{d}\tau = F(x, t)$. In the theory of generalized differential equations theorems are proved under assumptions concerning the function $F(x, t)$ and not $f(x, t) = \frac{\partial F}{\partial t}$; continuous partial derivative $\frac{\partial F}{\partial t}$ need not exist.

Several results contained in the present paper, relating to generalized differential equations, shall now be stated in a simplified form, for classical equations only.

Let the class $\varphi(K, \alpha, \beta)$ of vector functions $f(x, t)$ be defined as follows:

$f(x, t) \in \varphi$, if the following conditions are fulfilled for $(x, t) \in G$, where G is an open set of E_{n+1},

a) $|\int_{t_1}^{t_2} f(x, t)\, dt| \leqq K|t_2 - t_1|^\alpha$,

b) $|\int_{t_1}^{t_2} [f(x_2, t) - f(x_1, t)]\, dt| \leqq |x_2 - x_1|\, K|t_2 - t_1|^\beta$,

K, α, β being positive numbers, $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$, $0 \leqq t_1 \leqq \leqq t_2 \leqq T$.

Theorem 0,1. *Let* $f(x, t)$, $\bar{f}(x, t) \in \varphi(K, \alpha, \beta)$,

$$|\int_{t_1}^{t_2} f[(x, t) - \bar{f}(x, t)]\, dt| \leqq \omega(|t_2 - t_1|) .$$

Let $x(t)$ *(resp.* $y(t)$*) be a solution of the equation*

$$\frac{dx}{dt} = f(x, t) \tag{0,1}$$

resp.

$$\frac{dx}{dt} = \bar{f}(x, t) , \tag{0,4}$$

satisfying the initial conditions $x(0) = y(0) = x_0$. *Then for every positive integer* m *the following inequality is true:*

$$|x(T) - y(T)| \leqq \left[c_1\left(\frac{T}{m}\right)^{\alpha+\beta} + \omega\left(\frac{T}{m}\right)\right] . \left\{\left[1 + K\left(\frac{T}{m}\right)^\beta\right]^m - 1\right\} . \frac{1}{K\left(\frac{T}{m}\right)^\beta} \tag{0,5}$$

where c_1 *is a positive constant independent of* m.

In the special case when $f(x, t)$ is bounded on G,

$$|f(x_2, t) - f(x_1, t)| \leqq L|x_2 - x_1|$$

and

$$|f(x, t) - \bar{f}(x, t)| \leqq M ,$$

then by passing to the limit for $m \to \infty$ in (0,5), we obtain a well known estimate

$$|x(T) - y(T)| \leqq \frac{M}{L}\, (e^{LT} - 1) .$$

If equation (0,1) has a constant solution $x(t) = x_0$, then evidently

$$\int_{t_1}^{t_2} f(x_0, t)\, dt = 0 \tag{0,6}$$

for $t_1, t_2 \in \langle 0, T\rangle$.

It has been proved in [2] that this solution is unique if $f(x, t) \in \varphi(K, \alpha, \beta)$. It is now plausible that the solution of equation (0,1) shall stay near to its initial value x_0, if equation (0,6) is approximately fulfilled. Thus we have

Theorem 0,2. *Let $f(x, t) \in \varphi(K, \alpha, \beta)$. Let for some $\varepsilon > 0$*

$$\left|\int_{t_1}^{t_2} f(x_0, t)\, \mathrm{d}t\right| \leqq \varepsilon(t_2 - t_1)^\alpha ,$$

whenever $t_1, t_2 \in \langle 0, T\rangle$. Let $x(t)$ be a solution of equation (0,1) satisfying the initial condition $x(0) = x_0$.

Then there exists a $T_1 \in \langle 0, T\rangle$ (which depends on the class $\varphi(K, \alpha, \beta)$ only) such that $|x(t) - x_0| \leqq 3\varepsilon t^\alpha$ for $0 \leqq t \leqq T_1$.

The following theorem may be useful in the case that in the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu\, f(x, t) \tag{0,7}$$

μ represents a small parameter.

Theorem 0,3. *Let $x(t)$ be a solution of equation (0,7) satisfying the initial condition $x(0) = x_0$. Let $\mu > 0$ and $f \in \varphi(K, \alpha, \beta)$. Then for $0 \leqq t \leqq T$*

$$\left|x(t) - x_0 - \mu \int_0^t f(x_0, \tau)\, \mathrm{d}\tau\right| \leqq \mu^2 c_2 t^{\alpha+\beta} ,$$

with c_2 depending on $\varphi(K, \alpha, \beta)$ only.

The significance of the theorem is apparent if $\int_0^t f(x_0, \sigma)\, \mathrm{d}\sigma = 0$.

The rest of this paper is devoted to a comparison of classical methods with those relating to the theory of generalized differential equations, the efficiency of the new method being examined on equations of a special type

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p(t, \lambda)\, f(x) + g(x, t) , \tag{0,8}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(y, t) . \tag{0,9}$$

The scalar function $p(t, \lambda)$ is defined and continuous for $t \in \langle 0, T\rangle$, $\lambda \in (0, 1)$ and has the following properties:

$$\left|\int_{t_1}^{t_2} p(\sigma, \lambda)\, \mathrm{d}\sigma\right| \leqq \lambda \quad \text{for} \quad 0 < \lambda < 1 ,$$

whenever $t_1, t_2 \in \langle 0, T\rangle$. (Evidently, e. g. the function $\frac{1}{2}\lambda^{-\alpha} \sin \frac{t}{\lambda^{1+\alpha}}$, $\alpha > 0$, may be taken for $p(t, \lambda)$.)

The $f(x)$, $g(x, t)$ are continuous vector functions, $f(x)$ has continuous derivatives of the second order and $g(x, t)$ fulfils Lipschitz condition in x with a constant independent of (x, t).

Theorem 0,4. *Let $x(t, \lambda)$, $y(t)$ be solutions of equations* (0,8), (0,9) *respectively with the initial condition $x(0) = y(0) = x_0$. Then there exists a positive constant k_1 such that*

$$|x(T, \lambda) - y(T)| \leqq k_1 \lambda .$$

The following theorem relates to a more general class of equations and, accordingly, the estimate proves to be coarser than in the previous case. In the equations

$$\frac{dx}{dt} = q(t, \lambda)\, f(x) + g(x, t) , \tag{0,10}$$

$$\frac{dy}{dt} = g(y, t) \tag{0,11}$$

q denotes a square matrix continuous in t,

$$|q(t, \lambda)| \leqq \lambda^{-\alpha},\ 0 < \alpha < 1,\ |\int_{t_1}^{t_2} q(t, \lambda)\, dt| \leqq \lambda .$$

The functions $f(x)$, $g(x, t)$ are continuous in all arguments and fulfil a Lipschitz condition with respect to x with a constant independent of (x, t).

Theorem 0,5. *Let $x(t, \lambda)$, $y(t)$ be solutions of* (0,10), (0,11) *respectively, satisfying the initial condition $x(0, \lambda) = y(0) = x_0$. Then there exists a constant k_2 (independent of λ) such that $|x(T, \lambda) - y(T)| \leqq k_2 \lambda^{1-\alpha}$.*

The following proposition, unlike theorems 0,4 and 0,5, has been proved on the basis of the theory of generalized differential equations.

Theorem 0,6. *Let $x(t, \lambda)$, $y(t)$ be solutions of equations*

$$\frac{dx}{dt} = \overline{p}(x, t, \lambda) + g(x, t) , \tag{0,12}$$

$$\frac{dy}{dt} = g(y, t) \tag{0,13}$$

respectively, satisfying the initial condition $x(0, \lambda) = y(0) = x_0$; here $\overline{p}(x, t, \lambda)$ is a vector function satisfying the following conditions:

$$|\int_{t_1}^{t_2} \overline{p}(x, t, \lambda)\, dt| \leqq A \min ((t_2 - t_1)^\beta, \lambda) ,$$

$$|\int_{t_1}^{t_2} [\overline{p}(x_2, t, \lambda) - \overline{p}(x_1, t, \lambda)]\, dt| \leqq L \min ((t_2 - t_1)^\beta, \lambda)$$

for $0 \leqq t_1 < t_2 \leqq T$, A, L, β constant, $\frac{1}{2} < \beta < 1$; and g is a continuous function of all arguments and satisfies a Lipschitz condition with a constant independent of (x, t).

Then there exists a constant k_3 such that $|x(T, \lambda) - y(T)| \leqq k_3 \lambda^{2-\frac{1}{\beta}}$.

If we put $\beta = \dfrac{1}{1+\alpha}$ in theorem 0,6, we obtain readily that theorem 0,6 gives, under more general assumptions, the same estimate as theorem 0,5.

An example is included proving that the estimate in theorems 0,5 and 0,6 cannot be improved essentially.

Finally, another example proves that theorem 0,6 gives a better estimate than methods based on theorem 0,5.

1. The theory of generalized differential equations is based on the concept of generalized integral which has been introduced in [1], [3]. The definition, existence theorem with an approximate formula for evaluating this integral will now be reviewed.

Let $U(\tau, t) = (U_1(\tau, t), \ldots, U_n(\tau, t))$ be a vector function defined for $\tau \in \langle \tau_*, \tau^* \rangle$, $t \in \langle \tau - \sigma, \tau + \sigma \rangle \cap \langle \tau_*, \tau^* \rangle$, where $\sigma > 0$. The set of all such points (τ, t) will be denoted by S_σ.

Definition 1,1. *Let $0 < \delta \leqq \sigma$ and let $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ be a subdivision of the segment $\langle \tau_*, \tau^* \rangle$ (i. e. $\tau_* = \alpha_0 < \alpha_1 < \ldots < \alpha_s = \tau^*$, $\alpha_0 \leqq \tau_1 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{s-1} \leqq \tau_s \leqq \alpha_s$) such that $\tau_j - \alpha_{j-1} < \delta$, $\alpha_j - \tau_j < \delta$, $j = 1, \ldots, s$.*

Let us put $B(U, A) = \sum_{j=1}^{s} [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})]$. If to every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|B(U, A_1) - B(U, A_2)\| < \varepsilon$ [1]) *whenever the subdivisions*

$$A_1 = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}, \quad A_2 = \{\alpha_0', \tau_1', \ldots, \tau_r', \alpha_r'\}$$

of the segment $\langle \tau_*, \tau^* \rangle$ fulfil the condition

$$\tau_j - \alpha_{j-1} < \delta, \quad \alpha_j - \tau_j < \delta, \quad j = 1, 2, \ldots, s,$$
$$\tau_j' - \alpha_{j-1}' < \delta, \quad \alpha_j' - \tau_j' < \delta, \quad j = 1, 2, \ldots, r,$$

---

[1]) By the norm $\|U\|$ of the vector $U = (U_1, \ldots, U_n)$ (or $n \times n$ matrix $V = \{V_{ik}\}$), we mean a non-negative number fulfilling the following conditions:

i) $\|U\| = 0$ if and only if $U_i = 0$, $i = 1, \ldots, n$ ($V_{ik} = 0$, $i, k = 1, \ldots, n$),

ii) If $U = W_1 + W_2$ ($V = Y_1 + Y_2$) then $\|U\| \leqq \|W_1\| + \|W_2\|$, ($\|V\| \leqq \|Y_1\| + \|Y_2\|$),

iii) If α is a real number, then $\|\alpha U\| = |\alpha| \cdot \|U\|$ ($\|\alpha V\| = |\alpha| \cdot \|V\|$),

iv) $\|(U_1, \ldots, U_n)\| = \|(|U_1|, \ldots, |U_n|)\|$ ($\|\{V_{ik}\}\| = \|\{|V_{ik}|\}\|$).

Thus any of the following norms are readily seen to fulfil these conditions:

$$\|U\|_1 = \max_{j=1,\ldots,n} |U_j|, \quad \|V\|_1 = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |V_{ij}|,$$

$$\|U\|_2 = \sum_{j=1}^{n} |U_j|, \quad \|V\|_2 = \sum_{i=1}^{n} \max_{j=1,\ldots,n} |V_{ij}|,$$

$$\|U\|_3 = \left(\sum_{j=1}^{n} U_j^2\right)^{\frac{1}{2}}, \quad \|V\|_3 = \left(\sum_{i,j} V_{ij}^2\right)^{\frac{1}{2}}.$$

then obviously there exists a vector B which is the limit of $B(U, A)$ as the norm of subdivision A tends to zero. B is called the generalized integral of $U(\tau, t)$ and denoted by $\int_{\tau_*}^{\tau^*} \mathrm{D}U$.

Let the real non-negative function $\psi(\eta)$ be defined, continuous and non-decreasing on $\langle 0, \sigma \rangle$, $\sum_{n=1}^{\infty} 2^n \psi\left(\frac{\sigma}{2^n}\right) < \infty$.

Theorem 1,1. *Let the vector function $U(\tau, t)$ fulfil the inequality*

$$\|U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)\| \leqq \psi(\eta)$$

whenever

$$\eta \in \langle 0, \sigma \rangle,\ (\tau + \eta, t + \eta),\ (\tau + \eta, t),\ (\tau, t + \eta),\ (\tau, t) \in S_\sigma .$$

Then the generalized integral $\int_{\tau_*}^{\tau^*} \mathrm{D}U(\tau, t)$ *exists and*

$$\left\| \int_{\tau_1}^{\tau_2} \mathrm{D}U - U(\tau_1, \tau_2) + U(\tau_1, \tau_1) \right\| \leqq \frac{\tau_2 - \tau_1}{2} \Psi(\tau_2 - \tau_1) \tag{1,1}$$

for

$$\tau_* \leqq \tau_1 < \tau_2 \leqq \tau_1 + \sigma,\ \tau_2 \leqq \tau^*,\ \Psi(\eta) = \sum_{i=1}^{\infty} \frac{2^i}{\eta} \psi\left(\frac{\eta}{2^i}\right).$$

The function $\Psi(\eta)$ is continuous, non-negative and non-decreasing for $\eta \in \langle 0, \sigma \rangle$, $\Psi(0) = 0$.

Let $\omega_1(\eta)$, $\omega_2(\eta)$ be real valued, continuous and non-decreasing functions, defined for $\eta \in \langle 0, \sigma \rangle$, $\omega_1(0) = \omega_2(0) = 0$, $\omega_i(\eta) \geqq c\eta$, $c > 0$, $i = 1, 2$, and satisfying the following condition:

$$\sum_{i=1}^{\infty} 2^i \psi\left(\frac{\sigma}{2^i}\right) < \infty ,$$

where $\psi(\eta) = 2\omega_1(\eta)\, \omega_2(\eta)$.

By $\boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$ we denote a set of functions $F(x, t)$ which have the following properties:

a) $F(x, t)$ is defined and continuous for $(x, t) \in G$, G being an open set in E_{n+1}, $F(x, t) \in E_n$.

b) $\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|)$ for $(x, t_1), (x, t_2) \in G$, $|t_2 - t_1| \leqq \sigma$.

c) $\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\, \omega_2(|t_2 - t_1|)$ for $(x_i, t_j) \in G$ $(i, j = 1, 2)$,

$$\|x_2 - x_1\| \leqq 2\omega_1(\sigma) , \quad |t_2 - t_1| \leqq \sigma .$$

Definition 1,2. *Let $F(x, t) \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$. The function $x(\tau) = (x_1(\tau), \ldots, x_n(\tau))$, $\tau_1 \leqq \tau \leqq \tau_2$, is called a solution of the generalized differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{1,2}$$

if $(x(\tau), \tau) \in G$ *for* $\tau \in \langle \tau_1, \tau_2 \rangle$ *and if* $x(\tau_4) - x(\tau_3) = \int_{\tau_3}^{\tau_4} \mathrm{D}F(x(\tau), t)$ *for* $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$.

The following note illustrates the relation of the generalized equations to the classical ones.

Note 1,1. If in the equation (1,2) the function $F(x, t) = \int_{t_0}^{t} f(x, \tau)\, \mathrm{d}\tau \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$, $f(x, t)$ continuous, and $x(\tau)$ is a solution of (1,2) defined on $\langle \tau_1, \tau_2 \rangle$, then $x(\tau)$ is also a solution of the classical equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau)$$

and vice versa.

In [1], [3] the following existence theorem has been proved:

Theorem 1,2. *Let* $F(x, t) \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$, *K a compact subset of G. Then there exists a positive number* σ^* *such that for every point* $(x_0, t_0) \in K$ *there exists a solution* $x(\tau)$ *of the generalized differential equation* (1,2) *which is defined on the segment* $\langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$, $x(\tau^0) = x_0$ *and satisfies the inequality*

$$\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|) \tag{1,3}$$

for $\tau_1, \tau_2 \in \langle \tau_0 - \sigma^*, \tau_0 + \sigma^* \rangle$.

Hence it is apparent why only solutions of generalized equations satisfying the relation (1,3) will be taken into account.

Note 1,2. Let G_1 be an open set, $\overline{G}_1$ compact, $K \subset G_1$, $\overline{G}_1 \subset G$. Let ϱ be the distance of the set K from the complement of the set G_1. It may be proved [1], [3] that for the σ^* in theorem 1,2 we may choose any positive number $\bar{\sigma}^*$ with the following properties: $0 \leqq \bar{\sigma}^* \leqq \frac{\sigma}{2}$, $\bar{\sigma}^* + \omega_1(\bar{\sigma}^*) < \varrho$ and $\eta\, \Psi(\eta) < \omega_1(\eta)$ for $0 < \eta < 2\bar{\sigma}^*$.

Let $F(x, t) \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$, $\overline{F}(x, t) \in \boldsymbol{F}(G, \bar{\omega}_1, \bar{\omega}_2, \sigma)$. Let $x(\tau)$ be a solution of (1,2) defined on $\langle 0, T \rangle$ and satisfying the initial condition $x(0) = x_0$. Let $y(\tau)$ be a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}\overline{F}(y, t) \tag{1,4}$$

with the initial condition $y(0) = y_0$.

We shall assume that

$$(x(\tau), t)\,, \quad (y(\tau), t) \in G \quad \text{for} \quad \tau, t \in \langle 0, T \rangle\,,$$

$$\|x(\tau_2) - x(\tau_1)\| \leqq 2\omega_1(|\tau_2 - \tau_1|)\,, \quad \|y(\tau_2) - y(\tau_1)\| \leqq 2\bar{\omega}_1(|\tau_2 - \tau_1|)$$

for

$$\tau_1, \tau_2 \in \langle 0, T \rangle\,, \quad |\tau_2 - \tau_1| \leqq \sigma\,.$$

Let us denote $\bar{\psi}(\eta) = 2[\omega_1(\eta)\,\omega_2(\eta) + \bar{\omega}_1(\eta)\,\bar{\omega}_2(\eta)]$,

$$\Psi(\eta) = \sum_{j=1}^{\infty} \frac{2^j}{\eta}\,\bar{\psi}\left(\frac{\eta}{2^j}\right) \quad \text{for} \quad \eta \in \langle 0, \sigma \rangle\,,$$

$$F(x, t) - \bar{F}(x, t) = A(x, t) \quad \text{for} \quad (x, t) \in G\,.$$

Suppose that

$$\|A(x, t_2) - A(x, t_1)\| \leqq \omega_3(|t_2 - t_1|) \tag{1,5}$$

for $(x, t_i) \in G$ $(i = 1, 2)$, $|t_2 - t_1| \leqq \sigma$.

This inequality characterizes the proximity of equations (1,2) and (1,4). (As $\|A(x, t_2) - A(x, t_1)\| \leqq \|F(x, t_2) - F(x, t_1)\| + \|\bar{F}(x, t_2) - \bar{F}(x, t_1)\|$, the inequality (1,5) is always fulfilled for $\omega_3(\eta) = \omega_1(\eta) + \bar{\omega}_1(\eta)$.

However, the most important case is that this function may be chosen „substantially smaller" than $\omega_1(\eta) + \bar{\omega}_1(\eta)$).

Theorem 1,3. *Let m be a positive integer such that $\frac{T}{m} \leqq \sigma$. Then*

$$\|x(T) - y(T)\| \leqq \left[\frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right)\right] \frac{\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^m - 1}{\omega_2\left(\frac{T}{m}\right)} + \tag{1,6}$$

$$+ \|x_0 - y_0\| \left[1 + \omega_2\left(\frac{T}{m}\right)\right]^m.$$

Proof. From the definition of solutions of generalized differential equations we have, for $t_1, t_2 \in \langle 0, T \rangle$,

$$x(t_2) - y(t_2) - x(t_1) + y(t_1) = \int_{t_1}^{t_2} \mathrm{D}\{F(x(\tau), t) - \bar{F}(y(\tau), t)\}\,.$$

From the assumptions of the theorem it follows that

$$\begin{aligned}
&\|\{F(x(t_2), t_2) - \bar{F}(y(t_2), t_2)\} - \{F(x(t_2), t_1) - \bar{F}(y(t_2), t_1)\} - \\
&- \{F(x(t_1), t_2) - \bar{F}(y(t_1), t_2)\} + \{F(x(t_1), t_1) - \bar{F}(y(t_1), t_1)\}\| \leqq \\
&\leqq \|x(t_2) - x(t_1)\|\,\omega_2(|t_2 - t_1|) + \|y(t_2) - y(t_1)\|\,\bar{\omega}_2(|t_2 - t_1|) \leqq \\
&\leqq 2\{\omega_1(|t_2 - t_1|)\,\omega_2(|t_2 - t_1|) + \bar{\omega}_1(|t_2 - t_1|)\,\bar{\omega}_2(|t_2 - t_1|)\} = \bar{\psi}(|t_2 - t_2|)\,.
\end{aligned}$$

If we apply theorem 1,1 to the function $F(x(\tau), t) - \bar{F}(y(\tau), t)$, we obtain

$$\begin{aligned}
&\|x(t_2) - y(t_2) - x(t_1) + y(t_1)\| \leqq \\
&\leqq |t_2 - t_1|\,\Psi(|t_2 - t_1|) + \|F(x(t_1), t_2) - \bar{F}(y(t_1), t_2) - F(x(t_1), t_1) + \\
&+ \bar{F}(y(t_1), t_1)\| = |t_2 - t_1|\,\Psi(|t_2 - t_1|) + \|F(x(t_1), t_2) - F(y(t_1), t_2) + \\
&+ A(y(t_1), t_2) - F(x(t_1), t_1) + F(y(t_1), t_1) - A(y(t_1), t_1)\| \leqq \\
&\leqq |t_2 - t_1|\,\Psi(|t_2 - t_1|) + \|x(t_1) - y(t_1)\|\,\omega_2(|t_2 - t_1|) + \omega_3(|t_2 - t_1|)\,.
\end{aligned}$$

Hence

$$(1,7)\qquad \|x(t_2) - y(t_2)\| \leqq |t_2 - t_1|\, \Psi(|t_2 - t_1|) + \omega_3(|t_2 - t_1|) + \\ + \|x(t_1) - y(t_1)\|\,[1 + \omega_2(|t_2 - t_1|)]$$

for $|t_2 - t_1| \leqq \sigma,\ t_1, t_2 \in \langle 0, T\rangle$.

The next part of the proof consists in dividing the segment $\langle 0, T\rangle$ into subintervals of length $\frac{T}{m}$, and in applying the relation (1,7) to each of these subintervals. By such means the increment of the function $x(\tau) - y(\tau)$ relating to each of these subintervals may be estimated. It is true that

$$(1,8)\quad \left\| x\left(\frac{T}{m}\right) - y\left(\frac{T}{m}\right)\right\| \leqq \frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right) + \|x(0) - y(0)\|\left[1 + \omega_2\left(\frac{T}{m}\right)\right].$$

Finally we shall prove that for $1 \leqq k \leqq m$

$$(1,9)\quad \left\| x\left(k\frac{T}{m}\right) - y\left(k\frac{T}{m}\right)\right\| \leqq \left[\frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right)\right]\frac{\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^k - 1}{\omega_2\left(\frac{T}{m}\right)} + \\ + \|x(0) - y(0)\|\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^k.$$

Indeed, for $k = 1$ the relation (1,9) holds by (1,8). Assume it holds for $k - 1$. Then from (1,7) we obtain

$$\left\| x\left(k\frac{T}{m}\right) - y\left(k\frac{T}{m}\right)\right\| \leqq \frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right) + \left\{\left[\frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right)\right]\cdot \\ \cdot\frac{\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^{k-1} - 1}{\omega_2\left(\frac{T}{m}\right)} + \|x(0) - y(0)\|\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^{k-1}\right\}\cdot\left[1 + \omega_2\left(\frac{T}{m}\right)\right] = \\ = \left\{\frac{T}{m}\,\Psi\left(\frac{T}{m}\right) + \omega_3\left(\frac{T}{m}\right)\right\}\left\{1 + \frac{\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^{k-1} - 1}{\omega_2\left(\frac{T}{m}\right)}\left[1 + \omega_2\left(\frac{T}{m}\right)\right]\right\} + \\ + \|x(0) - y(0)\|\left[1 + \omega_2\left(\frac{T}{m}\right)\right]^k,$$

whence (1,9) follows immediately.

Theorem 1,4. *Let* $\omega_2(\eta) = L\eta,\ L > 0,\ \omega_3(\eta) = M\eta,\ M \geqq 0$. *Then*

$$(1,10)\qquad \|x(T) - y(T)\| \leqq \frac{M}{L}\,(e^{LT} - 1) + \|x_0 - y_0\|\, e^{LT}.$$

The proof follows readily by substituting into (1,6) and passing to the limit for $m \to \infty$, since $\Psi\left(\frac{T}{m}\right) \to 0$ for $m \to \infty$.

Note 1,3. In the classical theory of differential equations the formula (1,10) is proved for the case that $\frac{\partial F}{\partial t}$, $\frac{\partial \bar{F}}{\partial t}$ are continuous. In this case the equations (1,2) and (1,4) are equivalent to the classical equations $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \bar{f}(y, t)$ respectively, where $f = \frac{\partial F}{\partial t}$, $\bar{f} = \frac{\partial \bar{F}}{\partial t}$.

As in this case $\omega_2(\eta) = L\eta$ and $\omega_3(\eta) = M\eta$, we have $\|f(x_2, t) - f(x_1, t)\| \leqq \leqq L\|x_2 - x_1\|$ for $(x_1, t), (x_2, t) \in G$ and $\|f(x, t) - \bar{f}(x, t)\| \leqq M$ for $(x, t) \in G$.

Note 1,4. If $\omega_2(\eta) = L\eta$, L a positive constant, we obtain from theorem 1,4 the uniqueness of solutions $x(\tau)$:

Indeed, if $x(\tau), y(\tau)$ are two solutions of equation (1,2), $x(0) = y(0)$, both defined on $\langle 0, T\rangle$, $\|x(t_2) - x(t_1)\| \leqq 2\omega_1(|t_2 - t_1|)$, $\|y(t_2) - y(t_1)\| \leqq 2\omega_1(|t_2 - t_1|)$, $t_1, t_2 \in \langle 0, T\rangle$, $|t_2 - t_1| < \sigma$, then from theorem 1,4, where we put $\bar{F}(x, t) \equiv F(x, t)$ we obtain $\|x(T) - y(T)\| = 0$, since in this case $M = 0$. Evidently, it is not a substantial limitation to take the fundamental interval to be $\langle 0, T\rangle$, and the uniqueness theorem may be proved for any point of the interval where the solution is defined.

The following theorem may be useful in the case that equation (1,2) has a solution which slightly differs from a constant.

If $x(\tau) = x_0 = \text{const}$ is a solution of (1,2) for $\tau \in \langle 0, T\rangle$, then

$$(1,11) \qquad \int_{t_1}^{t_2} \mathrm{D}F(x_0, t) = 0$$

for $t_1, t_2 \in \langle 0, T\rangle$, and x_0 is unique by [2]. Hence it is plausible that if the equation (1,11) is fulfilled approximately, the solution $x(\tau)$ will remain near to its initial value x_0.

Theorem 1,5. *Let* $0 \leqq t_1 < t_2 \leqq T \leqq \sigma$, *and let the following conditions be fulfilled:*

a) $F(x, t) \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$.

b) $2\omega_1(T)\,\omega_2(t_2 - t_1) \leqq \frac{1}{3}\omega_1(t_2 - t_1)$.

c) *There exists a non-negative integer* N *such that*

$$\|F(x_0, t_2) - F(x_0\, t_1)\| \leqq \frac{1}{2^N}\frac{1}{3}\,\omega_1(t_2 - t_1)$$

d) $(t_2 - t_1)\,\Psi(t_2 - t_1) \leqq \frac{1}{3}\omega_1(t_2 - t_1)$, $\Psi(\eta) = \sum_{1}^{\infty} \frac{2^i}{\eta}\,\psi\left(\frac{\eta}{2^i}\right)$, $\psi(\eta) = 2\omega_1(\eta)\,\omega_2(\eta)$.

If $x(\tau)$ is a solution of equation (1,2) *which satisfies the conditions* $x(0) = x_0$, $\|x(t_2) - x(t_1)\| \leqq 2\omega_1(t_2 - t_1)$, $0 \leqq t_1 < t_2 \leqq T$, $(x(\tau), t) \in G$ *for* $\tau, t \in \langle 0, T\rangle$, *then* $\|x(T_1) - x_0\| \leqq \frac{1}{2^N}\omega_1(T_1)$ *for* $0 \leqq T_1 \leqq T$.

Proof. Let $0 \leqq t_1 < t_2 \leqq T$. Suppose that for some non-negative integer $k \leqq N$

$$\|x(t_2) - x(t_1)\| \leqq \frac{1}{2^k} 2\omega_1(t_2 - t_1). \tag{1,12}$$

Then

$$\|x(t_2) - x(t_1)\| = \|\int_{t_1}^{t_2} \mathrm{D}F(x(\tau), t)\| \leqq \|F(x(t_1), t_2) - F(x(t_1), t_1)\| +$$
$$+ (t_2 - t_1)\frac{1}{2^k}\Psi(t_2 - t_1) \leqq \|F(x(t_1), t_2) - F(x(t_1), t_1) - F(x_0, t_2) +$$
$$+ F(x_0, t_1)\| + \|F(x_0, t_2) - F(x_0, t_1)\| + (t_2 - t_1)\frac{1}{2^k}\Psi(t_2 - t_1) \leqq$$
$$\leqq \|x(t_1) - x_0\|\,\omega_2(t_2 - t_1) + \|F(x_0, t_2) - F(x_0, t_1)\| +$$
$$+ \frac{1}{2^k}(t_2 - t_1)\Psi(t_2 - t_1).$$

As $\|x(t_1) - x_0\| \leqq \frac{1}{2^k} 2\omega_1(T)$,

$$\|x(t_2) - x(t_1)\| \leqq \frac{1}{2^k} 2\omega_1(T)\,\omega_2(t_2 - t_1) + \|F(x_0, t_2) - F(x_0, t_1)\| + \tag{1,13}$$
$$+ \frac{1}{2^k}(t_2 - t_1)\Psi(t_2 - t_1).$$

From (1,13) and conditions a), b), c) of our theorem we obtain

$$\|x(t_2) - x(t_1)\| \leqq \frac{1}{2^k}\frac{1}{3}\omega_1(t_2 - t_1) + \frac{1}{2^k}\frac{1}{3}\omega_1(t_2 - t_1) + \tag{1,14}$$
$$+ \frac{1}{2^k}\frac{1}{3}\omega_1(t_2 - t_1) = \frac{1}{2^{k+1}} 2\omega_1(t_2 - t_1).$$

Since (1,12) holds for $k = 0$, (1,14) holds for any $k \leqq N$ and the theorem is thus proved.

Note 1,5. It will be shown that the uniqueness theorem in [2] is a consequence of theorem 1,5. Indeed, if $x(\tau) = x_0$ is a solution of equation (1,2) for $\tau \in \langle 0, T\rangle$, then $\int_{t_1}^{t_2} \mathrm{D}F(x_0, t) = F(x_0, t_2) - F(x_0, t_1) = 0$, and condition c) of theorem 1,5 is thus fulfilled for any N. Conditions b), d) are evidently satisfied whenever T is sufficiently small. If $y(\tau)$ is a solution of equation (1,2) for $\tau \in \langle 0, T\rangle$, $\|y(t_2) - y(t_1)\| \leqq 2\omega_1(t_2 - t_1)$ for $0 \leqq t_1 < t_2 \leqq T$, $y(0) = x_0$, then $\|y(T_1) - x_0\| = 0$

for $0 \leqq T_1 \leqq T$ from theorem 1,5. Consequently, in this sense, a constant solution of equation (1,2) is uniquely defined by its initial condition.

Theorem 1,6. *Let $F(x, t) \in \boldsymbol{F}(G, \omega_1, \omega_2, \sigma)$, $\mu > 0$. Let $x(\tau)$ be a solution of $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\mu\, F(x, t)$, defined on $\langle 0, T \rangle$, $T < \sigma$, and satisfying the following conditions: $x(0) = x_0$, $\|x(t_2) - x(t_1)\| \leqq 2\mu\, \omega_1(t_2 - t_1)$ for $0 \leqq t_1 < t_2 \leqq T$, $(x(\tau), t) \in G$ for $\tau, t \in \langle 0, T \rangle$.*

Then $\|x(T_1) - x_0 - \mu[F(x_0, T_1) - F(x_0, 0)]\| \leqq \mu^2 T_1\, \Psi(T_1)$ for $0 \leqq T_1 \leqq T$, where $\Psi(\eta) = \sum_1^\infty \frac{2^i}{\eta}\, \psi\left(\frac{\eta}{2^i}\right)$, $\psi(\eta) = 2\omega_1(\eta)\, \omega_2(\eta)$.

Proof. Let $0 \leqq t_1 \leqq t_2 \leqq T$. Then

$$\mu\|F(x(t_2), t_2) - F(x(t_2), t_1) - F(x(t_1), t_2) + F(x(t_1), t_1)\| \leqq$$
$$\leqq \mu \,.\, 2\mu\, \omega_1(t_2 - t_1)\, \omega_2(t_2 - t_1) = \mu^2\, \psi(t_2 - t_1)\,.$$

Hence the theorem follows immediately from theorem 1,1.

Note 1,6. For classical equations an analogical result may be obtained by means of the method of successive approximations.

2. Let us consider the vector equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p(t, \lambda)\, f(x) + g(x, t)\,, \tag{2,1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(y, t)\,. \tag{2,2}$$

The scalar function $p(t, \lambda)$ is defined and continuous for $0 \leqq t \leqq T$, $0 < \lambda < 1$, and

$$\left|\int_{t_1}^{t_2} p(\sigma, \lambda)\, \mathrm{d}\sigma\right| \leqq \lambda \quad \text{for} \quad 0 < \lambda < 1,\ t_1, t_2 \in \langle 0, T \rangle\,.$$

The vector function $f(x) \in E_n$ has continuous partial derivatives of the second order in H, H being an open subset of E_n.

The function $g(x, t) \in E_n$ is defined and continuous for $(x, t) \in H \times \langle 0, T \rangle$ and satisfies the following conditons:

$$\|g(x, t)\| \leqq A_g\,, \quad \|g(x_2, t) - g(x_1, t)\| \leqq L_g\|x_2 - x_1\|$$

for $x, x_1, x_2 \in H$, $t \in \langle 0, T \rangle$.

Theorem 2,1. *Let $y(t)$ be a solution of (2,2) which satisfies the following conditions: $y(t) \in H_1$ for $0 \leqq t \leqq T$, $y(0) = y_0$, where H_1 is an open subset of H whose closure $\overline{H}_1$ is a compact subset of H. Let there be given a function $x_0(\lambda)$ defined for $0 < \lambda < 1$, $x_0(\lambda) \in E_n$ with $\lim_{\lambda \to 0} x_0(\lambda) = y_0$.*

Then there exist positive numbers λ_0, k_1, k_2 such that the following proposition is true:

a) *There exists a solution $x(t, \lambda)$ of equation* (2,1) *satisfying the condition $x(0, \lambda) = x_0(\lambda)$, $x(t, \lambda) \in H$ for $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$, $\lambda_0 \in (0, 1)$.*

b) $\|x(t, \lambda) - y(t)\| \leqq k_1\lambda + k_2\|x_0(\lambda) - y_0\|$ *for* $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$.

Proof. The method of proof consists in a transformation of coordinates which transforms the vector $f(x) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ into a constant vector. To preserve the one-to-one correspondence of the mapping, the transformation is carried out in an $(n + 1)$ — dimensional space. Accordingly, we shall consider differential equations

$$(2,3) \qquad \frac{\mathrm{d}X_i}{\mathrm{d}t} = f_i(X_1, \ldots, X_n)\,, \quad i = 1, \ldots, n\,,$$

$$\frac{\mathrm{d}X_{n+1}}{\mathrm{d}t} = 1$$

for $(X_1, \ldots, X_n) \in H$.

By our assumptions on the function $f(x)$ there exists a positive number $\Delta \leqq 1$ such that in every point $(\xi_1, \ldots, \xi_n) \in H_1$ there exists precisely one solution $X_i(\xi_1, \ldots, \xi_n, t)$, $i = 1, \ldots, n$, of the system of n equations (2,3) satisfying the following condition: $X_i(\xi_1, \ldots, \xi_n, t) \in H$ for $-\Delta \leqq t \leqq \Delta$, $X_i(\xi_1, \ldots, \xi_n, 0) = \xi_i$, $i = 1, \ldots, n$. As the solution of the last equation of the system (2,3) we choose $X_{n+1}(\xi_1, \ldots, \xi_n, t) = t$.

Let $(\xi_1, \ldots, \xi_n) \in \overline{H}_1$, $\xi_{n+1} \in \langle -\Delta, \Delta \rangle$. Let a mapping T^* of $\overline{H}_1 \times \langle -\Delta, \Delta \rangle$ into $H \times \langle -\Delta, \Delta \rangle$ be defined by the relations

$$(2,4) \qquad x_i = X_i(\xi_1, \ldots, \xi_n, \xi_{n+1})\,, \quad (i = 1, \ldots, n + 1)\,.$$

Since the solutions of equations (2,3) are uniquely defined by their initial conditions, to every point of the set $\overline{H}_1 \times \langle -\Delta, \Delta \rangle$ there corresponds one and only one point of the set

$$T^*(\overline{H}_1 \times \langle -\Delta, \Delta \rangle) \subset H \times \langle -\Delta, \Delta \rangle$$

and vice versa. Let the inverse mapping T^*_{-1} of the set $T^*(\overline{H}_1 \times \langle -\Delta, \Delta \rangle)$ onto $\overline{H}_1 \times \langle -\Delta, \Delta \rangle$ be defined by the relations

$$(2,5) \qquad \xi_i = \Xi_i(x_1, \ldots, x_{n+1})\,, \quad i = 1, \ldots, n + 1\,.$$

Then evidently

$$\frac{\mathrm{D}(X_1, \ldots, X_{n+1})}{\mathrm{D}(\Xi_1, \ldots, \Xi_{n+1})} \cdot \frac{\mathrm{D}(\Xi_1, \ldots, \Xi_{n+1})}{\mathrm{D}(X_1, \ldots, X_{n+1})} = 1$$

for $(X_1, \ldots, X_{n+1}) \in T^*(\overline{H}_1 \times \langle -\Delta, \Delta \rangle)$, $(\Xi_1, \ldots, \Xi_{n+1}) \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$.

Since the mapping T^* relates to $n + 1$ coordinates, it is convenient to consider in place of the equations (2,1), (2,2), the equations

$$(2,6) \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = p(t, \lambda) f_i(x_1, \ldots, x_n) + g_i(x_1, \ldots, x_n, t), \quad i = 1, \ldots, n,$$

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}t} = p(t, \lambda)$$

with the initial conditions $x_i(0, \lambda) = x_{i0}(\lambda)$, $i = 1, \ldots, n$, $x_{n+1}(0, \lambda) = 0$, and

$$(2,7) \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = g_i(y_1, \ldots, y_n, t), \quad i = 1, \ldots, n,$$

$$\frac{\mathrm{d}y_{n+1}}{\mathrm{d}t} = 0.$$

with the initial conditions $y_i(0) = y_{i0}$, $i = 1, \ldots, n$, $y_{n+1}(0) = 0$.

Evidently the functions $y_i(t)$, $i = 1, \ldots, n$, $y_{n+1}(t) \equiv 0$ are solutions of the system (2,7) and $(y_1(t), \ldots, y_n(t)) \in K_1$ for $0 \leqq t \leqq T$, where K_1 is a compact subset of H_1. By means of the mapping (2,5) the solution $(y_1(t), \ldots, y_{n+1}(t))$ is transformed into a vector function $(\eta_1(t), \ldots, \eta_{n+1}(t))$ for $0 \leqq t \leqq T$. As $\xi_i = \Xi_i(x_1, \ldots, x_n, 0) = x_i$, $i = 1, \ldots, n$, $\xi_{n+1} = \Xi_{n+1}(x_1, \ldots, x_n, 0) = 0$, we have $y_i(t) = \eta_i(t)$ for $i = 1, \ldots, n + 1$, $0 \leqq t \leqq T$.

If we substitute ξ_i for x_i, $i = 1, \ldots, n + 1$, with respect to (2,4), the equations (2,6) are transformed into

$$(2,8) \qquad \sum_{j=1}^{n+1} \frac{\partial X_i}{\partial \xi_j} \dot{\xi}_j = p(t, \lambda) f_i(X_1(\xi_1, \ldots, \xi_{n+1}), \ldots, X_n(\xi_1, \ldots, \xi_{n+1})) +$$

$$+ g_i(X_1(\xi_1, \ldots, \xi_{n+1}), \ldots, X_n(\xi_1, \ldots, \xi_{n+1}), t), \quad i = 1, \ldots, n,$$

$$\dot{\xi}_{n+1} = p(t, \lambda)$$

for $(\xi_1, \ldots, \xi_{n+1}) \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$, $0 \leqq t \leqq T$.

As from (2,3), (2,4) it follows that $\dfrac{\partial X_i}{\partial \xi_{n+1}} = \dfrac{\partial X_i}{\partial t} = f_i(X_1, \ldots, X_n)$ for $i = 1, \ldots, n$, we can easily verify from (2,8) that

$$(2,9) \qquad \dot{\xi}_j = \frac{J_j}{J}, \quad j = 1, \ldots, n,$$

$$\dot{\xi}_{n+1} = p(t, \lambda),$$

where J is the Jacobian $\dfrac{\mathrm{D}(X_1, \ldots, X_{n+1})}{\mathrm{D}(\xi_1, \ldots, \xi_{n+1})}$ of the system (2,4) and J_j arises from J by replacing the j^{th} column of J by the vector $(g_1(X_1, \ldots, X_n, t), \ldots, g_n(X_1, \ldots, X_n, t), 0)$.

Thus we have shown that the solution $(x_1(t, \lambda), \ldots, x_{n+1}(t, \lambda))$ of the system (2,6) which satisfies the initial conditions $x_i(0, \lambda) = x_{i0}(\lambda)$, $i = 1, \ldots, n$, $x_{n+1}(0, \lambda) = 0$, is transformed into the solution $(\xi_1(t, \lambda), \ldots, \xi_{n+1}(t, \lambda))$ of the system (2,9), while the solution $(y_1(t), \ldots, y_{n+1}(t))$ of (2,7) is not changed by the mapping (2,5).

Let us introduce the following notation:

$$\frac{J_j}{J} = \varphi_j(\xi_1, \ldots, \xi_{n+1}, t) \quad \text{for} \quad j = 1, \ldots, n\,, \tag{2,10}$$

$$\xi = (\xi_1, \ldots, \xi_{n+1})\,, \quad \varphi(\xi, t) = (\varphi_1(\xi, t), \ldots, \varphi_n(\xi, t), 0)$$

for $(\xi_1, \ldots, \xi_n) \in \overline{H}_1$, $\xi_{n+1} \in \langle -\Delta, \Delta \rangle$, $t \in \langle 0, T \rangle$.

Since $X_i(\xi_1, \ldots, \xi_n, 0) = \xi_i$ for $i = 1, \ldots, n$, $\dfrac{\partial X_{n+1}(\xi_1, \ldots, \xi_{n+1})}{\partial \xi_{n+1}} = 1$, we conclude $\dfrac{\partial X_i(\xi_1, \ldots, \xi_n, 0)}{\partial \xi_j} = \delta_{ij}$ $(\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1)$, $i, j = 1, \ldots, n+1$. Hence

$$\varphi_j(\xi_1, \ldots, \xi_n, 0, t) = g_j(\xi_1, \ldots, \xi_n, t) \tag{2,11}$$

for $(\xi_1, \ldots, \xi_n) \in \overline{H}_1$, $t \in \langle 0, T \rangle$, $j = 1, \ldots, n$.

Now let us consider the system (2,9) which, with respect to (2,10), may be written in the form

$$\begin{aligned} \dot{\xi}_j &= \varphi_j(\xi_1, \ldots, \xi_{n+1}, t)\,, \quad j = 1, \ldots, n\,, \\ \dot{\xi}_{n+1} &= p(t, \lambda)\,. \end{aligned} \tag{2,12}$$

As the functions φ_j are continuous and as $\xi_{n+1}(t, \lambda) = \int_0^t p(\tau, \lambda)\,\mathrm{d}\tau$, by (2,11) we obtain that

$$\lim_{\lambda \to 0} \varphi_j\left(\xi_1, \ldots, \xi_n, \int_0^t p(\tau, \lambda)\,\mathrm{d}\tau, t\right) = g_j(\xi_1, \ldots, \xi_n, t) \tag{2,13}$$

uniformly for $(\xi_1, \ldots, \xi_n) \in \overline{H}_1$, $t \in \langle 0, T \rangle$, $j = 1, \ldots, n$.

As the functions X_i in (2,4) have continuous partial derivatives of the second order and $J \neq 0$ for $\xi \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$, it follows that there exists a positive constant L_1 such that

$$\|\varphi(\xi'', t) - \varphi(\xi', t)\| \leqq L_1 \|\xi'' - \xi'\| \tag{2,14}$$

for $\xi', \xi'' \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$.

The solution $y(t)$ of equation (2,2) satisfies the condition $y(t) \in K_1$ for $0 \leqq t \leqq T$. Let us denote by ϱ the distance from K_1 to the boundary of the set H_1. Evidently $\varrho > 0$. By continuity of the functions $\varphi_j(\xi_1, \ldots, \xi_{n+1}, t)$, $j = 1,$

..., n, it follows from (2,13) and (2,14) that the theorem on the continuous dependence of solution $(\xi_1(t, \lambda), \ldots, \xi_n(t, \lambda))$ of the equations

$$\dot{\xi}_j = \varphi_j(\xi_1, \ldots, \xi_n, \int_0^t p(\tau, \lambda)\, d\tau, t), \quad j = 1, \ldots, n, \tag{2,15}$$

on the parameter λ may be used for $\lambda = 0$. Hence there exists a $\lambda_0 \leqq \Delta$ such that the solution $u(t, \lambda) = (u_1(t, \lambda), \ldots, u_n(t, \lambda))$ of equations (2,15) which satisfies the initial condition $u(0, \lambda) = x_0(\lambda)$ is defined on $\langle 0, T\rangle$ and satisfies the inequality $\|u(t, \lambda) - y(t)\| \leqq \varrho$ whenever $0 < \lambda < \lambda_0$.

Let us denote $\bar{u}(t, \lambda) = (u_1(t, \lambda), \ldots, u_n(t, \lambda), \int_0^t p(\tau, \lambda)\, d\tau)$ and $w(t, \lambda) = T^*(\bar{u}(t, \lambda))$ for $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$. From (2,4) it follows that $w(t, \lambda) \in H \times \langle -\Delta, \Delta\rangle$ for $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$. With respect to the properties of the mapping T^* it is obvious that $w(t, \lambda)$ is a solution of (2,6). If we denote $(w_1(t, \lambda), \ldots, w_n(t, \lambda)) = x(t, \lambda)$, the first part of the theorem is proved.

In order to prove the proposition b), let us denote by $\bar{p}(t, \lambda)$ a vector $(0, 0, \ldots, 0, p(t, \lambda)) \in E_{n+1}$. The system (2,12) may be written in the form

$$\dot{\xi} = \bar{p}(t, \lambda) + \varphi(\xi, t) \tag{2,16}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n, 0)$.

The solution $\bar{u}(t, \lambda)$ of the equations (2,16) will be compared with the solution $\bar{y}(t) = (y_1(t), \ldots, y_n(t), 0)$ of the equations (2,7), which may be written in virtue of (2,11) in the vector form

$$\dot{\bar{y}} = \varphi(\bar{y}, t) \tag{2,17}$$

where $\bar{y} = (y_1, \ldots, y_n, 0)$.

By substituting their respective solutions into (2,16), (2,17), subtracting (2,17) from (2,16) and using (2,14) we obtain

$$\|\bar{u}(t, \lambda) - \bar{y}(t)\| \leqq \|\int_0^t \bar{p}(\sigma, \lambda)\, d\sigma\| + \|\bar{u}(0, \lambda) - \bar{y}(0)\| +$$
$$+ \|\int_0^t [\varphi(\bar{u}(\sigma,\lambda), \sigma) - \varphi(\bar{y}(\sigma), \sigma)]\, d\sigma\| \leqq k_3\lambda + \|\bar{u}(0, \lambda) - \bar{y}(0)\| +$$
$$+ L_1\|\int_0^t [\bar{u}(\sigma, \lambda) - \bar{y}(\sigma)]\, d\sigma\|$$

for $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$.

Hence it follows that

$$\|\bar{u}(t, \lambda) - \bar{y}(t)\| \leqq [k_3\lambda + \|u(0, \lambda) - \bar{y}(0)\|] \exp (L_1 t)$$

for $0 \leqq t \leqq T$, $0 < \lambda < \lambda_0$.

Since the functions $X_i(\xi)$, $i = 1, \dots, n + 1$ in (2,4) have continuous partial derivatives of the second order, evidently there exists a positive number L_2 such that

$$|X_i(\xi'') - X_i(\xi')| \leqq L_2 \|\xi'' - \xi'\|$$

whenever $\xi', \xi'' \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$. Thence and from the fact that $\overline{u}(t, \lambda)$, $\overline{y}(t) \in \overline{H}_1 \times \langle -\Delta, \Delta \rangle$ for $t \in \langle 0, T \rangle$, $\lambda \in (0, \lambda_0)$, the proof of the theorem 2,1 can be completed readily.

Now we will consider the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = q(t, \lambda)\, f(x) + g(x, t)\,, \tag{2,18}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x, t)\,. \tag{2,19}$$

Here $q(t, \lambda)$ is an $n \times n$ matrix, defined for $0 \leqq t \leqq T$, $0 < \lambda < 1$, continuous in t, with the following properties:

$$\|q(t, \lambda)\| \leqq \lambda^{-\alpha}\,, \quad 0 \leqq \alpha < 1\,, \quad \left\| \int_{t_1}^{t_2} q(t, \lambda)\, \mathrm{d}t \right\| \leqq \lambda$$

for $0 \leqq t \leqq T$, $0 \leqq t_1 \leqq t_2 \leqq T$, $0 < \lambda < 1$. $f(x) = (f_1(x), \dots, f_n(x))$ is defined for $x \in H$ and satisfies

$$\|f(x)\| \leqq A_f\,, \quad \|f(x_2) - f(x_1)\| \leqq L_f \|x_2 - x_1\|$$

for $x_1, x_2 \in H$, where H is an open set in E_n.

The function $g(x, t) = (g_1(x, t), \dots, g_n(x, t))$ is defined and continuous for $(x, t) \in H \times \langle 0, T \rangle$, $\|g(x, t)\| \leqq A_g$, $\|g(x_2, t) - g(x_1, t)\| \leqq L_g \|x_2 - x_1\|$, $x, x_1, x_2 \in H$, $0 \leqq t \leqq T$.

Lemma 2,1. *Let $u(t)$ be a vector function defined on $\langle 0, T \rangle$ which satisfies the following conditions: $u(t) \in H$ for $t \in \langle 0, T \rangle$, $\|u(t_2) - u(t_1)\| \leqq A(t_2 - t_1)$ for $0 \leqq t_1 \leqq t_2 \leqq T$.*

Then $f(u(t)) = f_P(t) - f_N(t)$ for $t \in \langle 0, T \rangle$, where the components f_{Pi}, f_{Ni} of the vectors $f_P(t), f_N(t)$ are non-decreasing and

$$\|f_P(t)\| \leqq \|f(u(0))\| + L_f A T\,, \quad \|f_N(t)\| \leqq L_f A T\,.$$

Proof. Since for every subdivision $\mathfrak{A}_m$: $0 = t_0 \leqq t_1 \leqq \dots \leqq t_m = T$, we have

$$\sum_{i=1}^{m} \|f(u(t_i)) - f(u(t_{i-1}))\| \leqq L_f A T\,,$$

each component f_i of f being of bounded variation on $\langle 0, T \rangle$. Consequently

$$f_i(u(t)) = f_i(u(0)) + P_i(t) - N_i(t)\,,$$

where $P_i(t)$ and $N_i(t)$ are non-negative and non-decreasing on $\langle 0, T \rangle$, $P_i(0) = N_i(0) = 0$, $i = 1, \dots, n$.

Let us denote

$$f_P(t) = (f_1(u(0)) + P_1(t), \ldots, f_n(u(0)) + P_n(t)),$$
$$f_N(t) = (N_1(t), \ldots, N_n(t)), \quad 0 \leqq t \leqq T,$$

$$\sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_i(u(t_j)) - f_i(u(t_{j-1}))| = V_i, \quad i = 1, \ldots, n.$$

Since $P_i(t) \leqq V_i$, $N_i(t) \leqq V_i$, it follows that

$$\|f_P(t)\| \leqq \|f(u(0))\| + \|P_1(t), \ldots, P_n(t)\| \leqq \leqq \|f(u(0))\| + \|(V_1, \ldots, V_n)\|, \tag{2,20}$$
$$\|f_N(t)\| \leqq \|(V_1, \ldots, V_n)\|.$$

Furthermore, we have

$$\|(V_1, \ldots, V_n)\| = \|(\sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_1(u(t_j)) - f_1(u(t_{j-1}))|, \ldots, \sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_n(u(t_j)) - f_n(u(t_{j-1}))|)\| = \sup_{\mathfrak{A}m} \|(\sum_{j=1}^{m} |f_1(u(t_j)) - f_1(u(t_{j-1}))|, \ldots, \sum_{j=1}^{m} |f_n(u(t_j) - f_n(u(t_{j-1}))|)\|. \tag{2,21}$$

The equation (2,21) may then be proved as follows: To every $\varepsilon > 0$ there exists a subdivision $\mathfrak{A}_{m1}$ such that

$$\sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_i(u(t_j)) - f_i(u(t_{j-1}))| - \sum_{j=1}^{m_1} |f_i(u(t_j)) - f_i(u(t_{j-1}))| < \varepsilon$$

for $i = 1, \ldots, n$.

Since all the norms introduced in footnote[1]) are equivalent to the Euclidean norm, there exists a constant γ (independent of ε) such that

$$\|(\sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_1(u(t_j)) - f_1(u(t_{j-1}))|, \ldots, \sup_{\mathfrak{A}m} \sum_{j=1}^{m} |f_n(u(t_j)) - f_n(u(t_{j-1}))|)\| - \|(\sum_{j=1}^{m_1} |f_1(u(t_j)) - f_1(u(t_{j-1}))|, \ldots, \sum_{j=1}^{m_1} |f_n(u(t_j)) - f_n(u(t_{j-1}))|)\| < \gamma\varepsilon.$$

In virtue of (2,21) and using the properties of f we have

$$\|(V_1, \ldots, V_n)\| \leqq \sup_{\mathfrak{A}m} \sum_{j=1}^{m} \|(|f_1(u(t_j)) - f_1(u(t_{j-1}))|, \ldots, |f_n(u(t_j)) - f_n(u(t_{j-1}))|)\| =$$
$$= \sup_{\mathfrak{A}m} \sum_{j=1}^{m} \|(f_1(u(t_j)) - f_1(u(t_{j-1})), \ldots, f_n(u(t_j)) - f_n(u(t_{j-1})))\| \leqq$$
$$\leqq \sup_{\mathfrak{A}m} \sum_{j=1}^{m} L_f A(t_j - t_{j-1}) = L_f AT.$$

Hence the lemma follows from (2,20).

Theorem 2,2. *Let $x(t, \lambda)$ be a solution of (2,18) for $0 < \lambda < 1$ defined on $\langle 0, T\rangle$ and satisfying the initial condition $x(0, \lambda) = x_0(\lambda)$. Let $y(t)$ be a solution of (2,19) defined on $\langle 0, T\rangle$, $y(0) = y_0$.*

Then there exist non-negative constants k_4, k_5 (independent of λ) such that

$$\|x(T, \lambda) - y(T)\| \leqq k_4\lambda^{1-\alpha} + k_5\|x_0(\lambda) - y_0\| .$$

Proof. Let $0 < \lambda < 1$. Consider the function $z(t, \lambda) = x(t, \lambda) - y(t)$. Substituting the solutions $x(t, \lambda)$, $y(t)$ into the equations (2,18), (2,19) respectively, and subtracting (2,19) from (2,18) we obtain

$$\frac{\mathrm{d}z(t, \lambda)}{\mathrm{d}t} = q(t, \lambda)\, f(x(t, \lambda)) + g(x(t, \lambda), t) - g(y(t), t) ,$$

whence

$$\|z(t, \lambda) - z(0, \lambda)\| \leqq \|\int_0^t q(\sigma, \lambda)\, f(x(\sigma, \lambda))\, \mathrm{d}\sigma\| + \\ + \|\int_0^t [g(x(\sigma, \lambda), \sigma) - g(y(\sigma), \sigma)]\, \mathrm{d}\sigma\| . \tag{2,22}$$

Obviously $\|x(t_2, \lambda) - x(t_1, \lambda)\| \leqq (A_f\lambda^{-\alpha} + A_g)(t_2 - t_1)$ for $0 \leqq t_1 \leqq t_2 \leqq T$. In virtue of lemma 2,1 there exist functions $f_P(t)$, $f_N(t)$ (depending on λ) such that $f(x(t, \lambda)) = f_P(t) - f_N(t)$, whose components $f_{Pi}(t)$, $f_{Ni}(t)$, $i = 1, \ldots, n$, are non-decreasing on $\langle 0, T\rangle$,

$$\|f_P(t)\| \leqq \|f(x_0(\lambda))\| + L_f(A_f\lambda^{-\alpha} + A_g)\, T ,$$
$$\|f_N(t)\| \leqq L_f(A_f\lambda^{-\alpha} + A_g)\, T .$$

The components of the vector $\int_0^t q(\sigma, \lambda)\, f(x(\sigma, \lambda))\, \mathrm{d}\sigma$ are $\int_0^t \sum_{i=1}^n q_{ji}(\sigma, \lambda)\, f_i(x(\sigma, \lambda))\, \mathrm{d}\sigma$, $f_i(x(\sigma, \lambda)) = f_{Pi}(\sigma) - f_{Ni}(\sigma)$ and, consequently, the second mean-value theorem may be used. Hence $\|\int_0^t q(\sigma, \lambda)\, f(x(\sigma, \lambda))\, \mathrm{d}\sigma\| \leqq \|\int_0^t q(\sigma, \lambda)\, f_P(\sigma)\, \mathrm{d}\sigma\| + \|\int_0^t q(\sigma, \lambda)\, f_N(\sigma)\, \mathrm{d}\sigma\| \leqq k_6\lambda^{1-\alpha}$.

From (2,22) we obtain

$$\|z(t, \lambda)\| \leqq \|z(0, \lambda)\| + k_6\lambda^{1-\alpha} + L_g\int_0^t \|z(\sigma, \lambda)\|\, \mathrm{d}\sigma$$

and

$$\|z(t, \lambda)\| \leqq (\|z(0, \lambda)\| + k_6\lambda^{1-\alpha}) \exp(L_g t)$$

for $0 \leqq t \leqq T$, $0 < \lambda < 1$. This completes the proof of theorem 2,2.

Note 2,1. The following more general proposition may be proved analogously: Theorem 2,2 remains true if in (2,18) the member $q(t, \lambda)\, f(x)$ is replaced by $\sum_{i=1}^s q_i(t, \lambda)\, f^{(i)}(x)$, where q_i are $n \times n$ matrices satisfying the conditions:

$$\max_{1\leqq i\leqq s} \|q_i(t, \lambda)\| \leqq \lambda^{-\alpha} , \quad \max_{1\leqq i\leqq s} \|\int_{t_1}^{t_2} q_i(t, \lambda)\, \mathrm{d}t\| \leqq \lambda ,$$
$$\|f^{(i)}(x)\| \leqq A_f , \quad \|f^{(i)}(x_2) - f^{(i)}(x_1)\| \leqq L_f\|x_2 - x_1\| \quad \text{for} \quad i = 1, \ldots, s .$$

Example 2,1. Some problems of particle motion in accelerators lead to the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X_0(x) + \sum_{i=1}^{s} \{B_{jk}^{(i)} \cos (\omega_{jk}^{(i)} t + C_{jk}^{(i)})\} A^{(i)}(x) , \tag{2,23}$$

$$X_0(x) , A^{(i)}(x) \in E_n \text{ for } x \in E_n , \quad \|X_0(x_2) - X_0(x_1)\| \leqq L_g\|x_2 - x_1\| ,$$
$$\|A^{(i)}(x_2) - A^{(i)}(x_1)\| \leqq L_f\|x_2 - x_1\| , \quad x_1, x_2 \in E_n ,$$

$B_{jk}^{(i)}, C_{jk}^{(i)}, \omega_{jk}^{(i)}$ are real numbers, $i = 1, \ldots, s$, $j, k = 1, \ldots, n$, $\min\limits_{i,j,k} \omega_{jk}^{(i)}$ is large in some sense.

Equation (2,23) is often solved by means of the approximate method of Kryloff-Bogolyuboff which consists in replacing the r. h. s. of equation (2,23) by the expression

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [\mathrm{X}_0(x) + \sum_{i=1}^{s} \{B_{jk}^{(i)} \cos (\omega_{jk}^{(i)} t + C_{jk}^{(i)})\} A^{(i)}(x)] \,\mathrm{d}t = \mathrm{X}_0(x) .$$

Note 2,1 makes it possible to estimate the error of this approximate solution. Indeed, let us put

$$X_0(x) = g(x, t) , \quad f^{(i)}(x) = A^{(i)}(x) ,$$
$$q_i(t, \lambda) = \{B_{jk}^{(i)} \cos (\omega_{jk}^{(i)} t + C_{jk}^{(i)})\} , \quad i = 1, \ldots, s .$$

Evidently

$$\|\int_{t_1}^{t_2} q_i(t, \lambda) \,\mathrm{d}t\| \leqq c_1(\min_{i,j,k} \omega_{jk}^{(i)})^{-1} , \quad \|q_i(c, \lambda)\| \leqq c_2 ,$$

$i = 1, \ldots, s$, $t, t_1, t_2 \in (-\infty, \infty)$, where c_1, c_2 are constants independent of $\omega_{jk}^{(i)}$. If we put $(\min\limits_{i,j,k} \omega_{jk}^{(i)})^{-1} = \lambda$, note 2,1 may be used for $\alpha = 0$ and we conclude that the order of error of this approximate solution is λ, i. e. $(\min\limits_{i,j,k} \omega_{jk}^{(i)})^{-1}$.

In the final part of this paragraph we will consider the equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\{P(x, t, \lambda) + G(x, t)\} , \tag{2,24}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}G(y, t) . \tag{2,25}$$

Here $P(x, t, \lambda) \in E_n$ is defined on $H \times \langle 0, T\rangle \times \langle 0, 1\rangle$, where H is an open subset of E_n. For $x, x_1, x_2 \in H$, $0 \leqq t_1 \leqq t_2 \leqq T$, $0 < \lambda < 1$ we assume

$$\|P(x, t_2, \lambda) - P(x, t_1, \lambda)\| \leqq A_3 \min [(t_2 - t_1)^\beta, \lambda] ,$$
$$\|P(x_2, t_2, \lambda) - P(x_2, t_1, \lambda) - P(x_1, t_2, \lambda) + P(x_1, t_1, \lambda)\| \leqq$$
$$\leqq \|x_2 - x_1\| L_3 \min [(t_2 - t_1)^\beta, \lambda] , \quad \tfrac{1}{2} < \beta \leqq 1 ,$$

A_3, L_3 positive.

$G(x, t) \in E_n$ is defined on $H \times \langle 0, T\rangle$ and satisfies the inequalities

$$\|G(x, t_2) - G(x, t_1)\| \leqq A_4(t_2 - t_1)\,, \quad A_4 = \text{const}\,,$$
$$\|G(x_2, t_2) - G(x_2, t_1) - G(x_1, t_2) + G(x_1, t_1)\| \leqq$$
$$\leqq \|x_2 - x_1\|\, L_4(t_2 - t_1)\,, \quad L_4 = \text{const}\,,$$

whenever $x, x_1, x_2 \in H$, $0 \leqq t_1 < t_2 \leqq T$.

Theorem 2,3. *Let k_7, k_8 be positive constants. Let $x(\tau, \lambda)$, $0 \leqq \lambda \leqq 1$, be a solution of the equation* (2,24) *defined on $\langle 0, T\rangle$ which satisfies the conditions*

$$\|x(t_2, \lambda) - x(t_1, \lambda)\| \leqq k_7\{A_3 \min\,[(t_2 - t_1)^\beta, \lambda] + A_4(t_2 - t_1)\} \tag{2,26}$$

for $0 \leqq t_1 < t_2 \leqq T$, $x(0, \lambda) = x_0(\lambda)$.

Let $y(\tau)$ be a solution of (2,25) *defined on $\langle 0, T\rangle$ and satisfying the inequality*

$$\|y(t_2) - y(t_1)\| \leqq k_8\, A_4(t_2 - t_1) \tag{2,27}$$

for $0 \leqq t_1 \leqq t_2 \leqq T$, $y(0) = y_0$.

Then there exist constants k_9, k_{10} (independent of λ) such that

$$\|x(\tau, \lambda) - y(\tau)\| \leqq k_9 \lambda^{2 - \frac{1}{\beta}} + k_{10}\|x_0(\lambda) - y_0\|$$

for $0 \leqq \tau \leqq T$, $0 \leqq \lambda \leqq 1$.

Proof. Let us denote $z(\tau, \lambda) = x(\tau, \lambda) - y(\tau)$ for $0 \leqq \lambda \leqq 1$, $\tau \in \langle 0, T\rangle$. Substituting the solutions $x(\tau, \lambda)$, $y(\tau)$ into the equations (2,24), (2,25) respectively and subtracting (2,25) from (2,24), we obtain

$$z(\tau, \lambda) - z(0, \lambda) = \int_0^\tau \mathrm{D}P(x(\sigma, \lambda), t, \lambda) + \int_0^\tau \mathrm{D}[G(z(\sigma, \lambda) + y(\sigma), t) - G(y(\sigma), t)] \tag{2,28}$$

for $0 \leqq \tau \leqq T$, $0 \leqq \lambda \leqq 1$.

The first integral on the right hand side of equation (2,28) may be estimated as follows: If $0 \leqq \tau_1 \leqq \tau_2 \leqq T$, $0 \leqq \lambda \leqq 1$, then

$$\|P(x(\tau_2, \lambda), \tau_2, \lambda) - P(x(\tau_2, \lambda), \tau_1, \lambda) - P(x(\tau_1, \lambda), \tau_2, \lambda) +$$
$$+ P(x(\tau_1, \lambda), \tau_1, \lambda)\| \leqq \|x(\tau_2, \lambda) - x(\tau_1, \lambda)\|\, L_3 \min\,[(\tau_2 - \tau_1)^\beta, \lambda] \leqq$$
$$\leqq k_7\{A_3 \min\,[(\tau_2 - \tau_1)^\beta, \lambda] + A_4(\tau_2 - \tau_1)\}\, L_3 \min\,[(\tau_2 - \tau_1)^\beta, \lambda] =$$
$$= \psi(\tau_2 - \tau_1)\,.$$

Evidently the function $\psi(\eta)$ is continuous and non-decreasing on $\langle 0, T\rangle$.

If $0 \leqq \eta \leqq \min [T, \lambda^{\frac{1}{\beta}}]$, we have

$$\Psi(\eta) = \sum_{i=1}^{\infty} \frac{2^i}{\eta} \psi\left(\frac{\eta}{2^i}\right) = k_7 L_3 \sum_{i=1}^{\infty} A_3 \left(\frac{\eta}{2^i}\right)^{2\beta-1} + k_7 L_3 A_4 \left(\frac{\eta}{2^i}\right)^{\beta} =$$
$$= k_7 L_3 A_3 \eta^{2\beta-1} \frac{1}{2^{2\beta-1}-1} + k_7 L_3 A_4 \, \eta^{\beta} \frac{1}{2^{\beta}-1} .$$

If $\lambda^{\frac{1}{\beta}} \leqq \eta \leqq T$, evidently there exists a positive integer ν (depending on η, λ) such that $\frac{\eta}{2^{\nu}} < \lambda^{\frac{1}{\beta}}$, $\frac{\eta}{2^{\nu-1}} \geqq \lambda^{\frac{1}{\beta}}$ and

$$\Psi(\eta) = \sum_{i=1}^{\nu-1} \frac{2^i}{\eta} \psi\left(\frac{\eta}{2^i}\right) + \sum_{i=\nu}^{\infty} \frac{2^i}{\eta} \psi\left(\frac{\eta}{2^i}\right) = 2k_7 L_3 A_3 \lambda^2 \frac{2^{\nu-1}-1}{\eta} + k_7 L_3 A_4 \lambda(\nu - 1) +$$
$$+ k_7 L_3 A_3 \frac{\eta^{2\beta-1}}{2^{\nu(2\beta-1)}} \frac{1}{2^{2\beta-1}-1} + k_7 L_3 A_4 \left(\frac{\eta}{2^{\nu}}\right)^{\beta} \frac{1}{2^{\beta}-1} .$$

Hence $\sum_{i=1}^{\infty} 2^i \psi\left(\frac{T}{2^i}\right) < \infty$ and in virtue of theorem 1,1

$$\left\| \int_0^{\tau} \mathrm{D}\mathrm{P}(x(\sigma, \lambda), t, \lambda) - \mathrm{P}(x_0(\lambda), \tau, \lambda) + \mathrm{P}(x_0(\lambda), 0, \lambda) \right\| \leqq \frac{\tau}{2} \Psi(\tau)$$

for $0 \leqq \tau \leqq T$, $0 \leqq \lambda \leqq 1$.

Thus it follows that there exists a positive constant c_3 such that

$$\left\| \int_0^{\tau} \mathrm{D}P(x(\sigma, \lambda), t, \lambda) \right\| \leqq c_3 \lambda^{2-\frac{1}{\beta}} \tag{2,29}$$

for $0 \leqq \tau \leqq T$, $0 \leqq \lambda \leqq 1$.

To estimate the second integral on the right hand side of (2,28) we shall proceed as follows: If $0 \leqq \tau_1 \leqq \tau_2 \leqq T$, $0 \leqq \lambda \leqq 1$, then

$$\|G(x(\tau_2, \lambda), \tau_2) - G(y(\tau_2), \tau_2) - G(x(\tau_2, \lambda), \tau_1) + G(y(\tau_2), \tau_1) -$$
$$- G(x(\tau_1, \lambda), \tau_2) + G(y(\tau_1), \tau_2) + G(x(\tau_1, \lambda), \tau_1) - G(y(\tau_1), \tau_1)\| \leqq$$
$$\leqq \|x(\tau_2, \lambda) - x(\tau_1, \lambda)\| \, L_4(\tau_2 - \tau_1) + \|y(\tau_2) - y(\tau_1)\| \, L_4(\tau_2 - \tau_1) \leqq$$
$$\leqq \{k_7[A_3 \min [(\tau_2 - \tau_1)^{\beta}, \lambda] + A_4(\tau_2 - \tau_1)] + k_8 A_4(\tau_2 - \tau_1)\} \, L_4(\tau_2 - \tau_1) .$$

Let us denote

$$L_4 \, \eta\{k_7[A_3 \min [\eta^{\beta}, \lambda] + A_4 \eta] + k_8 A_4 \eta\} = \psi_1(\eta) ,$$
$$\Psi_1(\eta) = \sum_{i=1}^{\infty} \psi_1\left(\frac{\eta}{2^i}\right) \frac{2^i}{\eta} \quad \text{for} \quad 0 \leqq \eta \leqq T, \quad 0 \leqq \lambda \leqq 1 .$$

(Analogically as in the case of $\psi(\eta)$ we verify that $\sum_{i=1}^{\infty} \psi_1\left(\frac{T}{2^i}\right) 2^i < \infty$.)

Let $\tau \in \langle 0, T \rangle$. In virtue of theorem 1,1 we have for every positive integer n_1 and for $0 = t_1 < t_2 < \ldots < t_{n_1} = \tau$, $t_{i+1} = t_i + \frac{\tau}{n_1}$:

$$\|\int_0^\tau \mathrm{D}[G(x(\sigma, \lambda), t) - G(y(\sigma), t)]\| \leqq \sum_{i=1}^{n_1-1} \|\int_{t_i}^{t_{i+1}} \mathrm{D}[G(x(\tau, \lambda), t) - G(y(\tau), t)]\| \leqq$$

$$\leqq \sum_{i=1}^{n_1-1} \|G(x(t_i, \lambda), t_{i+1}) - G(y(t_i), t_{i+1}) - G(x(t_i, \lambda), t_i) + G(y(t_i), t_i)\| +$$

$$+ (t_{i+1} - t_i)\, \Psi_1(t_{i+1} - t_i) \leqq \sum_{i=1}^{n_1-1} \|x(t_i, \lambda) - y(t_i)\|\, L_4(t_{i+1} - t_i) +$$

$$+ (t_{i+1} - t_i)\, \Psi_1(t_{i+1} - t_i) \leqq \sum_{i=1}^{n_1-1} \|z(t_i, \lambda)\|\, L_4(t_{i+1} - t_i) + \tau\, \Psi_1\left(\frac{\tau}{n_1}\right).$$

Since $z(\tau, \lambda)$ is continuous in τ and $\lim_{\eta \to 0} \Psi_1(\eta) = 0$, it follows that

$$\|\int_0^\tau \mathrm{D}[G(x(\sigma, \lambda), t) - G(y(\sigma), t)]\| \leqq L_4 \int_0^\tau \|z(t, \lambda)\|\, \mathrm{d}t\,.$$

Thence and from (2,28), (2,29) we obtain

$$\|z(\tau, \lambda)\| \leqq \|z(0, \lambda)\| + c_3 \lambda^{2-\frac{1}{\beta}} + L_4 \int_0^\tau \|z(t, \lambda)\|\, \mathrm{d}t\,,$$

i. e.

$$\|z(\tau, \lambda)\| \leqq [c_3 \lambda^{2-\frac{1}{\beta}} + \|z(0, \lambda)\|] \exp (L_4\, \tau)\,.$$

The proof of theorem 2,3 is thus complete.

Note 2,2. The assumptions (2,26), (2,27) in theorem 2,3 represent no substantial limitation. Indeed, using theorem 1,2 and note 1,2 we conclude that conditions (2,26), (2,27) are fulfilled for sufficiently large constants k_7, k_8, if the number σ^* in theorem 1,2 can be chosen independent of λ. The latter condition is satisfied if

$$\eta\, \Psi_2(\eta) < \omega_1(\eta) \tag{2,30}$$

for $0 < \eta < 2\sigma^*$, where $\omega_1(\eta) = A_3 \min (\eta^\beta, \lambda) + A_4 \eta$, $\omega_2(\eta) = L_3 \min (\eta^\beta, \lambda) + L_4 \eta)$, $\Psi_2(\eta) = \sum_{i=1}^{\infty} \frac{2^{i+1}}{\eta}\, \omega_1\left(\frac{\eta}{2^i}\right) \omega_2\left(\frac{\eta}{2^i}\right)$, $0 \leqq \lambda \leqq 1$ (cf. note 1,2).

We shall now prove that there exists a $\sigma^* > 0$ such that (2,30) is fulfilled for $0 \leqq \lambda \leqq 1$. It can be readily verified that $\Psi_2(\eta) \leqq c_4 \eta^{2\beta-1}$ for $0 \leqq \eta \leqq \lambda^{\frac{1}{\beta}}$, $\Psi_2(\eta) \leqq c_5 \lambda^{2-\frac{1}{\beta}} + c_6 \eta$ for $\lambda^{\frac{1}{\beta}} \leqq \eta \leqq T$, where c_4, c_5, c_6 are constants independent of λ. It follows that the relation (2,30) may be written in the form

$$\begin{aligned} & c_4 \eta^{2\beta-1} < A_3 \eta^{\beta-1} + A_4 && \text{for } 0 < \eta \leqq \lambda^{\frac{1}{\beta}}\,, \\ & c_5 \lambda^{2-\frac{1}{\beta}} + c_6 \eta < A_3 \lambda \eta^{-1} + A_4 && \text{for } \lambda^{\frac{1}{\beta}} \leqq \eta \leqq T\,, \\ & && 0 \leqq \lambda \leqq 1\,. \end{aligned} \tag{2,31}$$

Let us choose $\lambda \in \langle 0, 1\rangle$. Evidently there exists an $\eta_1 > 0$ such that the relation (2,31) holds for $0 < \eta \leqq \eta_1$. Denote the upper bound of these η_1 by η_2 (η_2 evidently depends on λ). Let $\eta_3 = \inf_{0 \leqq \lambda \leqq 1} \eta_2$. We will prove that $\eta_3 > 0$. Indeed, suppose that $\eta_3 = 0$. Then there exist two sequences $\{\eta^{(i)}\}$, $\{\lambda^{(i)}\}$, $i = 1, 2, \ldots$, $\eta^{(i)} \to 0$, $\lambda^{(i)} \to \lambda_0$ for $i \to \infty$, $0 < \eta^{(i)} \leqq T$, $0 \leqq \lambda^{(i)} \leqq 1$, such that (2,31) is not true for $\eta = \eta^{(i)}$, $\lambda = \lambda^{(i)}$, $i = 1, 2, \ldots$ But this is a contradiction, as the first of inequalities (2,31) is always fulfilled for η sufficiently small and the second one is evidently satisfied if $\lambda_0 = 0$. (If $\lambda_0 > 0$, then $\eta^{(i)} < (\lambda^{(i)})^{\frac{1}{\beta}}$ for i sufficiently large and (2,31) reduces to the first inequality.)

Note 2,3. In [1] the notion of a regular solution of a generalized differential equation was introduced, and all fundamental theorems in the theory of generalized equations were proved for regular solutions only. The assumption of regularity corresponds, in essence, to the assumptions (2,26), (2,27).

Note 2,4. If we put $\beta = \dfrac{1}{1 + \alpha}$ in theorem 2,3, the assumptions of theorem 2,3 are fulfilled whenever the conditions of theorem 2,2 are satisfied. Consequently, theorem 2,3 relating to the theory of generalized differential equations yields, under more general conditions, substantially the same estimate as theorem 2,2.

The following example shows that the estimate in theorem 2,3 cannot be improved.

Example 2,2. Let us consider the scalar equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\lambda^{-\alpha} \cos \frac{t}{\lambda^{1+\alpha}} + \lambda^{-\alpha} \sin \frac{t}{\lambda^{1+\alpha}} \tag{2,32}$$

for $0 < \lambda < 1$, $0 \leqq \alpha < 1$ with the initial condition $x(0, \lambda) = 0$.

We obtain

$$x\,(t,\lambda) = \exp\left(\lambda \sin \frac{t}{\lambda^{1+\alpha}}\right) \int_0^t \lambda^{-\alpha} \sin \frac{\sigma}{\lambda^{1+\alpha}} \exp\left(-\lambda \sin \frac{\sigma}{\lambda^{1+\alpha}}\right) \mathrm{d}\sigma =$$

$$= \exp\left(\lambda \sin \frac{t}{\lambda^{1+\alpha}}\right) \left[\lambda\left(1 - \cos \frac{t}{\lambda^{1+\alpha}}\right) - \int_0^t \lambda^{1-\alpha} \sin^2 \frac{\sigma}{\lambda^{1+\alpha}} \mathrm{d}\sigma + O(\lambda^{2-\alpha})\right] =$$

$$= \exp\left(\lambda \sin \frac{t}{\lambda^{1+\alpha}}\right) \left[\lambda\left(1 - \cos \frac{t}{\lambda^{1+\alpha}}\right) - \lambda^{1-\alpha}\left(\frac{t}{2} - \frac{\lambda^{1+\alpha}}{4} \sin \frac{2t}{\lambda^{1+\alpha}}\right) +\right.$$

$$\left. + O(\lambda^{2-\alpha})\right] = -\lambda^{1-\alpha} \frac{t}{2} + O(\lambda)\,.$$

If we write equation (2,32) in the form

$$\frac{dx}{dt} = D\left(x\lambda \cos \frac{t}{\lambda^{1+\alpha}} + \lambda \sin \frac{t}{\lambda^{1+\alpha}}\right),$$

it can be seen readily that (2,32) is a special case of (2,24) where we have put $\beta = \frac{1}{1+\alpha}$.

The following example shows that theorem 2,3 yields, even in the case of classical equations, a better estimate than note 2,1.

Example 2,3. Let there be given a scalar equation

$$\frac{dx}{dt} = f_1(x)\, \lambda^{-\gamma} \cos \lambda^{-(1+\gamma)}t + f_2(x)\, \lambda^{-\varkappa\gamma} \cos \lambda^{-\varkappa(1+\gamma)}t\,, \tag{2,33}$$

$0 \leqq \gamma < 1$, $\varkappa > 1$, $0 < \lambda < 1$, $f_1(x)$, $f_2(x)$ are bounded and fulfil a Lipschitz condition on $(-\infty, \infty)$. Let $x(t, \lambda)$ be a solution of (2,33) defined for $t \in \langle 0, T\rangle$ and satisfying the condition $x(0, \lambda) = 0$ for $0 < \lambda < 1$. From note 2,1 we have $|x(t, \lambda)| \leqq k_{11}\lambda^{1-\varkappa\gamma}$ for $0 \leqq t \leqq T$, $0 < \lambda < 1$. From theorem 2,3 we obtain $|x(t, \lambda)| \leqq k_{12}\lambda^{1-\gamma}$.

References

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Výtah

ODHADY VZDÁLENOSTI DVOU ŘEŠENÍ ZALOŽENÉ NA TEORII ZOBECNĚNÝCH DIFERENCIÁLNÍCH ROVNIC

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Článek navazuje na práci [1], kde byl zaveden pojem zobecněné diferenciální rovnice a dokázána věta o spojité závislosti na parametru, a je jejím doplněním v tom smyslu, že obsahuje odhady vzdálenosti řešení dvou rovnic, zatím co v [1] jsou věty limitního charakteru.

Definujme třídu $\varphi(K, \alpha, \beta)$ vektorových funkcí $f(x, t)$ takto: $f(x, t) \in \varphi$, jsou-li pro $(x, t) \in G$, kde G je otevřená množina v E_{n+1}, splněny tyto podmínky:

a) $|\int_{t_1}^{t_2} f(x, t)\, dt| \leqq K|t_2 - t_1|^\alpha$,

b) $|\int_{t_1}^{t_2} [f(x_2, t) - f(x_1, t)]\,dt| \leqq |x_2 - x_1|\, K|t_2 - t_1|^\beta$,

K, α, β jsou kladná čísla, $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$, $0 \leqq t_1 \leqq t_2 \leqq T$.

Věta 0,1. *Budiž* $f(x, t), \bar{f}(x, t) \in \varphi(K, \alpha, \beta)$, $|\int_{t_1}^{t_2} [f(x, t) - \bar{f}(x, t)]\,dt| \leqq \omega(|t_2 - t_1|)$. *Budiž* $x(t)$ *resp.* $y(t)$ *řešením rovnice* (0, 1) *resp.* (0, 4) *splňující počáteční podmínky* $x(0) = y(0) = x_0$.

Potom pro každé přirozené m *platí nerovnost* (0, 5), *kde* c_1 *je kladná konstanta nezávislá na* m.

Speciálně, je-li $f(x, t)$ omezená na G a platí $|f(x_2, t) - f(x_1, t)| \leqq L|x_2 - x_1|$, $|f(x, t) - \bar{f}(x, t)| \leqq M$, limitním přechodem pro $m \to \infty$ v (0,5) dostaneme známý odhad

$$|x(T) - y(T)| \leqq \frac{M}{L}\,(e^{LT} - 1)\,.$$

Má-li rovnice (0,1) konstantní řešení $x(t) = x_0$, potom zřejmě platí (0,6) pro $t_1, t_2 \in \langle 0, T\rangle$.

V práci [2] bylo dokázáno, že toto konstantní řešení je jediné, jestliže $f \in \varphi$. Lze nahlédnout, že řešení rovnice (0,1) se příliš nevzdálí od své počáteční hodnoty x_0, je-li rovnice (0,6) přibližně splněna.

Věta 0,2. *Budiž* $f(x, t) \in \varphi(K, \alpha, \beta)$. *Nechť existuje* $\varepsilon > 0$ *tak, že* $|\int_{t_1}^{t_2} f(x_0, t)\,dt| \leqq \varepsilon(t_2 - t_1)^\alpha$ *pro* $t_1, t_2 \in \langle 0, T\rangle$. *Budiž* $x(t)$ *řešením rovnice* (0, 1) *splňujícím počáteční podmínku* $x(0) = x_0$.

Potom existuje $T_1 \in \langle 0, T\rangle$ *(které závisí pouze na třídě* $\varphi(K, \alpha, \beta)$*) takové, že* $|x(t) - x_0| \leqq 3\varepsilon t^\alpha$ *pro* $0 \leqq t \leqq T_1$.

Následující věta může být užitečná v případě, že v rovnici (0, 7) μ představuje malý parametr.

Věta 0,3. *Budiž* $x(t)$ *řešením rovnice* (0, 7) *splňujícím počáteční podmínku* $x(0) = x_0$. *Budiž* $\mu > 0$ *a* $f(x, t) \in \varphi(K, \alpha, \beta)$.

Potom platí pro $0 \leqq t \leqq T$

$$|x(t) - x_0 - \mu \int_0^t f(x_0, \tau)\,d\tau| \leqq \mu^2 c_2 t^{\alpha+\beta}\,,$$

kde c_2 *závisí pouze na třídě* $\varphi(K, \alpha, \beta)$.

Smysl věty vynikne pro $\int_0^t f(x_0, \sigma)\,d\sigma = 0$.

Druhá část práce je věnována srovnání metod klasických a metod založených na teorii zobecněných rovnic. Účinnost nové metody je prověřována na rovnicích speciálního typu (0,8) a (0,9).

$p(t, \lambda)$ je skalární funkce definovaná a spojitá pro $t \in \langle 0, T \rangle$, $\lambda \in (0, 1)$ a splňuje podmínku $\int_{t_1}^{t_2} |p(\sigma, \lambda)\, d\sigma| \leqq \lambda$ pro $0 < \lambda < 1$, jakmile $t_1, t_2 \in \langle 0, T \rangle$. (Za funkci $p(t, \lambda)$ můžeme např. vzít $\frac{\lambda^{-\alpha}}{2} \sin \frac{t}{\lambda^{1+\alpha}}$, $\alpha > 0$.)

$f(x)$, $g(x, t)$ jsou spojité, $f(x)$ má spojité derivace druhého řádu a $g(x, t)$ splňuje Lipschitzovu podmínku s konstantou nezávislou na (x, t).

Věta 0,4. *Nechť $x(t, \lambda)$ $(y(t))$ je řešením rovnice (0, 8) ((0, 9)) s počáteční podmínkou $x(0) = y(0) = x_0$. Potom existuje kladná konstanta k_1 tak, že $|x(T, \lambda) - y(T)| \leqq k_1 \lambda$.*

Další věta se týká obecnější třídy rovnic, zato odhad vychází větší než v předchozím případě.

V rovnicích (0,10), (0,11) q značí čtvercovou matici spojitou v t, $|q(t, \lambda)| \leqq \lambda^{-\alpha}$, $0 < \alpha < 1$, $|\int_{t_1}^{t_2} q(t, \lambda)\, dt| \leqq \lambda$.

Funkce $f(x)$, $g(x, t)$ jsou spojité a splňují Lipschitzovu podmínku vzhledem k x s konstantou nezávislou na (x, t).

Věta 0,5. *Budiž $x(t, \lambda)$ $(y(t))$ řešením (0, 10) ((0, 11)) splňujícím počáteční podmínku $x(0) = y(0) = x_0$. Potom existuje konstanta k_2 (nezávislá na λ) tak, že $|x(T, \lambda) - y(T)| \leqq k_2 \lambda^{1-\alpha}$.*

Následující věta, na rozdíl od vět 0,4 a 0,5, byla dokázána na základě teorie zobecněných diferenciálních rovnic.

Věta 0,6. *Nechť $x(t, \lambda)$ $(y(t))$ je řešením rovnice (0, 12) ((0, 13)) splňujícím počáteční podmínku $x(0, \lambda) = y(0) = x_0$.*

$\bar{p}(x, t, \lambda)$ je vektorová funkce splňující podmínky:

$$|\int_{t_1}^{t_2} \bar{p}(x, t, \lambda)\, dt| \leqq A \min((t_2 - t_1)^\beta, \lambda),$$

$$|\int_{t_1}^{t_2} [\bar{p}(x_2, t, \lambda) - \bar{p}(x_1, t, \lambda)]\, dt| \leqq L \min((t_2 - t_1)^\beta, \lambda)$$

pro $0 \leqq t_1 < t_2 \leqq T$, A, L, β konstantní, $\frac{1}{2} < \beta < 1$. g je spojitá a splňuje Lipschitzovu podmínku s konstantou nezávislou na (x, t).

Potom existuje konstanta k_3 tak, že $|x(T, \lambda) - y(T)| \leqq k_3 \lambda^{2 - \frac{1}{\beta}}$.

Položíme-li $\beta = \frac{1}{1 + \alpha}$ ve větě 0,6, vidíme, že věta 0,6 dává za obecnějších předpokladů stejný odhad jako věta 0,5.

Je uveden příklad ukazující, že odhad ve větě 0,5 a 0,6 nelze již řádově zlepšit. Na dalším příkladu je ukázáno, že věta 0,6 dává podstatně lepší odhad než metoda založená na větě 0,5.

Резюме

ОЦЕНКИ РАССТОЯНИЯ МЕЖДУ ДВУМЯ РЕШЕНИЯМИ, ОСНОВАННЫЕ НА ТЕОРИИ ОБОБЩЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

ЗДЕНЕК ВОРЕЛ (Zdeněk Vorel), Прага

Статья примыкает к работе [1], где было введено понятие обобщенного дифференциального уравнения и была доказана теорема о непрерывной зависимости от параметра, и является ее дополнением в том смысле, что содержит оценки расстояния между решениями двух уравнений, тогда как в [1] приводятся теоремы предельного характера.

Определим класс $\varphi(K, \alpha, \beta)$ векторных функций $f(x, t)$ следующим образом: $f(x, t) \in \varphi$, если для $(x, t) \in G$, где G — открытое множество в E_{n+1}, выполняются следующие условия:

а) $|\int_{t_1}^{t_2} f(x, t)\,\mathrm{d}t| \leqq K|t_2 - t_1|^\alpha$,

б) $|\int_{t_1}^{t_2} [f(x_2, t) - f(x_1, t)]\,\mathrm{d}t \leqq |x_2 - x_1|\, K|t_2 - t_1|^\beta$,

K, α, β — положительные числа, $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$, $0 \leqq \leqq t_2 \leqq t_2 \leqq T$.

Теорема 0,1. *Пусть* $f(x, t)$, $\bar{f}(x, t) \in \varphi(K, \alpha, \beta)$, $|\int_{t_1}^{t_2} f(x, t) - \bar{f}(x, t)\,\mathrm{d}t \leqq \leqq \omega(|t_2 - t_1|)$.

Пусть $x(t)$ и $y(t)$ являются, соответственно, решениями уравнений (0,1) *и* (0,4), *удовлетворяющими начальным условиям* $x(0) = y(0) = x_0$.

Тогда для любого натурального m справедливо неравенство (0,5), *где c_1 — положительная постоянная, не зависящая от m.*

В частности, цсли $f(x, t)$ ограничена на Γ и $|f(x_2, t) - f(x_1, t)| \leqq \leqq L|x_2 - x_1|$, $|f(x, t) - \bar{f}(x, t)| \leqq M$, то предельным переходом для $m \to \to \infty$ в (0,5) мы получим известную оценку

$$|x(T) - y(T)| \leqq \frac{M}{L}(e^{LT} - 1)\,.$$

Если уравнение (0,1) обладает постоянным рещением $x(t) = x_0$, то, очевидно, справедливо (0,6) для t_1, $t_2 \in \langle 0, T \rangle$.

В работе [2] было доказано, что это постоянное решение является единственным, если $f \in \varphi$. Нетрудно видеть, что решение уравнения (0, 1) не слишком удалится от своего начального значения x_0, если уравнение (0,6) приближенно выполняется.

Теорема 0,2. *Пусть* $f(x, t) \in \varphi(K, \alpha, \beta)$. *Пусть существует* $\varepsilon > 0$ *так, что* $|\int_{t_1}^{t_2} f(x_0, t)\,\mathrm{d}t| \leqq \varepsilon |t_2 - t_1|^\alpha$, *для* $t_1, t_2 \in \langle 0, T\rangle$. *Пусть* $x(t)$ — *решение уравнения* (0,1), *удовлетворяющее начальному условию* $x(0) = x_0$.

Тогда существует $T_1 \in \langle 0, T\rangle$ (*зависящее только от класса* $\varphi(K, \alpha, \beta)$) *такое, что* $|x(t) - x_0| \leqq 3\varepsilon t^\alpha$ *для* $0 \leqq t \leqq T_1$.

Следующая теорема может быть полезна в том случае, когда в уравнении (0,7) μ является малым параметром.

Теорема 0,3. *Пусть* $x(t)$ — *решение уравнения* (0,7), *удовлетворяющее начальному условию* $x(0) = x_0$. *Пусть* $\mu > 0$ *и* $f(x, t) \in \varphi(K, \alpha, \beta)$.

Тогда для $0 \leqq t \leqq T$

$$|x(t) - x_0 - \mu \int_0^t f(x_0, \tau)\,\mathrm{d}\tau| \leqq \mu^2 c_2 t^{\alpha+\beta},$$

где c_2 *зависит только от класса* $\varphi(K, \alpha, \beta)$.

Смысл теоремы выступает более ясно для $\int_0^t f(x_0, \sigma)\,\mathrm{d}\sigma = 0$.

Вторая часть работы посвящается сравнению классических методов и методов, основанных на теории обобщенных уравнений. Эффективность нового метода проверяется на уравнениях специального типа (0,8) и (0,9).

$p(t, \lambda)$ — скалярная функция, определенная и непрерывная для $t \in \langle 0, T\rangle$, $\lambda \in (0, 1)$ и выполняющая условие $|\int_{t_1}^{t_2} p(\sigma, \lambda)\,\mathrm{d}\sigma| \leqq \lambda$ для $0 < \lambda < 1$, как только $t_1, t_2 \in \langle 0, T\rangle$. (В качестве функции $p(t, \lambda)$ можно, например, взять $\frac{\lambda^{-\alpha}}{2} \sin \frac{t}{\lambda^{1+\alpha}}$, $\alpha > 0$.)

$f(x)$, $g(x, t)$ непрерывны, $f(x)$ обладает непрерывными производными второго порядка, а $g(x, t)$ удовлетворяет условию Липшица с постоянной, не зависящей от $x(t)$.

Теорема 0,4. *Пусть* $x(t, \lambda)$ $(y(t))$ *является решением уравнения* (0,8) ((0,9)) *с начальным условием* $x(0) = y(0) = x_0$. *Тогда существует положительная постоянная* k_1 *так, что* $|x(T, \lambda) - y(T)| \leqq k_1 \lambda$.

Следующая теорема касается более общего класса уравнений, зато оценка получается больше, чем в предыдущем случае.

В уравнениях (0,10), (0,11) q означает квадратную матрицу, непрерывную относительно t,

$$|q(t, \lambda)| \leqq \lambda^{-\alpha}, \quad 0 < \alpha < 1, \quad |\int_{t_1}^{t_2} q(t, \lambda)\,\mathrm{d}t| \leqq \lambda .$$

Функции $f(x)$, $g(x, t)$ непрерывны и удовлетворяют условию Липшица по отношению к x с постоянной, не зависящей от (x, t).

Теорема 0,5. *Пусть* $x(t, \lambda)(y(t))$ *является решением* (0,10) ((0,11)), *удовлетворяющим начальному условию* $x(0) = y(0) = x_0$, *Тогда существует постоянная* k_2 *(не зависящая от* λ*) так, что* $|x(T, \lambda) - y(T)| \leqq k_2\lambda^{1-\alpha}$.

Следующая теорема, в отличие от теорем 0,4 и 0,5, была доказана на основании теории обобщенных дифференциальных уравнений.

Теорема 0,6. *Пусть* $x(t, \lambda)(y(t))$ *является решением уравнения* (0,12) ((0,13)) *удовлетворяющим начальному условию* $x(0, \lambda) = y(0) = x_0$.

$\bar{p}(x, t, \lambda)$ *есть векторная функция, удовлетворяющая улсовиям*:

$$\left|\int_{t_1}^{t_2} \bar{p}(x, t, \lambda)\, \mathrm{d}t\right| \leqq A \min\left((t_2 - t_1)^\beta, \lambda\right),$$

$$\left|\int_{t_1}^{t_2} [\bar{p}(x_2, t, \lambda) - \bar{p}(x_1, t, \lambda)]\, \mathrm{d}t\right| \leqq L \min\left((t_2 - t_1)^\beta, \lambda\right)$$

для $0 \leqq t_1 < t_2 \leqq T$, A, L, β *— постоянные,* $\frac{1}{2} < \beta < 1$. g *— непрерывная функция, удовлетворяющая условию Липшица с постоянной, не зависящей от* (x, t).

Тогда существует постоянная k_3 так, что

$$|x(T, \lambda) - y(T)| \leqq k_3\lambda^{2-1/\beta}.$$

Если положить $\beta = 1/1 + \alpha$ в теореме 0,6, то видно, что теорема 0,6 дает в более общих предположениях ту же оценку, как и теорема 0,5.

Приводится пример, показывающий, что порядок оценки в теоремах 0,5 и 0,6 уже нельзя улучшить. На следующем примере показано, что теорема 0,6 дает существенно лучшую оценку, чем метод, основанный на теореме 0,5.