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DERIVATIONS ON THE ALGEBRA OF DIFFERENTIAL FORMS OF INFINITE ORDER ON A MANIFOLD

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Summary. The paper deals with the algebra of differential forms of higher orders on a differentiable manifold. All derivations on this algebra are described, and their structure is investigated.

Keywords: Differentiable manifold, p-form of higher order, derivation on an algebra.

AMS Classification: 58A10.

The goal of this paper is to describe the structure of all derivations on the algebra of differential forms of order $\leqq \infty$ on a differentiable manifold. The structure of derivations on the algebra of differential forms of a fixed finite order was studied in [3]. It is well known that any differential form of order $\leqq \infty$ on a compact manifold has finite order, so that forms of order ∞ appear only on noncompact manifold. Thus in the case of a compact manifold the algebra of differential forms of order $\leqq \infty$ coincides with the algebra of differential forms of all finite orders.

Throughout this paper differentiable will mean differentiable of class C^∞. Together with differentiable manifolds we shall need differentiable $\mathbb{R}^\infty$-manifolds in the sense of [1]. Let us recall that $\mathbb{R}^\infty$ is a topological vector space arising as the projective limit $\varprojlim \mathbb{R}^k$ of the projective system $\{\mathbb{R}^k; p_k^l\}$, where $\mathbb{R}^k$ has its standard topology, and the projections $p_k^l: \mathbb{R}^l \to \mathbb{R}^k$, $0 \leqq k \leqq l < \infty$ are defined by the formula $p_k^l(x_1, \ldots, x_k, x_{k+1}, \ldots, x_l) = (x_1, \ldots, x_k)$. There is a natural way (see [1]) how to define the notion of a differentiable mapping $f: U \to \mathbb{R}^\infty$, where $U \subset \mathbb{R}^\infty$ is an open subset. Especially we can define the notion of a local diffeomorphism on $\mathbb{R}^\infty$. Then a differentiable $\mathbb{R}^\infty$-manifold is defined along the same lines as a differentiable manifold, using only $\mathbb{R}^\infty$ instead of $\mathbb{R}^k$. Further, together with (differentiable) vector bundles we shall need (differentiable) $\mathbb{R}^\infty$-vector bundles. They are again defined in the same way as vector bundles, only we must take $\mathbb{R}^\infty$ instead of $\mathbb{R}^k$ as the standard fibre.

Let M be a connected paracompact differentiable manifold, $\dim M = m < \infty$. For $x \in M$ we denote by $T_x = T_xM$ the tangent space of M at x, and by $T = TM = \bigcup_{x \in M} T_xM$ the tangent bundle of M. For any $0 \leqq r \leqq \infty$ we denote by J_x^rT the vector space of all r-jets of local differentiable sections of T at x. Obviously $J_x^0T = T_x$.

If $0 \leqq r < \infty$ then $J^r_x T$ is a vector space of finite dimension and we shall provide it with its standard topology and differentiable manifold structure. For $0 \leqq r \leqq s < < \infty$ we have the natural projection $\pi^s_r\colon J^s_x T \to J^r_x T$. We get the projective system $\{J^r_x T; \pi^s_r\}$ of vector spaces, and obviously $J^\infty_x T = \varprojlim J^r_x T$. This enables us to endow $J^\infty_x T$ with the projective limit topology. With this topology $J^\infty_x T$ is a topological vector space. Moreover, it can be easily seen that $J^\infty_x T$ can be provided with a natural $\mathbb{R}^\infty$-manifold structure and that with this structure $J^\infty_x T$ is linearly diffeomorphic with $\mathbb{R}^\infty$.

As usual we set $J^r T = \bigcup_{x \in M} J^r_x T$ for $0 \leqq r \leqq \infty$. We have $J^0 T = T$. It is well known that $J^r T$ for $0 \leqq r < \infty$ carries a natural structure of a differentiable manifold and a natural structure of a vector bundle over M. In the case of $J^\infty T$ the situation is not much different. For any $0 \leqq r \leqq s < \infty$ we have the natural projection $\pi^s_r\colon J^s T \to \to J^r T$, and it is obvious that $\{J^r T; \pi^s_r\}$ is a projective system of vector bundles. In the appropriate category of families of vector spaces we have

$$J^\infty T = \varprojlim J^r T .$$

This enables us first to define the topology on $J^\infty T$ as the projective limit topology. We can easily check that the topology of $J^\infty_x T$ introduced above coincides with the topology induced from $J^\infty T$. Taking any open subset $U \subset M$ such that $T \mid U$ is trivial, we easily find that $(J^r T) \mid U$ is trivial for any $0 \leqq r < \infty$. The obvious formula $(J^\infty T) \mid U = \varprojlim (J^r T) \mid U$ allows us to introduce on $J^\infty T$ a differentiable $\mathbb{R}^\infty$-manifold structure and a structure of an $\mathbb{R}^\infty$-vector bundle.

There is an obvious way in which we can extend the notion of a (differentiable) homomorphism between vector bundles to a (differentiable) homomorphism between a vector bundle and an $\mathbb{R}^\infty$-vector bundle or to a (differentiable) homomorphism between $\mathbb{R}^\infty$-vector bundles. In all these cases we shall simply speak about a homomorphism. For example the natural projections $\pi^\infty_r\colon J^\infty T \to J^r T$, $0 \leqq r < \infty$ are homomorphisms.

One can easily prove the following lemma.

1. Lemma. *Let N be a differentiable manifold. A mapping $f\colon N \to J^\infty TM$ is differentiable if and only if for each $0 \leqq r < \infty$ the mapping $\pi^\infty_r \circ f\colon N \to J^r TM$ is differentiable.*

2. Definition. Let $p \geqq 0$ be an integer, and let $0 \leqq r \leqq \infty$. A *p-form ω of order $\leqq r$ on M* is a family $\omega = \{\omega_x\}_{x \in M}$ of p-forms with ω_x being a p-form on $J^r_x T$ for each $x \in M$.

Let ω be a p-form of order $\leqq r$ on M, and let $X_1, \ldots, X_p$ be differentiable vector fields defined on an open subset $U \subset M$. We can define a function $\omega(j^r X_1, \ldots, j^r X_p)$ on U by the formula

$$\omega(j^r X_1, \ldots, j^r X_p)(x) = \omega_x(j^r_x X_1, \ldots, j^r_x X_p) .$$

However, instead of $\omega(j^r X_1, \ldots, j^r X_p)$ we will use a simpler notation $\omega(X_1, \ldots, X_p)$.

3. Definition. A p-form ω of order $\leqq r$ on M is called *differentiable* if for any open subset $U \subset M$ and any differentiable vector fields $X_1, \ldots, X_p$ on U the function $\omega(X_1, \ldots, X_p)$ is differentiable on U.

Following the same lines as with the $\mathbb{R}^\infty$-vector bundle $J^\infty T$ we can define its p-th exterior power $\Lambda^p(J^\infty T)$. In fact we have here at least two possibilities. Either we define $\Lambda^p(J^\infty T)$ as $\bigcup_{x \in M} \Lambda^p(J_x^\infty T)$ or we use the projective system $\{\Lambda^p(J^r T); \Lambda^p \pi_r^s\}$ of vector bundles and define $\Lambda^p(J^\infty T) = \varprojlim \Lambda^p(J^r T)$. Both methods yield the same result, and it is easy to see that $\Lambda^p(J^\infty T)$ carries a natural structure of an $\mathbb{R}^\infty$-vector bundle.

We shall denote by E the trivial line bundle $M \times \mathbb{R}$ over M.

4. Lemma. *Let ω be a differentiable p-form of order $\leqq \infty$ on M, and let $x \in M$. Then there exists an open neighborhood U of x, an integer $0 \leqq r < \infty$, and a differentiable p-form $\tilde{\omega}$ of order $\leqq r$ on M such that $\omega \mid U = (\pi_r^\infty)^* (\tilde{\omega} \mid U)$.*

Proof. The mapping

$$(X_1, \ldots, X_p) \mapsto \omega(X_1, \ldots, X_p)$$

is obviously a multilinear mapping

$$\underbrace{\Gamma T \times \ldots \times \Gamma T}_{p\times} \to \Gamma E\,,$$

where Γ denotes the functor of all differentiable sections. It can be immediately seen that this multilinear mapping is local. Now it suffices to use the multilinear version of Peetre's theorem (see e.g. [2]), and the lemma follows.

Let ω be a p-form of order $\leqq r$ on M. This form in fact represents a mapping

$$\omega\colon \Lambda^p(J^r T) \to E\,.$$

Obviously $\omega_x\colon \Lambda^p(J_x^r T) \to E_x$ is a linear mapping for any $x \in M$, so that ω behaves almost like a homomorphism. Of course, ω need not be differentiable.

5. Lemma. *A p-form ω of order $\leqq r$ on M is differentiable if and only if the mapping $\omega\colon \Lambda^p(J^r T) \to E$ is a homomorphism.*

Proof. The assertion is well known if $0 \leqq r < \infty$. Thus it remains to prove it for $r = \infty$. If $\omega\colon \Lambda^p(J^\infty T) \to E$ is a homomorphism, then ω is obviously a differentiable p-form of order $\leqq \infty$. Conversely, let us suppose that ω is a differentiable p-form of order $\leqq \infty$, and let $x \in M$ be a point. Then by virtue of Lemma 4 there exists an open neighbourhood U of x, an integer $0 \leqq r < \infty$, and a differentiable p-form $\tilde{\omega}$ of order $\leqq r$ on M such that $\omega \mid U = (\pi_r^\infty)^* (\tilde{\omega} \mid U)$. But this shows clearly that the mapping $\omega\colon \Lambda^p(J^\infty T) \to E$ is a homomorphism, which completes the proof.

It is obvious that differentiable p-forms of order $\leqq r$, $0 \leqq r \leqq \infty$ on M form a real vector space which we denote by $\Phi_p^{(r)} = \Phi_p^{(r)}(M)$. We define $\Phi^{(r)} = \bigoplus_{p=0}^{\infty} \Phi_p^{(r)}$. With the usual $\wedge$-multiplication $\Phi^{(r)}$ is a graded algebra. The standard formula

$$(d\omega)_x (j_x^r X_1, \ldots, j_x^r X_{p+1}) =$$
$$= \{\sum_{i=1}^{p+1} (-1)^{i-1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j],$$
$$X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})\} (x),$$

where $X_1, \ldots, X_{p+1}$ are differentiable vector fields defined on an open neighborhood of x, defines a linear mapping d: $\Phi_p^{(r)} \to \Phi_{p+1}^{(r)}$. This is well known and can be easily verified for $r < \infty$. In the case $r = \infty$ it suffices to use the result for $r < \infty$ and the above mentioned multilinear version of Peetre's theorem. It can be easily checked that d is a differential on $\Phi^{(r)}$, and that $\Phi^{(r)}$ together with d is a differential graded algebra.

Let $0 \leqq r \leqq s \leqq \infty$. The homomorphism π_r^s: $J^sT \to J^rT$ induces a mapping π_r^{s*}: $\Phi^{(r)} \to \Phi^{(s)}$. It is easy to verify that this mapping is an injective homomorphism of differential graded algebras. Using the injective homomorphism $\pi_r^{\infty*}$, we shall identify $\Phi^{(r)}$ with $\pi_r^{\infty*}\Phi^{(r)}$. Under this identification $\Phi^{(r)}$, for any $0 \leqq r < \infty$, is a differential graded subalgebra of $\Phi^{(\infty)}$. Moreover, for $0 \leqq r \leqq s < \infty$ $\Phi^{(r)}$ is a differential graded subalgebra in $\Phi^{(s)}$.

6. Definition. Let $0 \leqq r$, $s \leqq \infty$. A *derivation of degree k on $\Phi^{(s)}$ with values in $\Phi^{(r)}$* is any real linear mapping D: $\Phi^{(s)} \to \Phi^{(r)}$ satisfying

(i) $D\Phi_p^{(s)} \subset \Phi_{p+k}^{(r)}$,

(ii) $D(\varphi_p \wedge \varphi_q) = D\varphi_p \wedge \varphi_q + (-1)^{kp} \varphi_p \wedge D\varphi_q$ for any $\varphi_p \in \Phi_p^{(s)}$, $\varphi_q \in \Phi_q^{(s)}$.

In the case $r = s$ we call D simply the derivation of degree k on $\Phi^{(r)}$. (We recall again that all elements appearing in this definition belong to $\Phi^{(\infty)}$. From this point of view we must understand the condition (ii).)

Using Lemma 4 we can, along the same lines as in [4], prove the following two lemmas.

7. Lemma. *Let D be a derivation on $\Phi^{(s)}$ with values in $\Phi^{(r)}$, $0 \leqq r$, $s \leqq \infty$, and let $\varphi, \psi \in \Phi_p^{(s)}$. If $\varphi \mid U = \psi \mid U$ with U being an open subset of M, then also $(D\varphi) \mid U = (D\psi) \mid U$.*

8. Lemma. *Any derivation on $\Phi^{(s)}$ with values in $\Phi^{(r)}$, $0 \leqq r$, $s \leqq \infty$, is uniquely determined by its values on $\Phi_0^{(s)}$ and $\Phi_1^{(s)}$.*

9. Corollary. *There are no nontrivial derivations of degree $\leqq -2$ on $\Phi^{(s)}$ with values in $\Phi^{(r)}$.*

10. Proposition. *If* $0 \leqq r < s \leqq \infty$ *and* $\dim M = m \geqq 2$, *then there are no nontrivial derivations on* $\Phi^{(s)}$ *with values in* $\Phi^{(r)}$.

Proof. Let D be a derivation of degree k on $\Phi^{(s)}$ with values in $\Phi^{(r)}$. According to the previous lemma it suffices to prove that $D \mid \Phi_0^{(s)} \oplus \Phi_1^{(s)} = 0$.

First let us consider an arbitrary function $f \in \Phi_0^{(s)}$, and let $x \in M$. We take a continuous 1-form α on $J_x^s T$ such that α is not a pullback $(\pi_r^s)^* \beta$ of any continuous 1-form β on $J_x^r T$. Obviously, there exists a 1-form $\varphi \in \Phi_1^{(s)}$ such that $\varphi_x = \alpha$. We have

$$D(f\varphi) = Df \wedge \varphi + f \, . \, D\varphi \, .$$

Because $D(f\varphi), f \, . \, D\varphi \in \Phi_{1+k}^{(r)}$, we have also $Df \, . \, \varphi \in \Phi_{1+k}^r$. But this obviously implies $(Df)_x = 0$. Because $x \in M$ was arbittary, we get $Df = 0$. (Let us notice that the assumption $\dim M \geqq 2$ have not been used here.)

Further, let $\varphi \in \Phi_1^{(s)}$ be arbitrary, and let $x \in M$. Because $\dim M \geqq 2$, we can find a continuous 1-form α on $J_x^s T$ such that α is not a pullback $(\pi_r^s)^* \beta$ of any continuous 1-form β on $J_x^r T$, and such that α and φ_x are linearly independent. Again there exists a 1-form $\psi \in \Phi_1^{(s)}$ such that $\psi_x = \alpha$. This time we have

$$D(\varphi \wedge \psi) = D\varphi \wedge \psi + (-1)^k \varphi \wedge D\psi \, ,$$

where $D(\varphi \wedge \psi) \in \Phi_{2+k}^{(r)}$, $D\varphi$, $D\psi \in \Phi_{1+k}^{(r)}$. If $\varphi_x \neq 0$ the 1-forms φ_x and ψ_x are linearly independent, hence the above equality yields $(D\varphi)_x = 0$. Because $x \in M$ was arbitrary, we get $D\varphi = 0$. This completes the proof.

The previous proposition is not valid if $m = 1$. Nevertheless, we shall see that even for $m = 1$ it holds very often.

If $m = 1$ then, because M is connected, we have only two possibilities, namely $M = \mathbb{R}$ or S^1. $\mathbb{R}$ and S^1 are commutative Lie groups, and therefore in both cases we can choose a nowhere vanishing invariant vector field X^0 on M. If X is a differentiable vector field on M, then there exists a unique differentiable function f on M such that $X = fX^0$. For any integer $0 \leqq r < \infty$ we shall define a 1-form $\omega^{(r)}$ of order $\leqq r$ as follows. Let $x \in M$, let $V \in J_x^r T$. We take any differentiable vector fied X on M such that $j_x^r X = V$. Then we define

$$\omega_x^{(r)}(V) = X_x^0 (\underbrace{X^0 \dots X^0}_{(r-1)\,\times}) f \, ,$$

where $X = fX^0$. It can be easily verified that this definition does not depend on the choice of X with the property $j_x^r X = V$, and that $\omega^{(r)}$ is a differentiable 1-form of order $\leqq r$ on M.

11. Proposition. *If* $0 \leqq r + 1 < s \leqq \infty$ *and* $m = 1$, *then there is no nontrivial derivation on* $\Phi^{(s)}$ *with values in* $\Phi^{(r)}$.

Proof. Let D be a derivation of degree k on $\Phi^{(s)}$ with values in $\Phi^{(r)}$. By virtue of the first part of the proof of Proposition 10 we have $Df = 0$ for any $f \in \Phi_0^{(s)}$. Further for any $0 \leqq i \leqq r$ we have

$$D(\omega^{(i)} \wedge \omega^{(s)}) = D\omega^{(i)} \wedge \omega^{(s)} + (-1)^k \omega^{(i)} \wedge D\omega^{(s)}$$

with $D(\omega^{(i)} \wedge \omega^{(s)})$, $\omega^{(i)} \wedge D\omega^{(s)} \in \Phi^{(r)}$. Consequently $D\omega^{(i)} \wedge \omega^{(s)} \in \Phi^{(r)}$, which implies

$$D\omega^{(i)} = 0\,, \quad 0 \leqq i \leqq r\,.$$

(Let us notice that this result holds under the assumption $0 \leqq r < s \leqq \infty$ only.)

Moreover, for any $r < j < s$ we have

$$D(\omega^{(j)} \wedge \omega^{(j+1)}) = D\omega^{(j)} \wedge \omega^{(j+1)} + (-1)^k \omega^{(j)} \wedge D\omega^{(j+1)}\,.$$

Because $D(\omega^{(j)} \wedge \omega^{(j+1)}) \in \Phi^{(r)}$, this equality implies $D\omega^{(j)} = D\omega^{(j+1)} = 0$. Thus we get

$$D\omega^{(j)} = 0\,, \quad r < j \leqq s \quad \text{if} \quad s < \infty\,,$$

$$D\omega^{(j)} = 0\,, \quad r < j \qquad \text{if} \quad s = \infty\,.$$

Any form $\varphi \in \Phi_1^{(s)}$ can be expressed (at least locally) as a finite linear combination of the forms $\omega^{(0)}, \omega^{(1)}, \ldots$ with coefficients from $\Phi_0^{(s)}$. Consequently $D\varphi = 0$, which completes the proof.

We denote by ${}^k\mathscr{R}^r_{r+1}$ the vector space of all derivations of degree k on $\Phi^{(r+1)}$ with values in $\Phi^{(r)}$. If $m = 1$, then for $D \in {}^k\mathscr{R}^r_{r+1}$ we define $\nu(D) \in \Phi^{(r)}_{k+1}$ by the formula $\nu(D) = D\omega^{(r+1)}$.

12. Proposition. *Let $m = 1$. Then $\nu\colon {}^k\mathscr{R}^r_{r+1} \to \Phi^{(r)}_{k+1}$ is an isomorphism of the vector space ${}^k\mathscr{R}^r_{r+1}$ of all derivations of degree k on $\Phi^{(r+1)}$ with values in $\Phi^{(r)}$ with the vector space $\Phi^{(r)}_{k+1}$ of all $(k+1)$-forms of degree $\leqq r$.*

Proof. We have proved in the first part of the proof of Proposition 10 that for any $D \in {}^k\mathscr{R}^r_{r+1}$ we have $Df = 0$ for $f \in \Phi_0^{(r+1)}$. Later on we have seen in the proof of Proposition 11 that for any $D \in {}^k\mathscr{R}^r_{r+1}$ we have $D\omega^{(0)} = \ldots = D\omega^{(r)} = 0$. Now using Lemma 8 we can immediately see that ν is injective.

Conversely, let $\varphi \in \Phi^{(r)}_{k+1}$. We define

$$Df = 0 \quad \text{for any} \quad f \in \Phi_0^{(r+1)}\,,$$

$$D(\omega^{(i_1)} \wedge \ldots \wedge \omega^{(i_p)}) = 0 \quad \text{for} \quad i_1 < \ldots < i_p < r + 1\,,$$

$$D(\omega^{(i_1)} \wedge \ldots \wedge \omega^{(i_p)}) = (-1)^{k(p-1)} \omega^{(i_1)} \wedge \ldots \wedge \omega^{(i_{p-1})} \wedge \varphi$$

for

$$i_1 < \ldots < i_{p-1} < i_p = r + 1.$$

It is easy to cheak that D can be uniquely extended to a derivation of order k on $\Phi^{(r+1)}$ with values in $\Phi^{(r)}$, and that $\nu(D) = \varphi$. This shows that ν is surjective.

Propositions 10, 11 and 12 give a complete description of derivations on $\Phi^{(s)}$ with values in $\Phi^{(r)}$ under the assumption $0 \leqq r < s \leqq \infty$. Therefore, from now on, we shall always assume that $0 \leqq s \leqq r \leqq \infty$. We denote by ${}^k\mathscr{R}^r_s$ the vector space of all derivations of degree k on $\Phi^{(s)}$ with values in $\Phi^{(r)}$. Setting $\mathscr{R}^r_s = \bigoplus_{k=-\infty}^{\infty} {}^k\mathscr{R}^r_s$ we obtain the graded vector space of all derivations on $\Phi^{(s)}$ with values in $\Phi^{(r)}$. If $r = s$ and $D \in {}^k\mathscr{R}^r_r$, $D' \in {}^l\mathscr{R}^r_r$ we define

$$[D, D'] = DD' - (-1)^{kl} D'D .$$

It is easy to verify that with this operation $\mathscr{R}^r_r$ is a graded Lie algebra.

Proceeding analogously as in [4] and using Lemma 4, we easily obtain the following lemma.

13. Lemma. *Any linear mapping* D: $\Phi^{(s)}_0 \oplus \Phi^{(s)}_1 \to \Phi^{(r)}$ *satisfying*

(i) $D\Phi^{(s)}_p \subset \Phi^{(r)}_{p+k}$, $p = 0, 1$,

(ii) $D(\varphi_p \wedge \varphi_q) = D\varphi_p \wedge \varphi_q + (-1)^{kp} \varphi_p \wedge D\varphi_q$ *for* $p + q \leqq 1$, $\varphi_p \in \Phi^{(s)}_p$, $\varphi_q \in \Phi^{(s)}_q$

can be extended to a derivation of degree k on $\Phi^{(s)}$ *with values in* $\Phi^{(r)}$.

Now we shall start the study of special derivations.

14. Definition. A derivation $D \in {}^k\mathscr{R}^r_s$ is called *a derivation of type* i_* if it satisfies $D(\Phi^{(s)}_0) = 0$.

We denote

$${}^k\mathscr{I}^r_s = \{D \in {}^k\mathscr{R}^r_s;\ D \text{ is of type } i_*\} .$$

${}^k\mathscr{I}^r_s$ is a subspace of ${}^k\mathscr{R}^r_s$. The graded vector space $\mathscr{I}^r_s = \bigoplus_{k=-\infty}^{\infty} {}^k\mathscr{I}^r_s$ is a homogeneous subspace of $\mathscr{R}^r_s$.

15. Definition. Let $p \geqq 0$ be an integer. *A* J^sT*-valued p-form* L *of order* $\leqq r$ *on* M is a family $L = \{L_x\}_{x \in M}$, where L_x is a J^s_xT-valued p-form on J^r_xT for each $x \in M$.

Let L be a J^sT-valued p-form of order $\leqq r$ on M, and let $X_1, \ldots, X_p$ be differentiable vector fields defined on an open subset $U \subset M$. We can define a section $L(j^rX_1, \ldots, j^rX_p)$ of J^sT on U by the formula

$$L(j^rX_1, \ldots, j^rX_p)(x) = L_x(j^r_xX_1, \ldots, j^r_xX_p) .$$

Instead of $L(j^rX_1, \ldots, j^rX_p)$ we shall again use a simpler notation $L(X_1, \ldots, X_p)$.

16. Definition. A J^sT-valued p-form L of order $\leqq r$ on M is called *differentiable* if for any open subset $U \subset M$ and any differentiable vector fields $X_1, \ldots, X_p$ on U the section $L(X_1, \ldots, X_p)$ is differentiable on U.

Any J^sT-valued p-form L of order $\leqq r$ on M defines a mapping

$$L\colon \Lambda^p(J^rT) \to J^sT.$$

The following lemma can be proved in the same way as Lemma 5.

17. Lemma. *A J^sT-valued p-form L of order $\leqq r$ on M is differentiable if and only if the mapping $L\colon \Lambda^p(J^rT) \to J^sT$ is a homomorphism.*

On the set of all differentiable J^sT-valued p-forms of order $\leqq r$ on M there is a natural vector space structure. We shall denote this vector space by $\mathscr{L}_p^{(r)(s)}$.

Let $\varphi \in \Phi_p^{(s)}$ and $L \in \mathscr{L}_{k+1}^{(r)(s)}$. We define a $(p + k)$-form $\varphi \barwedge L$ of order $\leqq r$ on M by the formula

$$(\varphi \barwedge L)_x (V_1, \dots, V_{p+k}) =$$

$$= \frac{1}{(p-1)!\,(k+1)!} \sum_\alpha \operatorname{sg} \alpha \,.\, \varphi_x(L_x(V_{\alpha_1}, \dots, V_{\alpha_{k+1}}), \pi V_{\alpha_{k+2}}, \dots, \pi V_{\alpha_{p+k}}).$$

Here $V_1, \dots, V_{p+k} \in J_x^rT$, and $\pi = \pi_s^r$. The sum is taken over all permutations α of $p + k$ elements, and $\operatorname{sg} \alpha$ denotes the sign of α. (If $p = 0$ we define $\varphi \barwedge L = 0$.) Obviously $\varphi \barwedge L$ is a differentiable $(p + k)$-form of order $\leqq r$ on M, i.e. $\varphi \barwedge L \in \Phi_{p+k}^{(r)}$. If $\varphi \in \Phi_p^{(s)}$, $\psi \in \Phi_q^{(s)}$ it can be proved by computation that

$$(\varphi \wedge \psi) \barwedge L = (\varphi \barwedge L) \wedge \pi_s^{r*}\psi + (-1)^{kp}\, \pi_s^{r*}\varphi \wedge (\psi \barwedge L).$$

(Due to our identifications we can omit the π_s^{r*}'s in this formula.) For $L \in \mathscr{L}_{k+1}^{(r)(s)}$ we define

$$i_L\varphi = \varphi \barwedge L.$$

The above formula shows that i_L is a derivation of degree k on $\Phi^{(s)}$ with values in $\Phi^{(r)}$.

18. Proposition. *Any derivation D of degree $k \geqq -1$ on $\Phi^{(s)}$ with values in $\Phi^{(r)}$ of type i_* can be uniquely expressed in the form $D = i_L$, where $L \in \mathscr{L}_{k+1}^{(r)(s)}$.*

Proof. We shall first consider the case $0 \leqq s < \infty$ and $0 \leqq r \leqq \infty$. Let $U \subset M$ be an open subset diffeomorphic with an open ball in $\mathbb{R}^m$, and let $u_1, \dots, u_a$ be a local basis of J^sT on U. Let $\varphi_1, \dots, \varphi_a$ be the corresponding dual basis of $(J^sT)^*$ on U. We define L on U by the formula

$$L = \sum_{i=1}^{a} D\varphi_i \,.\, u_i\,.$$

One can check that this definition does not depend on the choice of the basis. Then it is easy to see that we get $L \in \mathscr{L}_{k+1}^{(r)(s)}$ which has the desired property.

It remains to consider the case $r = s = \infty$. Let again $U \subset M$ be an open subset diffeomorphic with an open ball in $\mathbb{R}^m$. For any integer $0 \leqq t < \infty$ we fix a local basis $u_1^{(t)}, \dots, u_{a_t}^{(t)}$ of J^tT on U in such a way that

$$\pi_t^{t+1} u_i^{(t+1)} = u_i^{(t)} \quad \text{for} \quad 1 \leqq i \leqq a_t,$$

$$\pi_t^{t+1} u_i^{(t+1)} = 0 \quad \text{for} \quad a_t + 1 \leqq i \leqq a_{t+1}.$$

We denote again by $\varphi_i^{(t)}$ the basis of $(J^t T)$ on U dual to $u_1^{(t)}, \ldots, u_{a_t}^{(t)}$. The sequence

$$\{L_t = \sum_{i=1}^{a_t} D\varphi_i^{(t)} \cdot u_i^{(t)}\}_{t=0}^{\infty}$$

obviously defines a $J^\infty T$-valued $(k + 1)$-form of order $\leqq \infty$ on U. Now it is easy to check that in this way we obtain an element $L \in \mathscr{L}_{k+1}^{(\infty)(\infty)}$ with the required property.

The next lemma can be proved along the same lines as Lemma 4.

19. Lemma. *Let L be a differentiable T-valued p-form of order $\leqq \infty$ on M, and let $x \in M$. Then there exists an open neighborhood U of x, an integer $0 \leqq r < \infty$, and a differentiable T-valued p-form $\tilde{L}$ of order $\leqq r$ on M such that $L \mid U = (\pi_r^\infty)^* (\tilde{L} \mid U)$.*

Let $L \in \mathscr{L}_{k+1}^{(r)(0)}$, and let $V_1, \ldots, V_{k+1} \in J_x^{r+l}$, where $0 \leqq r$, $l \leqq \infty$. We take differentiable vector fields $X_1, \ldots, X_{k+1}$ defined on an open neighborhood U of x such that $j_x^{r+l} X_i = V_i$, $1 \leqq i \leqq k + 1$. Obviously $L(X_1, \ldots, X_{k+1})$ is a differentiable vector field on U. We define a $J^l T$-valued $(k + 1)$-form $(\tilde{\mu}^{(l)} L)_x$ on $J_x^{r+l} T$ by the formula

$$(\tilde{\mu}^{(l)} L)_x (V_1, \ldots, V_{k+1}) = j_x^l (L(X_1, \ldots, X_{k+1})).$$

It can be easily verified (using the above lemma if $r = \infty$) that this definition does not depend on the choice of the vector fields $X_1, \ldots, X_{k+1}$. Moreover, it is obvious that the family $\tilde{\mu}^{(l)} L = \{(\tilde{\mu}^{(l)} L)_x\}_{x \in M}$ is a differentiable $J^l T$-valued $(k + 1)$-form of order $\leqq r + l$ on M. In this way we obtain a linear mapping

$$\tilde{\mu}^{(l)} \colon \mathscr{L}_{k+1}^{(r)(0)} \to \mathscr{L}_{k+1}^{(r+l)(l)}$$

$$L \mapsto \tilde{\mu}^{(l)} L.$$

It can be easily seen that this linear mapping is injective. Using Proposition 18 we can modify $\tilde{\mu}^{(l)}$ in order to obtain an injective linear mapping

$$\mu^{(l)} \colon {}^k\mathscr{I}_0^r \to {}^k\mathscr{I}_l^{r+l}$$

$$i_L \mapsto i_{\tilde{\mu}^{(l)} L}.$$

20. Definition. Let $D \in {}^k\mathscr{R}_s^r$. We shall say that D is *a derivation of type d_** if it satisfies $Dd - (-1)^k dD = 0$. (We shall abbreviate $[D, d] = Dd - (-1)^k dD$. The differential d was defined after Lemma 5.)

It is easy to see that the set ${}^k\mathscr{D}_s^r$ of all derivations of degree k and type d_* on $\Phi^{(s)}$ with values in $\Phi^{(r)}$ carries a natural structure of a vector space. In the sequel we denote

$$\mathscr{D}_s^r = \bigoplus_{k=-\infty}^{\infty} {}^k\mathscr{D}_s^r.$$

21. Lemma. *Any derivation of type d_* on $\Phi^{(0)}$ with values in $\Phi^{(r)}$ is uniquely determined by its values on $\Phi_0^{(0)}$.*

Proof is easy.

22. Corollary. *There are no nontrivial derivations of degree $k \leqq -1$ and type d_* on $\Phi^{(0)}$ with values in $\Phi^{(r)}$.*

23. Lemma. *Any linear mapping $D: \Phi_0^{(0)} \to \Phi^{(r)}$ satisfying*

(i) $D\Phi_0^{(0)} \subset \Phi_k^{(r)}$,

(ii) $D(fg) = Df \,.\, g + f \,.\, Dg$ *for* $f, g \in \Phi_0^{(0)}$

can be extended to a derivation of degree k and type d_* on $\Phi^{(0)}$ with values in $\Phi^{(r)}$.

Proof follows the lines of the proof of an analogous lemma in [4].

24. Lemma. *Let $D \in {}^k\mathscr{R}_s^r$ be a derivation. Then $[D, d] \in {}^{k+1}\mathscr{R}_s^r$ is a derivation of type d_*.*

Proof is straightforward.

Let D be a derivation of degree k on $\Phi^{(0)}$ with values in $\Phi^{(r)}$. For $f \in \Phi_0^{(0)}$ and $V_1, \ldots, V_k \in J_x^r T$ we define

$$\sigma(D)(V_1, \ldots, V_k) f = (Df)(V_1, \ldots, V_k) .$$

Obviously $\sigma(D)(V_1, \ldots, V_k) \in T_x$. In this way we obtain a differentiable T-valued k-form of order $\leqq r$, which we denote by $\sigma(D)$. Thus we have a homomorphism

$$\sigma: {}^k\mathscr{R}_0^r \to \mathscr{L}_k^{(r)(0)}$$
$$D \mapsto \sigma(D) .$$

We can now define a homomorphism

$$\eta: {}^k\mathscr{R}_0^r \to {}^k\mathscr{D}_0^r$$

by the formula $\eta(D) = [i_{\sigma(D)}, d]$.

25. Proposition. *η is a projector and* $\ker \eta = {}^k\mathscr{I}_0^r$, $\operatorname{im} \eta = {}^k\mathscr{D}_0^r$. *Consequently* ${}^k\mathscr{R}_0^r = {}^k\mathscr{I}_0^r \oplus {}^k\mathscr{D}_0^r$.

Proof. Let $D \in {}^k\mathscr{R}_0^r$, $f \in \Phi_0^{(0)}$, and let $V_1, \ldots, V_k \in J_x^r T$. We get

$$\begin{aligned}(\eta(D) f)(V_1, \ldots, V_k) &= ([i_{\sigma(D)}, d] f)(V_1, \ldots, V_k) = \\ &= (i_{\sigma(D)} df)(V_1, \ldots, V_k) = df(\sigma(D)(V_1, \ldots, V_k)) = \\ &= \sigma(D)(V_1, \ldots, V_k) f = Df(V_1, \ldots, V_k) .\end{aligned}$$

If $D \in {}^k\mathscr{D}_0^r$ then by virtue of Lemma 21 this implies that $\eta(D) = D$. We have thus proved that η is a projector and that $\operatorname{im} \eta = {}^k\mathscr{D}_0^r$. Moreover, the above formula also shows that $\ker \eta = {}^k\mathscr{I}_0^r$. The rest of the proof is obvious.

The previous result enables us to extend the linear mapping $\mu^{(\infty)}: {}^k\mathscr{I}_0^\infty \to {}^k\mathscr{I}_\infty^\infty$ defined before. Let $D \in {}^k\mathscr{R}_0^\infty$. Then there are uniquely determined $D_1 \in {}^k\mathscr{I}_0^\infty$, $D_2 \in {}^k\mathscr{D}_0^\infty$ such that $D = D_1 + D_2$. Moreover, $D_2 = [i_{\sigma(D_2)}, d]$. We define a linear mapping $\hat{\mu}^{(\infty)}: {}^k\mathscr{R}_0^\infty \to {}^k\mathscr{R}_\infty^\infty$ by the formula

$$\hat{\mu}^{(\infty)}(D) = \mu^{(\infty)}(D_1) + [i_{\hat{\mu}^{(\infty)}\sigma(D_2)}, d] .$$

Obviously $\hat{\mu}^{(\infty)} \mid {}^k\mathscr{I}_0^\infty = \mu^{(\infty)}$ and $\hat{\mu}^{(\infty)}({}^k\mathscr{D}_0^\infty) \subset {}^k\mathscr{D}_\infty^\infty$. Furthermore, it can be easily seen that for any $D \in {}^k\mathscr{R}_0^\infty$ we have

$$\hat{\mu}^{(\infty)}(D) \mid \Phi^{(0)} = D ,$$

which shows that the linear mapping $\hat{\mu}^{(\infty)}$ is injective. From now on we shall write simply $\mu^{(\infty)}$ instead of $\hat{\mu}^{(\infty)}$.

26. Definition. A derivation $D \in {}^k\mathscr{R}_\infty^\infty$ is called *a simple derivation* if $D \in \operatorname{im} \mu^{(\infty)}$. A derivation $D \in {}^k\mathscr{R}_\infty^\infty$ is called *a lifting derivation* if $D \mid \Phi^{(0)} = 0$.

Let us notice that there exists a natural projection $\pi_0^\infty: \mathscr{L}_{k+1}^{(\infty)(\infty)} \to \mathscr{L}_{k+1}^{(\infty)(0)}$. Using this projection we can formulate the following proposition.

27. Proposition. *A derivation $D \in {}^k\mathscr{R}_\infty^\infty$ is a lifting derivation if and only if D is a derivation of type i_* and $D = i_L$, $L \in \mathscr{L}_{k+1}^{(\infty)(\infty)}$ with $\pi_0^\infty L = 0$.*

Proof is obvious.

28. Theorem. *Any derivation $D \in {}^k\mathscr{R}_\infty^\infty$ can be uniquely expressed in the form*

$$D = \mu^{(\infty)}(D_1^{(0)}) + \mu^{(\infty)}(D_2^{(0)}) + D^{(\infty)} ,$$

where $D_1^{(0)} \in {}^k\mathscr{I}_0^\infty$, $D_2^{(0)} \in {}^k\mathscr{D}_0^\infty$, and $D^{(\infty)}$ is a lifting derivation.

Proof follows from the previous considerations.

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Souhrn

DERIVACE NA ALGEBŘE DIFERENCIÁLNÍCH FOREM NEKONEČNÉHO ŘÁDU NA VARIETĚ

JAROSLAV CARBOL, JIŘÍ VANŽURA

V práci se uvažuje algebra diferenciálních forem vyšších řádů na diferencovatelné varietě. Jsou popsány všechny derivace na této algebře a je zkoumána jejich struktura.

Резюме

ДИФФЕРЕНЦИРОВАНИЯ НА АЛГЕБРЕ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ БЕСКОНЕЧНОГО ПОРЯДКА НА МНОГООБРАЗИИ

JAROSLAV CARBOL, JIŘÍ VANŽURA

В работе рассматривается алгебра дифференциальных форм высших порядков на дифференцируемом многообразии. Описаны все дифференцирования на этой алгебре и изучена их структура.

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